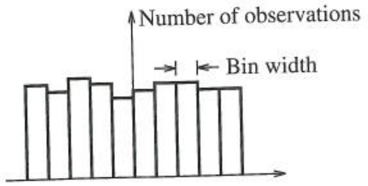
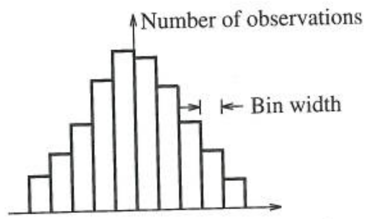
f a

nd a

ndom



Histogram of a uniform random signal



Histogram of a Gaussian random signal

Pseudorandom Signals

In many situations, we use artificially generated signals (which can never be truly random) with prescribed statistical features called **pseudorandom signals**. Such signals are actually periodic (with a *very* long period), but over one period their statistical features approximate those of random signals.

2.7.11 Random Signal Analysis

If a random signal forms the input to a system, the best we can do is to develop features that describe the output *on the average* and **estimate** the response of a system under the influence of random signals. Such estimates may be developed either in the time domain or in the frequency domain.

Signal-to-Noise Ratio

For a noisy signal $x(t) = s(t) + An(t)$ with a signal component $s(t)$ and a noise component $An(t)$ (with noise amplitude A), the **signal-to-noise ratio** (SNR) is the ratio of the signal power σ_s^2 and noise power $A^2\sigma_n^2$ and is usually defined in **decibels** (dB) as

$$\text{SNR} = 10\log\left(\frac{\sigma_s^2}{A^2\sigma_n^2}\right) \text{ dB} \tag{2.52}$$

The decibel value of α is defined as $20\log\alpha$. We can adjust the SNR by varying the noise amplitude A.

Application: Coherent Signal Averaging

Coherent signal averaging is a method of extracting signals from noise and assumes that the experiment can be repeated and the noise corrupting the signal is random (and uncorrelated). Averaging the results of many runs tends to average out the noise to zero, and the signal quality (or signal-to-noise ratio) improves. The more the number of runs, the smoother and less noisy the averaged signal. We often remove the mean or any linear trend before averaging. Figure 2.11 shows one realization of a noisy sine wave and the much smoother results of averaging 8 and 48 such realizations. This method is called **coherent** because it requires *time coherence* (time alignment of the signal for each run). It relies, for its success, on perfect synchronization of each run and on the statistical independence of the contaminating noise.

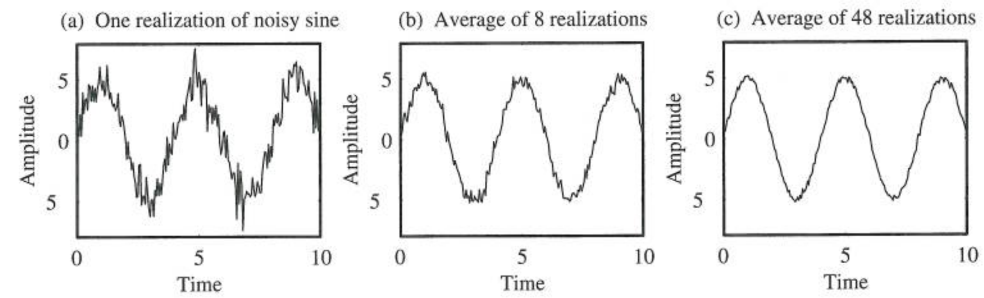


FIGURE 2.11 Coherent averaging of a noisy sine wave. Notice how the signal quality improves as the number of realizations to be averaged increases

2.8 Problems

2.1. **(Discrete Signals)** Sketch each signal and find its energy or average power as appropriate.

(a) $x[n] = \{\overset{\Downarrow}{6}, 4, 2, 2\}$ **(b)** $y[n] = \{-3, -2, \overset{\Downarrow}{-1}, 0, 1\}$

(c) $f[n] = \{\overset{\Downarrow}{0}, 2, 4, 6\}$ **(d)** $g[n] = u[n] - u[n-4]$

(e) $p[n] = \cos(n\pi/2)$ **(f)** $q[n] = 8(0.5)^n u[n]$

[**Hints and Suggestions:** Only $p[n]$ is a power signal. The rest have finite energy.]

2.2. **(Signal Duration)** Use examples to argue that the product of a right-sided and a left-sided discrete-time signal is always time-limited or identically zero.

[**Hints and Suggestions:** Select simple signals that either overlap or do not overlap.]

