3.17.1 The z-Transform

For an input $x[n] = r^n e^{j2\pi nF} = (re^{j2\pi F})^n = z^n$, where z is complex, with magnitude $|z| = r$, the response may be written as

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} z^{n-k}h[k] = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = x[n]H(z) \qquad (3.62)$$

The response equals the input (eigensignal) modified by the system function $H(z)$, where

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k} \quad \text{(two-sided z-transform)} \qquad (3.63)$$

The complex quantity $H(z)$ describes the **z-transform** of $h[n]$ and is not, in general, periodic in z. Denoting the z-transform of $x[n]$ and $y[n]$ by $X(z)$ and $Y(z)$, we write

$$Y(z) = \sum_{k=-\infty}^{\infty} y[k]z^{-k} = \sum_{k=-\infty}^{\infty} x[k]H(z)z^{-k} = H(z)X(z) \qquad (3.64)$$

Convolution in the time domain thus corresponds to multiplication in the z-domain.

3.17.2 The Discrete-Time Fourier Transform

For the harmonic input $x[n] = e^{j2\pi nF}$, the response $y[n]$ equals

$$y[n] = \sum_{k=-\infty}^{\infty} e^{j2\pi(n-k)F}h[k] = e^{j2\pi nF} \sum_{k=-\infty}^{\infty} h[k]e^{-j2\pi kF} = x[n]H(F) \qquad (3.65)$$

This is just the input modified by the system function $H(F)$, where

$$H(F) = \sum_{k=-\infty}^{\infty} h[k]e^{-j2\pi kF} \qquad (3.66)$$

The quantity $H(F)$ describes the **discrete-time Fourier transform** (DTFT) or **discrete-time frequency response** or **spectrum** of $h[n]$. Any signal $x[n]$ may similarly be described by its DTFT $X(F)$. The response $y[n] = x[n]H[F]$ may then be transformed to its DTFT $Y[n]$ to give

$$Y(F) = \sum_{k=-\infty}^{\infty} y[k]e^{-j2\pi Fk} = \sum_{k=-\infty}^{\infty} x[k]H(F)e^{-j2\pi Fk} = H(F)X(F) \qquad (3.67)$$

Once again, convolution in the time domain corresponds to multiplication in the frequency domain. Note that we obtain the DTFT of $h[n]$ from its z-transform $H(z)$ by letting $z = e^{j2\pi F}$ or $|z| = 1$ to give

$$H(F) = H(z)|_{z=\exp(j2\pi F)} = H(z)|_{|z|=1} \qquad (3.68)$$

The DTFT is thus the z-transform evaluated on the unit circle $|z| = 1$. The system function $H(F)$ is also periodic in F with a period of unity because $e^{-j2\pi kF} = e^{-j2\pi k(F+1)}$. This periodicity is a direct consequence of the discrete nature of $h[n]$.

3.18 Problems

3.1. **(Operators)** Which of the following describe linear operators?

(a) $\mathcal{O}\{\ \} = 4\{\ \}$ **(b)** $\mathcal{O}\{\ \} = 4\{\ \} + 3$ **(c)** $\mathcal{O}\{\ \} = \alpha^{\{\ \}}$

3.2. **(System Classification)** In each of the systems below, $x[n]$ is the input and $y[n]$ is the output. Check each system for linearity, shift invariance, memory, and causality.

(a) $y[n] - y[n-1] = x[n]$ **(b)** $y[n] + y[n+1] = nx[n]$
(c) $y[n] - y[n+1] = x[n+2]$ **(d)** $y[n+2] - y[n+1] = x[n]$
(e) $y[n+1] - x[n]y[n] = nx[n+2]$ **(f)** $y[n] + y[n-3] = x^2[n] + x[n+6]$
(g) $y[n] - 2^n y[n] = x[n]$ **(h)** $y[n] = x[n] + x[n-1] + x[n-2]$

3.3. **(System Classification)** Classify the following systems in terms of their linearity, time invariance, memory, causality, and stability.

(a) $y[n] = 3^n x[n]$ **(b)** $y[n] = e^{jn\pi}x[n]$
(c) $y[n] = \cos(0.5n\pi)x[n]$ **(d)** $y[n] = [1 + \cos(0.5n\pi)]x[n]$
(e) $y[n] = e^{x[n]}$ **(f)** $y[n] = x[n] + \cos[0.5(n+1)\pi]$

3.4. **(System Classification)** Classify the following systems in terms of their linearity, time invariance, memory, causality, and stability.

