UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	INF3470/4470 — Digital signal processing
Day of examination:	December 12th, 2013
Examination hours:	14.30 – 18.30

This problem set consists of 7 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 The z-transform

a) Find the z-transform and Region of Convergence (ROC) for the two-sided data sequence:

$$x[n] = \begin{cases} a^n & \text{for} \quad n \geq 0, \\ -b^n & \text{for} \quad n < 0, \end{cases} \qquad n \in \mathbb{Z},$$

where $\mathbb{Z}$ represents integers. 1 p.

b) An impulse response $h[n]$ has the z-transform:

$$H(z) = \frac{1 + z^{-1}}{(1 - 2z^{-1})(1 - 0.5z^{-1})}.$$

Draw a pole-zero plot for this transform. 1 p.

c) $H(z)$ in **b)** can be causal, anti-causal or two-sided. Sketch the four (and not three) possibilities this leads to for the ROC, and determine stability in each case. 2 p.

d) Why is stability so important in filter design? 1 p.

e) One of the z-transform properties is the one on time reversal:

$$\begin{array}{llll} \text{If} & x[n] \overset{Z}{\longleftrightarrow} X(z) & \text{with} & \text{ROC:} \quad r_1 < |z| < r_2 \\ \text{then} & x[-n] \overset{Z}{\longleftrightarrow} X(z^{-1}) & \text{with} & \text{ROC:} \ 1/r_2 < |z| < 1/r_1. \end{array}$$

Use the z-transform definition to show that this is correct. How is $X(z)$ affected when $x[n]$ is symmetric? 1 p.

(Continued on page 2.)

Problem 2 Conversion of sample rate

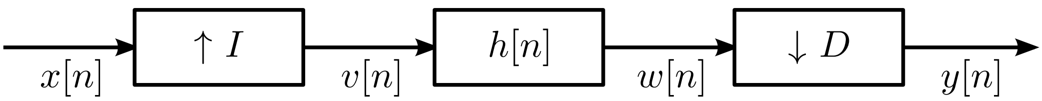

Figure 1: A system for rate conversion

Figure 1 shows a system for up- and down-sampling of discrete signals, also called rate conversion. Let us assume that $x[n]$ is sampled from a continuous signal. If for example $I = 2$ then $v[n]$ will contain an extra sample in between each of the samples in $x[n]$.

$h[n]$ is a perfect low-pass filter with a cut-off frequency f_c as high as is possible without causing convolution error (frequency aliasing). This means that f_c depends on I.

The last filter down-samples the signal. If for example $D = 3$ then $y[n]$ will only contain every third sample from $w[n]$.

In short the system shown in Figure 1 will change the sampling frequency f_s to $(I/D)f_s$ where f_s has the unit Hz. Assume that the input signal $x[n]$ has a magnitude response as shown in Figure 2 (note that we are not using a normalized frequency here) and that $f_s = 12$ kHz.

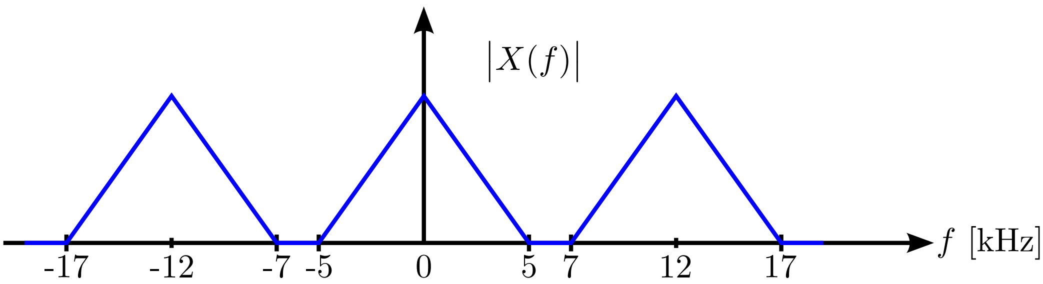

Figure 2: The magnitude response of $x[n]$

2a

Sketch frequency spectra for $v[n]$, $w[n]$ and $y[n]$, and the magnitude response (including the cut-off frequency) for $h[n]$ in the following two cases:

1. $I = 3$, $D = 2$. 1 p.
2. $I = 2$, $D = 3$. 1 p.

2b

What is the condition that the ratio I/D has to satisfy in order for $x[n]$ to be perfectly reproducible from $y[n]$? 1 p.

(Continued on page 3.)