2.3. **(Operations)** Let $x[n] = \{\overset{\Downarrow}{6}, 4, 2, 2\}$. Sketch the following signals and find their signal energy.

(a) $y[n] = x[n-2]$ **(b)** $f[n] = x[n+2]$

(c) $g[n] = x[-n+2]$ **(d)** $h[n] = x[-n-2]$

[**Hints and Suggestions:** Note that $g[n]$ is a time-reversed version of $f[n]$.]

2.4. **(Operations)** Let $x[n] = 8(0.5)^n(u[n+1] - u[n-3])$. Sketch the following signals.

(a) $y[n] = x[n-3]$ **(b)** $f[n] = x[n+1]$

(c) $g[n] = x[-n+4]$ **(d)** $h[n] = x[-n-2]$

[**Hints and Suggestions:** Note that $x[n]$ contains five samples (from $n = -1$ to $n = 3$). To display the marker for $y[n]$ (which starts at $n = 2$), we include two zeros at $n = 0$ (the marker) and $n = 1$.]

2.5. **(Energy and Power)** Classify the following as energy signals, power signals, or neither and find the energy or average power as appropriate.

(a) $x[n] = 2^n u[-n]$ **(b)** $y[n] = 2^n u[-n-1]$ **(c)** $f[n] = \cos(n\pi)$

(d) $g[n] = \cos(n\pi/2)$ **(e)** $p[n] = \frac{1}{n}u[n-1]$ **(f)** $q[n] = \frac{1}{\sqrt{n}}u[n-1]$

(g) $r[n] = \frac{1}{n^2}u[n-1]$ **(h)** $s[n] = e^{jn\pi}$ **(i)** $d[n] = e^{jn\pi/2}$

(j) $t[n] = e^{(j+1)n\pi/4}$ **(k)** $v[n] = j^{n/4}$ **(l)** $w[n] = (\sqrt{j})^n + (\sqrt{j})^{-n}$

[**Hints and Suggestions:** For $x[n]$ and $y[n]$, $2^{2n} = 4^n = (0.25)^{-n}$. Sum this from $n = -\infty$ to $n = 0$ (or $n = -1$) using a change of variable ($n \to -n$) in the summation. For $p[n]$, sum $1/n^2$ over $n = 1$ to $n = \infty$ using tables. For $q[n]$, the sum of $1/n$ from $n = 1$ to $n = \infty$ does not converge! For $t[n]$, separate the exponentials. To compute the average power for $s[n]$ and $d[n]$, note that $|s[n]| = |d[n]| = 1$. For $v[n]$, use $j = e^{j\pi/2}$. For $w[n]$, set $\sqrt{j} = e^{j\pi/4}$ and use Euler's relation to convert to a sinusoid.]

2.6. (Energy and Power) Sketch each of the following signals, classify as an energy signal or power signal, and find the energy or average power as appropriate.

(a) $x[n] = \sum_{k=-\infty}^{\infty} y[n - kN]$, where $y[n] = u[n] - u[n-3]$ and $N = 6$

(b) $f[n] = \sum_{k=-\infty}^{\infty} (2)^{n-5k}(u[n-5k] - u[n-5k-4])$

[**Hints and Suggestions:** The period of $x[n]$ is $N = 6$. With $y[n] = u[n] - u[n-3]$, one period of $x[n]$ (starting at $n = 0$) is $\{1, 1, 1, 0, 0, 0\}$. The period of $f[n]$ is $N = 5$. Its one period (starting at $n = 0$) contains four samples from $2^n(u[n] - u[n-4])$ and one trailing zero.]

2.7. (Decimation and Interpolation) Let $x[n] = \{4,\ 0,\ \overset{\Downarrow}{2},\ -1,\ 3\}$. Find and sketch the following signals and compare their signal energy with the energy in $x[n]$.

(a) The decimated signal $d[n] = x[2n]$
(b) The *zero-interpolated* signal $f[n] = x^{\uparrow}[\frac{n}{2}]$
(c) The *step-interpolated* signal $g[n] = x^{\uparrow}[\frac{n}{2}]$
(d) The *linearly interpolated* signal $h[n] = x^{\uparrow}[\frac{n}{2}]$

[**Hints and Suggestions:** To get $d[n]$, retain the samples of $x[n]$ at $n = 0, \pm 2, \pm 4, \ldots$. Assuming that the interpolated signals will be twice the length of $x[n]$, the last sample will be 0 for $f[n]$, 3 for $g[n]$ and 1.5 (the linearly interpolated value with $x[n] = 0$, $n > 2$) for $h[n]$.]

