Homework #10

For Tuesday, *March 27*

⋆ 1. Consider the equivalence

$$\exists x\ (\varphi(x) \wedge \psi(x)) \leftrightarrow (\exists x\ \varphi(x) \wedge \exists x\ \psi(x)).$$

You can assume that φ and ψ have no free variable other than x.

a. Show that one direction of this equivalence is valid (i.e. true in every structure). Prove this carefully; you can use Lemma 2.4.5.

b. Find examples of φ and ψ where the other direction is not valid (and justify this claim).

⋆ 2. Fix a language, L, which has one binary relation symbol, R. Which of the following statements are true and which are false? Justify your answers.

a. If φ is any sentence, either $\models \varphi$ or $\models \neg\varphi$.

b. If φ is any sentence and $\mathfrak{A}$ is any structure, either $\mathfrak{A} \models \varphi$ or $\mathfrak{A} \models \neg\varphi$.

c. If φ is any sentence and Γ is any set of sentences, then either $\Gamma \models \varphi$ or $\Gamma \models \neg\varphi$.

3. Do problems 6 and 7 on page 72.

4. Do problems 1, 2, and 3 on page 80.

5. (**Oblig**) Consider the equivalence

$$\forall x \exists y\ \varphi(x, y) \leftrightarrow \exists y\ \forall x \varphi(x, y).$$

You can assume that φ has no free variables other than x and y.

a. Show that one direction of this equivalence is valid (i.e. true in every structure). Prove this carefully.

b. Show that the other direction is not valid (construct a counterexample, be careful and precise).

6. Do problem 5 on page 80. Note that this relies on the convention that we do not consider structures with empty universes.

7. Do problem 6 on page 80.

8. Do problem 12 on page 81. You can use any of the lemmas and theorems in section 2.5 to make your argument as clear as possible. Note that the barber paradox reads as follows: "In a certain town the barber shaves all and exactly those people who do not shave themselves. Who shaves the barber?"

$\star$ 9. Do problem 14c on page 81, assuming φ, ψ, and σ are quantifier-free.

10. Do problem 15 on page 81.

11. Consider the language of orderings with a single binary relation symbol $\leq$. Use $x < y$ as shorthand for $x \leq y \wedge \neg(x = y)$. An element a in a partial ordering $\mathcal{P}$ is *minimal* if there is no element smaller than it; that is, $\mathcal{P} \models \forall x\ \neg(x < \bar{a})$. An element a is *minimum* if it is at least as small as every other element; that is, $\mathcal{P} \models \forall x\ (\bar{a} \leq x)$.

    a. Describe a partial ordering with a minimal element, but no minimum element. (You can use a diagram.)

    b. Show that in a linear ordering, an element is minimum iff it is minimal.

    c. Describe a linear ordering with no minimal element.

    You can argue informally about what is "true in $\mathcal{P}$," without having to use Lemma 2.4.5 explicitly; but your argument should be mathematically rigorous.