Problem 3 Signals in time and frequency

We sample a periodic signal $x_p(t)$ to obtain $x_p[n]$, and we sample an aperiodic signal $x_a(t)$ to obtain $x_a[n]$. The sampling frequency in each case is 1 Hz. All signals are shown in Figure 3:

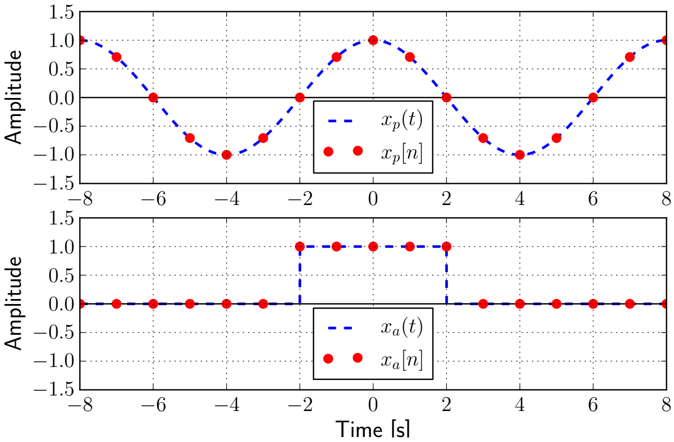

Figure 3: To analoge signaler samples.

a) Sketch the magnitude response for each of these four signals, and place a value on both the frequency axis and magnitude axis (in addition to a 0 in the origin). What symbol and unit would it be reasonable to use on the frequency axis in each case? 2 p.

In this course we have mainly discussed these 3 transforms for digital temporal signals:

z-transform: $$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Discrete time Fourier transform (DTFT): $$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

Discrete Fourier transform (DFT): $$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi \frac{k}{N} n}$$

b) Which transform(s) would you use in these cases (make sure to explain why): 1 p.

1. You have sampled a signal from a microphone and want to find the frequency contents of this signal. A computer is available.
2. You need to design an infinite impulse response (IIR) filter. You lack a computer but have pen and paper.
3. You need to analytically evaluate the frequency response of a window function.

c) How can we obtain the DTFT if we already have an expression for the z-transform? 1 p.

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Problem 4 Filters

A linear and time invariant (LTI) system is filtering the sum of three sinusoids:

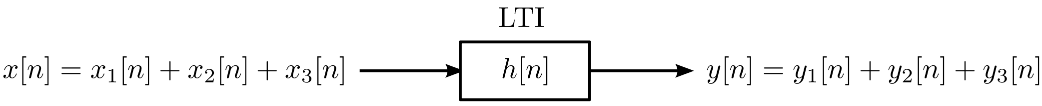

Figure 4: An LTI system.

How these signals look like before and after they pass through the filter is shown in Figures 5 and 6:

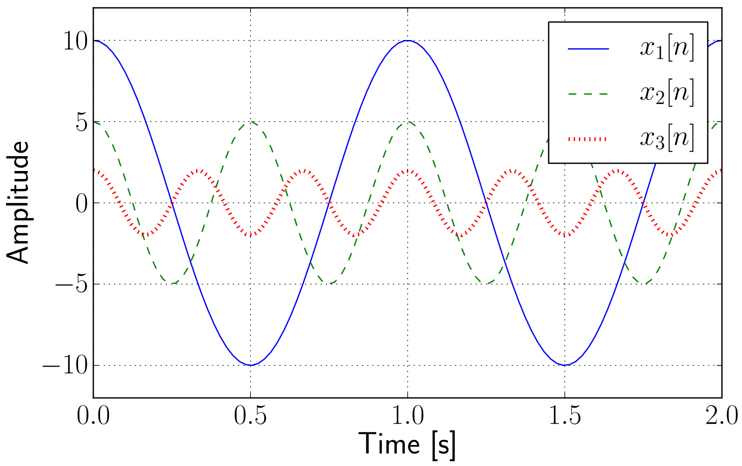

Figure 5: Plot of $x_1[n]$, $x_2[n]$, and $x_3[n]$.

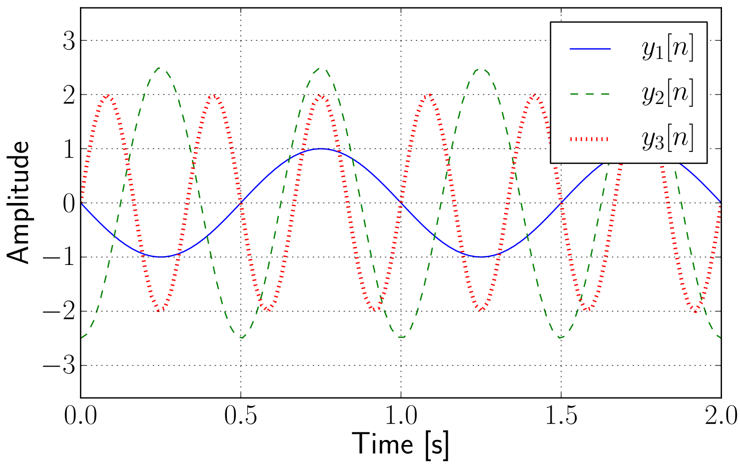

Figure 6: Plot of $y_1[n]$, $y_2[n]$, and $y_3[n]$.

a) Sketch the magnitude- and phase response for this filter. 1 p.

b) Explain the difference between phase- and group delay. Feel free to use Figures 5 and 6 as an aid. Find the group delay of the filter below: 1 p.

$$H(z) = \frac{\frac{1}{2}z + 1 + \frac{1}{2}z^{-1}}{z}$$

c) Another filter has an impulse response as shown in Figure 7 and a frequency response as shown in Figure 8. Is this a FIR or IIR filter? Make well founded arguments to support your answer. 1 p.