2.8. (Interpolation and Decimation) Let $x[n] = 4\,\text{tri}(n/4)$. Sketch the following signals and describe how they differ.

(a) $x[\frac{2}{3}n]$ (using zero interpolation followed by decimation)
(b) $x[\frac{2}{3}n]$ (using step interpolation followed by decimation)
(c) $x[\frac{2}{3}n]$ (using decimation followed by zero interpolation)
(d) $x[\frac{2}{3}n]$ (using decimation followed by step interpolation)

2.9. (Fractional Delay) Starting with $x[n]$, we can generate the signal $x[n-2]$ (using a delay of 2) or $x[2n-3]$ (using a delay of 3 followed by decimation). However, to generate a *fractional delay* of the form $x[n - \frac{M}{N}]$ requires a delay, interpolation, and decimation!

(a) Describe the sequence of operations required to generate $x[n - \frac{2}{3}]$ from $x[n]$.
(b) Let $x[n] = \{\overset{\Downarrow}{1}, 4, 7, 10, 13\}$. Sketch $x[n]$ and $x[n - \frac{2}{3}]$. Use *linear* interpolation where required.
(c) Generalize the results of part (a) to generate $x[n - \frac{M}{N}]$ from $x[n]$. Any restrictions on M and N?

[**Hints and Suggestions:** In part (a) the sequence of operations requires interpolation, delay (by 2) and decimation. The interpolation and decimation factors are identical.]

2.10. (Symmetry) Sketch each signal and its even and odd parts.

(a) $x[n] = 8(0.5)^n u[n]$ **(b)** $y[n] = u[n]$ **(c)** $f[n] = 1 + u[n]$
(d) $g[n] = u[n] - u[n-4]$ **(e)** $p[n] = \text{tri}(\frac{n-3}{3})$ **(f)** $q[n] = \{6,\ \overset{\Downarrow}{4},\ 2,\ 2\}$

[**Hints and Suggestions:** Confirm the appropriate symmetry for each even part and each odd part. For each even part, the sample at $n = 0$ must equal the original sample value. For each odd part, the sample at $n = 0$ must equal zero.]

2.11. (Sketching Discrete Signals) Sketch each of the following signals:

(a) $x[n] = r[n+2] - r[n-2] - 4u[n-6]$ **(b)** $y[n] = \text{rect}(\frac{n}{6})$
(c) $f[n] = \text{rect}(\frac{n-2}{4})$ **(d)** $g[n] = 6\,\text{tri}(\frac{n-4}{3})$

[**Hints and Suggestions:** Note that $f[n]$ is a rectangular pulse centered at $n = 2$ with five samples. Also, $g[n]$ is a triangular pulse centered at $n = 4$ with seven samples (including the zero-valued end samples).]

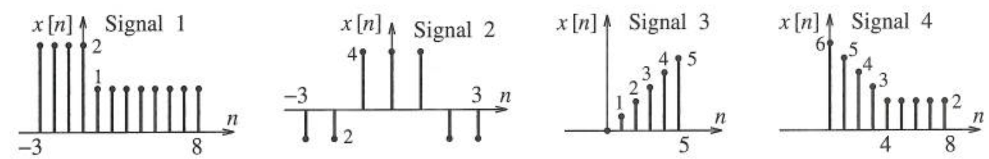
2.12. (Sketching Signals) Sketch the following signals and describe how they are related.

(a) $x[n] = \delta[n]$ **(b)** $f[n] = \text{rect}(n)$ **(c)** $g[n] = \text{tri}(n)$ **(d)** $h[n] = \text{sinc}(n)$

2.13. (Signal Description) For each signal shown in Figure P2.13,

(a) Write out the numeric sequence, and mark the index $n = 0$ by an arrow.
(b) Write an expression for each signal using impulse functions.
(c) Write an expression for each signal using steps and/or ramps.
(d) Find the signal energy.
(e) Find the average power, assuming that the sequence shown repeats itself.

FIGURE P.2.13 Signals for Problem 2.13



[**Hints and Suggestions:** In part (c), all signals must be turned off (by step functions) and any ramps must be first flattened out (by other ramps). For example, signal 3 = $r[n] - r[n-5] - 5u[n-6]$. The second term flattens out the first ramp and last term turns the signal off after $n = 5$.]

2.14. (Discrete Exponentials) A causal discrete exponential has the form $x[n] = \alpha^n u[n]$.

