Homework #7

For Tuesday, March 6

Problem 1 is Oblig. Problems 2–6 have to do with a more explicit proof of the restricted version of the completeness theorem: if $\models \varphi$, then $\vdash \varphi$. Problem 7 is relevant for the next lecture, so have a look at it.

1. (**Oblig**) The purpose of this exercise is to show that the RAA rule is equivalent to the Law of Excluded Middle (LEM) and to Double Negation Elimination (DNE). Use our usual system where we treat $\bot$, $\vee$, $\wedge$, $\rightarrow$ as primitive and $\neg$, $\leftrightarrow$ as defined.
   a. DNE is the rule that from $\neg\neg\phi$ you can conclude ϕ. Use this rule together with the other rules (but not RAA) to show RAA, that is, show that if you have a deduction with $\neg\phi$ as a premise and $\bot$ as the conclusion, then you can produce a proof using the new DNE rule which has $\neg\phi$ as a *cancelled* premise and ϕ as the conclusion.
   b. Using the RAA rule, show $\vdash \phi \vee \neg\phi$ (Hint: Show first that
   $$\{\neg(\phi \vee \neg\phi), \phi\} \vdash \neg\phi)$$
   .
   c. LEM is the rule that from anything (or nothing) you can conclude $\phi \vee \neg\phi$. Show that from LEM we can get DNE by showing, without using the RAA rule, that $\phi \vee \neg\phi \vdash \neg\neg\phi \rightarrow \phi$.

∘ 2. Using the proof of Theorem 1.3.8 in van Dalen, describe an algorithm that converts any formula φ to formulas $\varphi^{\vee}$ and $\varphi^{\wedge}$, in disjunctive and conjunctive normal form, respectively.

3. a. Remember that a formula φ is *satisfiable* if there is a truth assignment v such that $[\![\varphi]\!]_v = 1$. φ is *unsatisfiable* if it is not satisfiable. Show that for any formula φ, φ is unsatisfiable if and only if $\models \neg\varphi$.
   b. Now suppose φ is of the form
   $$\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_k$$
   in disjunctive normal form, so that each formula φ_i is a conjunction of atomic formulas and their negations. Show that φ is satisfiable if and only if one of the conjunctions φ_i does not contain an atomic formula p_j together with its negation $\neg p_j$.

c. Use this, together with the previous problem, to give an algorithm to determine whether or not a formula φ is valid.

∘ 4. This problem outlines a more constructive approach to the following special case of the completeness theorem: if $\models \varphi$, then $\vdash \varphi$.

a. Modify the algorithm of problem 5 so that given φ, it outputs not only a formula $\varphi^\wedge$, but also a proof of $\varphi \leftrightarrow \varphi^\wedge$.

b. Show that if $\varphi^\wedge$ is valid, it is easy to prove. (The previous problem is relevant.)

∘ 5. If φ is any formula, show that

$$length(\varphi^\vee) \leq 2^{length(\varphi)+3}.$$

(Use the definition of $\varphi^\vee$ implicit in Theorem 1.3.9 on page 26.)

∘ 6.

a. Assuming φ has length n, what is the worst-case running time of the algorithm in part (c) of the problem 6?

b. Another way to determine if a formula φ is a tautology is to compute its value on every truth assignment (the "truth table" method). What is the worst-case running time of this algorithm?

c. Come up with a polynomial-time algorithm for determining if a propositional formula φ is a tautology or not, or prove that no such algorithm exists. (Note: a successful solution to this problem amounts to settling the famous open question, $P = NP$?.)

7. Treating the connectives $\wedge$, $\vee$, $\to$, and $\bot$ as basic (i.e. $\vee$ is not defined from other connectives, but $\neg$, $\leftrightarrow$, and $\top$ are), we get intuitionistic (propositional) logic by deleting RAA from our set of inference rules. The remaining rules, then, are $\wedge$ intro and elim; $\vee$ intro and elim; $\to$ intro and elim; and $\bot$ elim (that from $\bot$ you can conclude anything). We show that RAA does not follow from these rules by constructing a sound semantics for intuitionistic logic in which RAA does not hold. Let I be the closed unit interval,

$$I = [0, 1] = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$$

a. Given a valuation

$$v : \{p_0, p_1, \ldots\} \to I$$

justify briefly that there is a unique function

$$[\![-]\!]_v : \text{PROP} \to I$$

such that

$$[\![\bot]\!]_v = 0 \tag{1}$$

$$[\![\phi \wedge \psi]\!]_v = min\{[\![\phi]\!]_v, [\![\psi]\!]_v\} \tag{2}$$

$$[\![\phi \vee \psi]\!]_v = max\{[\![\phi]\!]_v, [\![\psi]\!]_v\} \tag{3}$$

$$[\![\phi \to \psi]\!]_v = \begin{cases} 1, & \text{if } [\![\phi]\!]_v \leq [\![\psi]\!]_v \\ [\![\psi]\!]_v, & \text{otherwise} \end{cases} \tag{4}$$

b. Define *semantic consequence with respect to valuations in* I by

$$\{\phi_1, \ldots, \phi_n\} \vDash^I \psi \Leftrightarrow \text{if } min\{\phi_1, \ldots, \phi_n\} \leq [\![\psi]\!]_v$$

for all valuations $v : \{p_0, p_1, \ldots\} \to I$, where Γ is a finite set of formulas. (Accordingly, ϕ is valid, $\vDash^I \phi$, iff $[\![\phi]\!]_v = 1$ for all valuations v.) Show by induction on derivations that this defines a sound semantics for intuitionistic logic. The induction step has one case for each derivation rule, do at least two such cases. (The $\vee$-elim case is tricky, so you might want to avoid that one.)

∘ c. Do all cases in the induction proof above.

d. Use the definition of $[\![-]\!]_v$ to compute $[\![\phi \vee \neg\phi]\!]_v$. Conclude that RAA does not follow from the other deduction rules.

e. Is this semantics complete? That is, is it the case that

$$\Gamma \vDash^I \phi \Rightarrow \Gamma \vdash \phi$$

for Γ a finite set of formulas? Justify your answer.