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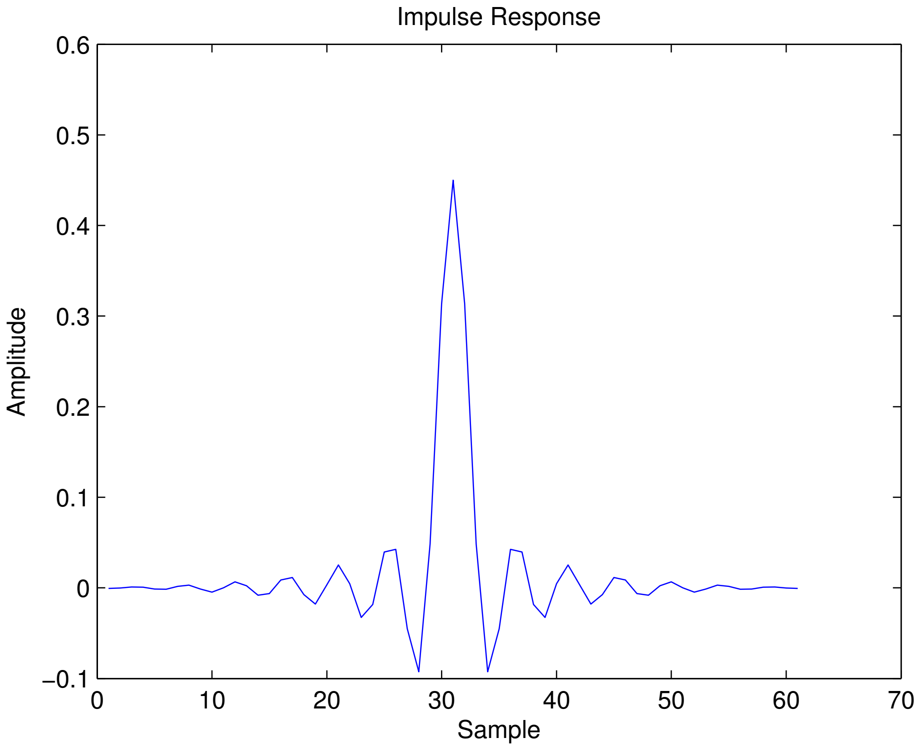

Figure 7: Filter impulse response.

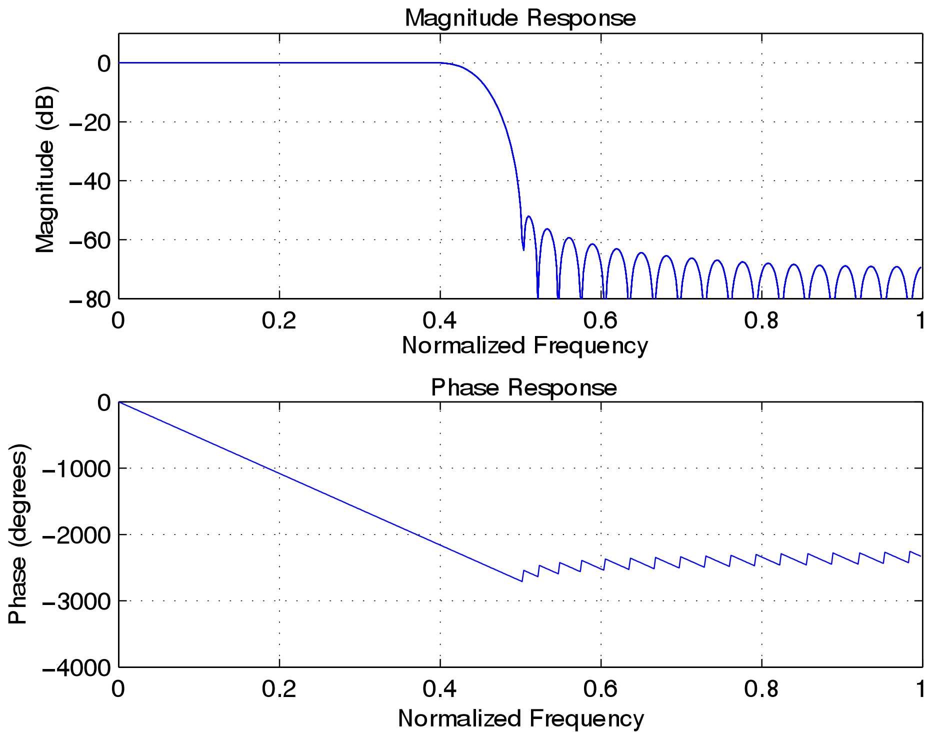

Figure 8: Filter frequency response.