(a) Assume that α is real and positive. Pick convenient values for $\alpha > 1$, $\alpha = 1$, and $\alpha < 1$; sketch $x[n]$; and describe the nature of the sketch for each choice of α.
(b) Assume that α is real and negative. Pick convenient values for $\alpha < -1$, $\alpha = -1$, and $\alpha > -1$; sketch $x[n]$; and describe the nature of the sketch for each choice of α.
(c) Assume that α is complex and of the form $\alpha = Ae^{j\theta}$, where A is a positive constant. Pick convenient values for θ and for $A < 1$, $A = 1$, and $A > 1$; sketch the real part and imaginary part of $x[n]$ for each choice of A; and describe the nature of each sketch.

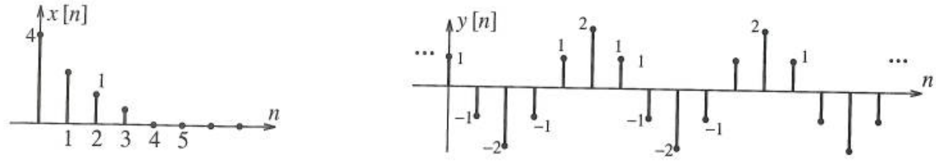
(d) Assume that α is complex and of the form $\alpha = Ae^{j\theta}$, where A is a positive constant. Pick convenient values for θ and for $A < 1$, $A = 1$, and $A > 1$; sketch the magnitude and imaginary phase of $x[n]$ for each choice of A; and describe the nature of each sketch.

2.15. (Signal Representation) The two signals shown in Figure P2.15 may be expressed as

(a) $x[n] = A\alpha^n(u[n] - u[n-N])$ **(b)** $y[n] = A\cos(2\pi Fn + \theta)$

Find the constants in each expression and then find the signal energy or average power as appropriate.

FIGURE P.2.15
Signals for Problem 2.15



[**Hints and Suggestions:** For $y[n]$, first find the period to compute F. Then, evaluate $y[n]$ at two values of n to get two equations for, say $y[0]$ and $y[1]$. These will yield θ (from their ratio) and A.]

2.16. (Discrete-Time Harmonics) Check for the periodicity of the following signals, and compute the common period N if periodic.

(a) $x[n] = \cos(\frac{n\pi}{2})$ **(b)** $y[n] = \cos(\frac{n}{2})$
(c) $f[n] = \sin(\frac{n\pi}{4}) - 2\cos(\frac{n\pi}{6})$ **(d)** $g[n] = 2\cos(\frac{n\pi}{4}) + \cos^2(\frac{n\pi}{4})$
(e) $p[n] = 4 - 3\sin(\frac{7n\pi}{4})$ **(f)** $q[n] = \cos(\frac{5n\pi}{12}) + \cos(\frac{4n\pi}{9})$
(g) $r[n] = \cos(\frac{8}{3}n\pi) + \cos(\frac{8}{3}n)$ **(h)** $s[n] = \cos(\frac{8n\pi}{3})\cos(\frac{n\pi}{2})$
(i) $d[n] = e^{j0.3n\pi}$ **(j)** $e[n] = 2e^{j0.3n\pi} + 3e^{j0.4n\pi}$
(k) $v[n] = e^{j0.3n}$ **(l)** $w[n] = (j)^{n/2}$

[**Hints and Suggestions:** There is no periodicity if F is not a rational fraction for any component. Otherwise, work with the periods and find their LCM. For $w[n]$, note that $j = e^{j\pi/2}$.]

2.17. (The Roots of Unity) The N roots of the equation $z^N = 1$ can be found by writing it as $z^N = e^{j2k\pi}$ to give $z = e^{j2k\pi/N}$, $k = 0, 1, \ldots, N-1$. What is the magnitude and angle of each root? The roots can be displayed as vectors directed from the origin whose tips lie on a circle.

(a) What is the length of each vector and the angular spacing between adjacent vectors? Sketch for $N = 5$ and $N = 6$.

(b) Extend this concept to find the roots of $z^N = -1$ and sketch for $N = 5$ and $N = 6$.

[**Hints and Suggestions:** In part (b), note that $z^N = -1 = e^{j\pi}e^{j2k\pi} = e^{j(2k+1)\pi}$.]

2.18. (Digital Frequency) Set up an expression for each signal, using a digital frequency $|F| < 0.5$, and another expression using a digital frequency in the range $4 < F < 5$.