(a) $y[n] = x[n/3]$ (zero interpolation)
(b) $y[n] = \cos(n\pi)x[n]$ (modulation)
(c) $y[n] = [1 + \cos(n\pi)]x[n]$ (modulation)
(d) $y[n] = \cos(n\pi x[n])$ (frequency modulation)
(e) $y[n] = \cos(n\pi + x[n])$ (phase modulation)
(f) $y[n] = x[n] - x[n-1]$ (differencing operation)
(g) $y[n] = 0.5x[n] + 0.5x[n-1]$ (averaging operation)
(h) $y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$ (moving average)
(i) $y[n] - \alpha y[n-1] = \alpha x[n]$, $0 < \alpha < 1$ (exponential averaging)
(j) $y[n] = 0.4(y[n-1] + 2) + x[n]$

3.5. **(Classification)** Classify each system in terms of its linearity, time invariance, memory, causality, and stability.

(a) the time-reversing system $y[n] = x[-n]$
(b) the decimating system $y[n] = x[2n]$
(c) the zero-interpolating system $y[n] = x[n/2]$
(d) the sign-inversion system $y[n] = \text{sgn}\{x[n]\}$
(e) the rectifying system $y[n] = |x[n]|$

3.6. **(Classification)** Classify each system in terms of its linearity, time invariance, causality, and stability.

(a) $y[n] = \text{round}\{x[n]\}$ **(b)** $y[n] = \text{median}\{x[n+1], x[n], x[n-1]\}$
(c) $y[n] = x[n]\,\text{sgn}(n)$ **(d)** $y[n] = x[n]\,\text{sgn}\{x[n]\}$

3.7. **(Realization)** Find the difference equation for each system realization shown in Figure P3.7.

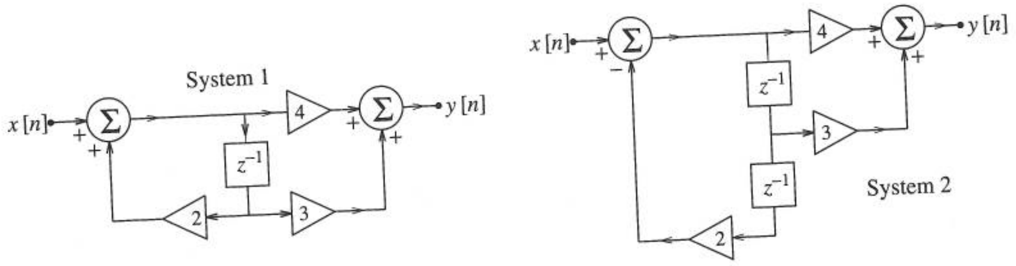


FIGURE P.3.7 Filter realizations for Problem 3.7

[Hints and Suggestions: Compare with the generic first-order and second-order realizations to get the difference equations.]

3.8. (Response by Recursion) Use recursion to find the response $y[n]$ of the following systems for the first few values of n and discern the general form for $y[n]$.

(a) $y[n] - ay[n-1] = \delta[n]\ y[-1] = 0$ **(b)** $y[n] - ay[n-1] = u[n]\ y[-1] = 1$
(c) $y[n] - ay[n-1] = u[n]\ y[-1] = 1$ **(d)** $y[n] - ay[n-1] = nu[n]\ y[-1] = 0$

[Hints and Suggestions: For parts (b) through (d), you may need to use a table of summations to simplify the results for the general form.]

3.9. (Response by Recursion) Let $y[n] + 4y[n-1] + 3y[n-2] = u[n-2]$ with $y[-1] = 0$, $y[-2] = 1$. Use recursion to compute $y[n]$ up to $n = 4$. Can you discern a general form for $y[n]$?

3.10. (Forced Response) Find the forced response of the following systems.

(a) $y[n] - 0.4y[n-1] = 3u[n]$ **(b)** $y[n] - 0.4y[n-1] = (0.5)^n$
(c) $y[n] + 0.4y[n-1] = (0.5)^n$ **(d)** $y[n] - 0.5y[n-1] = \cos(n\pi/2)$

[Hints and Suggestions: For part (a), $3u[n] = 3,\ n \ge 0$ and implies that the forced response (or its shifted version) is constant. So, choose $y_F[n] = C = y_F[n-1]$. For part (c), pick $y_F[n] = A\cos(0.5n\pi) + B\sin(0.5n\pi)$, expand terms like $\cos[0.5(n-1)\pi]$ using trigonometric identities, and compare the coefficients of $\cos(0.5n\pi)$ and $\sin(0.5n\pi)$ to generate two equations to solve for A and B.]

3.11. (Zero-State Response) Find the zero-state response of the following systems.