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Problem 5 DFT og DTFT

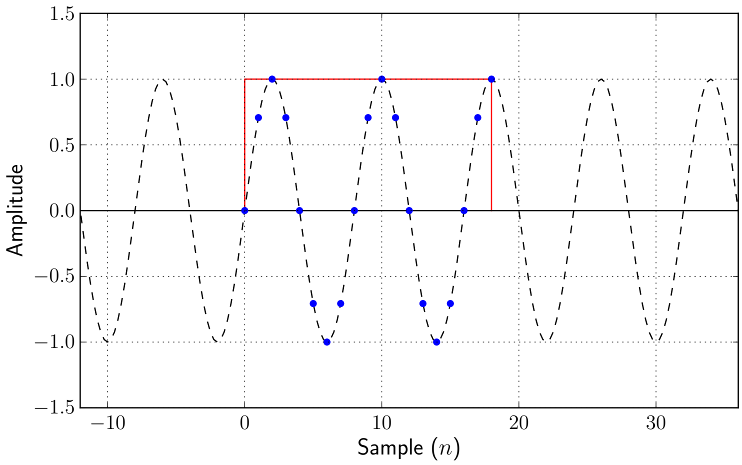

Figure 9: A sampled signal.

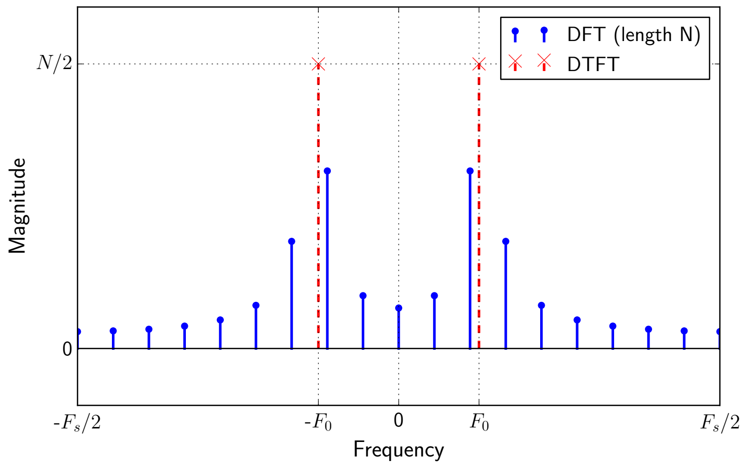

Figure 10: The DFT and DTFT of the signal in Figure 9. Only the magnitude is shown here.

We attempt to use the DFT to compute the magnitude response of the stippled signal in Figure 9. The result is shown in Figure 10, where the DTFT is superimposed as reference (because it produces the correct result).

a) Why does the DTF yield an incorrect magnitude response when the DTFT produces a correct one? 1 p.

b) In Figure 9 we used a rectangular window. What would happen if we substitute it with a triangular window of equal length? 1 p.

c) Let us say that the window in Figure 9 has length L. We may then describe it as follows:

$$w[n] = u[n] - u[n - L]$$

Find the DTFT $W(\Omega)$ of this window. How will the mainlobe width and the sidelobes be affected if the length L goes to infinity? 1 p.

(Continued on page 7.)

Formulas

Basic relations:

$$
\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos^2\alpha + \sin^2\alpha &= 1 \\
\cos\alpha &= \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}) \\
\sin\alpha &= \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}) \\
\sum_{n=0}^{N-1} a^n &= \begin{cases} N & \text{for } a = 1 \\ \frac{1-a^N}{1-a} & \text{otherwise} \end{cases} \\
ax^2 + bx + c = 0 &\Leftrightarrow x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Discrete convolution:

$$
y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = h[n] * x[n]
$$

Discrete-time Fourier transformation (DTFT):

$$
\begin{aligned}
\text{Analysis:} \qquad X(\Omega) &= = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\
\text{Synthesis:} \qquad x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega
\end{aligned}
$$

Discrete Fourier transformation (DFT):

$$
\begin{aligned}
\text{Analysis:} \qquad X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad 0 \le k \le N-1 \\
\text{Synthesis:} \qquad x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \quad 0 \le k \le N-1
\end{aligned}
$$

z-transformation:

$$
\text{Analysis:} \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
$$