(a) $x[n] = \cos(\frac{4n\pi}{3})$ **(b)** $x[n] = \sin(\frac{4n\pi}{3}) + 3\sin(\frac{8n\pi}{3})$

[**Hints and Suggestions:** First find the digital frequency of each component in the principal range ($-0.5 < F \leq 0.5$). Then, add 4 or 5 as appropriate to bring each frequency into the required range.]

2.19. **(Digital Sinusoids)** Find the period N of each signal if periodic. Express each signal using a digital frequency in the principal range ($|F| < 0.5$) and in the range $3 \le F \le 4$.

(a) $x[n] = \cos(\frac{7n\pi}{3})$ **(b)** $x[n] = \cos(\frac{7n\pi}{3}) + \sin(0.5n\pi)$ **(c)** $x[n] = \cos(n)$

2.20. **(Sampling and Aliasing)** Each of the following sinusoids is sampled at $S = 100$ Hz. Determine if aliasing has occurred and set up an expression for each sampled signal using a digital frequency in the principal range ($|F| < 0.5$).

(a) $x(t) = \cos(320\pi t + \frac{\pi}{4})$ **(b)** $x(t) = \cos(140\pi t - \frac{\pi}{4})$ **(c)** $x(t) = \sin(60\pi t)$

[**Hints and Suggestions:** Find the frequency f_0. If $S > 2f_0$ there is no aliasing and $F < 0.5$. Otherwise, bring F into the principal range to write the expression for the sampled signal.]

2.21. **(Aliasing and Signal Reconstruction)** The signal $x(t) = \cos(320\pi t + \frac{\pi}{4})$ is sampled at 100 Hz, and the sampled signal $x[n]$ is reconstructed at 200 Hz to recover the analog signal $x_r(t)$.

(a) Has aliasing occurred? What is the period N and the digital frequency F of $x[n]$?
(b) How many full periods of $x(t)$ are required to generate one period of $x[n]$?
(c) What is the analog frequency of the recovered signal $x_r(t)$?
(d) Write expressions for $x[n]$ (using $|F| < 0.5$) and for $x_r(t)$.

[**Hints and Suggestions:** For part (b), if the digital frequency is expressed as $F = k/N$ where N is the period and k is an integer, it takes k full cycles of the analog sinusoid to get N samples of the sampled signal. In part (c), the frequency of the reconstructed signal is found from the aliased frequency in the principal range.]

2.22. **(Digital Pitch Shifting)** One way to accomplish *pitch shifting* is to play back (or reconstruct) a sampled signal at a *different* sampling rate. Let the analog signal $x(t) = \sin(15800\pi t + 0.25\pi)$ be sampled at a sampling rate of 8 kHz.

(a) Find its sampled representation with digital frequency $|F| < 0.5$.
(b) What frequencies are heard if the signal is reconstructed at a rate of 4 kHz?
(c) What frequencies are heard if the signal is reconstructed at a rate of 8 kHz?
(d) What frequencies are heard if the signal is reconstructed at a rate of 20 kHz?

[**Hints and Suggestions:** The frequency of the reconstructed signal is found from the aliased digital frequency in the principal range and the appropriate reconstruction rate.]

2.23. **(Discrete-Time Chirp Signals)** Consider the signal $x(t) = \cos[\phi(t)]$, where $\phi(t) = \alpha t^2$. Show that its instantaneous frequency $f_i(t) = \frac{1}{2\pi}\phi'(t)$ varies linearly with time.

(a) Choose α such that the frequency varies from 0 Hz to 2 Hz in 10 seconds, and generate the sampled signal $x[n]$ from $x(t)$, using a sampling rate of $S = 4$ Hz.
(b) It is claimed that, unlike $x(t)$, the signal $x[n]$ is periodic. Verify this claim, using the condition for periodicity ($x[n] = x[n + N]$), and determine the period N of $x[n]$.
(c) The signal $y[n] = \cos(\pi F_0 n^2/M)$, $n = 0, 1, \ldots, M - 1$, describes an M-sample chirp whose digital frequency varies linearly from 0 to F_0. What is the period of $y[n]$ if $F_0 = 0.25$ and $M = 8$?

[**Hints and Suggestions:** In part (b), if $x[n] = \cos(\beta n^2)$, periodicity requires $x[n] = x[n + N]$ or $\cos(\beta n^2) = \cos[\beta(n^2 + 2nN + N^2)]$. Thus $2nN\beta = 2m\pi$ and $N^2\beta = 2k\pi$ where m and k are integers. Satisfy these conditions for the smallest integer N.]