Homework #4

For Friday, February 19

1. Read through section 1.2 of van Dalen and section 3.5 of Avigad. Read section (1.3 and) 1.4 of the Enderton handout, which discusses unique readability for propositional formulas.

2. Write down explicit definitions of the functions f and g, where
   a. f is defined recursively by $f(0) = 0$, $f(n+1) = 3 + f(n)$, and
   b. g is defined recursively by $g(0) = 1$, $g(n+1) = (n+1)^2 g(n)$. (Hint: use "factorial" notation: $m! = 1 \times 2 \times \ldots \times m$.)

⋆ 3. Write down an explicit definition of the function h, where $h(0) = 0$ and $h(n+1) = 3 \cdot h(n) + 1$. (Hint: compare to the sequence $1, 3, 9, 27, 81, \ldots$.) Use induction to prove that your formula is correct.

4. Suppose g is a function from $\mathbb{N}$ to $\mathbb{N}$. Write down a recursive definition of the function $f(n)$, defined by $f(n) = \sum_{i=0}^{n} g(i)$.

5. Do problem 1 on page 30 of the Enderton handout.

∘ 6. Do problem 3 on page 30 of the Enderton handout.

⋆ 7. Suppose, as in Section 2.2 of the notes, we are given a set U, a subset $B \subseteq U$, and some functions $f_1, \ldots, f_k$. Say a set is *inductive* if it contains B and is closed under the f_i's, and let C^* be the intersection of all the inductive subsets of U. Show C^* is inductive.

⋆ 8. Let U, B, and $f_1, \ldots, f_k$ be as in the previous problem. Recall that a *formation sequence* (or, in Enderton's terminology, a *construction sequence*), is a sequence of elements $\langle a_1, a_2, \ldots, a_k \rangle$ of elements of U, such that for each i, either a_i is in B, for there is a function f_l on the list and indices $j_1, \ldots, j_m$ less than i such that $a_i = f_l(a_{j_1}, \ldots, a_{j_m})$. A formation sequence for an element a is simply a formation sequence ending with a.

   Show, carefully, that if f_j is any function on the list and there are formation sequences for $a_1, \ldots, a_m$, then there is a formation sequence for $f_j(a_1, \ldots, a_m)$. You may use any basic properties of concatenation of sequences, etc.

⋆ 9. Define the set of "babble-strings" inductively, as follows:

- “ba” is a babble-string
- if s is a babble-string, so is “ab”ˆs
- if s and t are babble-strings, so is sˆt

Prove by induction that every babble-string has the same number of a's and b's, and that every babble-string ends with an “a”. Is the set of babble-strings freely generated? (Justify your answer.)

10. Consider the following inductive definition of the set of all “AB-strings”:

    - $\emptyset$, the empty string, is an ab-string
    - if s is an AB-string, so is $f_1(s)$
    - if s is an AB-string, so is $f_2(s)$.

    In the “correct” interpretation, the underlying set U is a set of strings, and f_1 and f_2 are functions that prepend the letters “A” and “B” respectively. However, if instead we take U' to be the set of strings of stars (e.g. “*****”), let f_1 be a function that prepends one star, and let f_2 be the function that prepends two stars, then the smallest subset of U' the contains $\emptyset$ and is closed under f_1 and f_2 is *not* generated freely.

    Come up with better functions f_1 and f_2, so that they still act on the underlying set U', but make the resulting set of “ab-strings” freely generated.

⋆ 11. Recall the definition of “arithmetic expressions” on p. 14 in Avigad:

    - any string of digits that doesn't start with “0” is an arithmetic expression
    - if s and t are arithmetic expressions, so is “$(s+t)$” (more precisely, “(”ˆsˆ“+”ˆtˆ“)”)
    - if s and t are arithmetic expressions, so is “$(s \times t)$”.

    Let $length(s)$ denote the length of s, and let $val(s)$ denote the evaluation function defined on p. 16 in Avigad. Prove by induction that for every expression s, the inequality $val(s) \leq 10^{length(s)}$ holds.

◦ 12. What would happen to the previous theorem if we were to add exponentiation, $a \uparrow b$?