Homework #2

For Thursday, February 2

Det er ikke meningen at man nødvendigvis skal gjøre alle, men kikk på alle, de kan dukke opp på eksamen. Det anbefales at man i det minste gjør de som er markert med stjerne ($\star$). Sirkel ($\circ$) betyr "mer for moro". De som står markert som Oblig gjennomgås ikke på gruppetimene, og kommer til å komme som obligoppgave. Hvis ingen har noe i mot det så kjører vi på engelsk (om man selv vil skrive besvarelsene sine på norsk eller engelsk er naturligvis valgfritt).

1. Read Avigad chapter 2 and the Enderton handout.

2. Use the least element principle to prove the induction principle, and vice-versa.

$\star$ 3. Prove by induction that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$. Keep in mind that $\sum_{i=0}^{n} 2^i$ is an abbrevation for $2^0 + 2^1 + 2^2 + \ldots + 2^n$.

$\circ$ 4. Can you find a formula for $\sum_{i=0}^{n} i^2$?

5. (**Oblig**) Prove by induction that whenever $n \geq 4$, $n! > 2^n$. Recall that $n!$, read "n factorial," is defined to be $n \cdot (n-1) \cdot \ldots \cdot 1$.

6. Prove by induction that whenever $n \geq 5$, $2^n > n^2$. (Hint: you will have to prove an auxilliary statement first.)

$\star$ 7. A "binary string of length n" is a sequence of n 0's and 1's; for example, 011101 is a binary string of length 6. Prove by induction that for every n there are 2^n binary strings of length n. How many binary strings are there having length *at most* n? Justify your answer.

8. Prove that there are 2^n subsets of a set having n elements. (Hint: you can use the preceding problem.)

9. (**Oblig**) Suppose, as in Section 2.2 of the notes, we are given a set U, a subset $B \subseteq U$, and some functions $f_1, \ldots, f_k$. Say a set is *inductive* if it contains B and is closed under the f_i's, and let C^* be the intersection of all the inductive subsets of U. Show C^* is inductive.

⋆ 10. Let U, B, and $f_1, \ldots, f_k$ be as in the previous problem. Recall that a *formation sequence* (or, in Enderton's terminology, a *construction sequence*), is a sequence of elements $\langle a_1, a_2, \ldots, a_k \rangle$ of elements of U, such that for each i, either a_i is in B, for there is a function f_l on the list and indices $j_1, \ldots, j_m$ less than i such that $a_i = f_l(a_{j_1}, \ldots, a_{j_m})$. A formation sequence for an element a is simply a formation sequence ending with a.

Show, carefully, that if f_j is any function on the list and there are formation sequences for $a_1, \ldots, a_m$, then there is a formation sequence for $f_j(a_1, \ldots, a_m)$. You may use any basic properties of concatenation of sequences, etc.

⋆ 11. Let "HiLo" be the following children's game: Player 1 picks a natural number between 1 and M (inclusive), and Player 2 tries to guess it. After each incorrect guess, Player 1 responds "higher" or "lower." Assuming Player 2 has n guesses, what is the largest value of M for which there is an algorithm that guarantees success? Describe the algorithm, and use induction to prove that it works.

∘ 12. Show that the algorithm you gave in response to the previous question is optimal, i.e. for larger values of M there will be numbers for which the algorithm fails to determine the correct number after n guesses.

13. Use the least element principle to prove that all numbers are interesting.