(a) $y[n] - 0.5y[n-1] = 2u[n]$ **(b)** $y[n] - 0.4y[n-1] = (0.5)^n u[n]$
(c) $y[n] - 0.4y[n-1] = (0.4)^n u[n]$ **(d)** $y[n] - 0.5y[n-1] = \cos(n\pi/2)$

[Hints and Suggestions: Here, zero-state implies $y[-1] = 0$. Part (c) requires $y_F[n] = Cn(0.4)^n$ because the root of the characteristic equation is 0.4.]

3.12. (Zero-State Response) Consider the system $y[n] - 0.5y[n-1] = x[n]$. Find its zero-state response to the following inputs.

(a) $x[n] = u[n]$ **(b)** $x[n] = (0.5)^n u[n]$ **(c)** $x[n] = \cos(0.5n\pi)u[n]$
(d) $x[n] = (-1)^n u[n]$ **(e)** $x[n] = j^n u[n]$ **(f)** $x[n] = (\sqrt{j})^n u[n] + (\sqrt{j})^{-n} u[n]$

[Hints and Suggestions: For part (e), pick the forced response as $y_F[n] = C(j)^n$. This will give a complex response because the input is complex. For part (f), $x[n]$ simplifies to a sinusoid by using $j = e^{j\pi/2}$ and Euler's relation.]

3.13. (Zero-State Response) Find the zero-state response of the following systems.

(a) $y[n] - 1.1y[n-1] + 0.3y[n-2] = 2u[n]$
(b) $y[n] + 0.7y[n-1] + 0.1y[n-2] = (0.5)^n$
(c) $y[n] - 0.9y[n-1] + 0.2y[n-2] = (0.5)^n$
(d) $y[n] - 0.25y[n-2] = \cos(n\pi/2)$

[Hints and Suggestions: Zero-state implies $y[-1] = y[-2] = 0$. For part (b), use $y_F[n] = C(0.5)^n$, but for part (c), pick $y_F[n] = Cn(0.5)^n$ because one root of the characteristic equation is 0.5.]

3.14. (System Response) Let $y[n] - 0.5y[n-1] = x[n]$, with $y[-1] = -1$. Find the response of this system due to the following inputs for $n \ge 0$.

(a) $x[n] = 2u[n]$ **(b)** $x[n] = (0.25)^n u[n]$ **(c)** $x[n] = n(0.25)^n u[n]$
(d) $x[n] = (0.5)^n u[n]$ **(e)** $x[n] = n(0.5)^n u[n]$ **(f)** $x[n] = (0.5)^n \cos(0.5n\pi)u[n]$

[Hints and Suggestions: For part (c), pick $y_F[n] = (C + Dn)(0.5)^n$ (and compare coefficients of like powers of n to solve for C and D). For part (d), pick $y_F[n] = Cn(0.5)^n$ because the root of the characteristic equation is 0.5. Part(e) requires $y_F[n] = n(C + Dn)(0.5)^n$ for the same reason.]

3.15. (System Response) For the system realization shown in Figure P3.15, find the response to the following inputs and initial conditions.

(a) $x[n] = u[n]$ $y[-1] = 0$ **(b)** $x[n] = u[n]$ $y[-1] = 4$
(c) $x[n] = (0.5)^n u[n]$ $y[-1] = 0$ **(d)** $x[n] = (0.5)^n u[n]$ $y[-1] = 6$
(e) $x[n] = (-0.5)^n u[n]$ $y[-1] = 0$ **(f)** $x[n] = (-0.5)^n u[n]$ $y[-1] = -2$

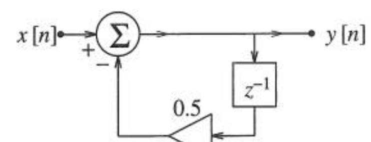


FIGURE P.3.15 System realization for Problem 3.15

[Hints and Suggestions: For part (e), pick the forced response as $y_F[n] = Cn(-0.5)^n$.]

3.16. (System Response) Find the response of the following systems.

(a) $y[n] - 0.4y[n-1] = 2(0.5)^{n-1}u[n-1]$ $y[-1] = 0$
(b) $y[n] - 0.4y[n-1] = (0.4)^n u[n] + 2(0.5)^{n-1}u[n-1]$ $y[-1] = 2.5$
(c) $y[n] - 0.4y[n-1] = n(0.5)^n u[n] + 2(0.5)^{n-1}u[n-1]$ $y[-1] = 2.5$

[Hints and Suggestions: Start with $y[n] - 0.4y[n-1] = 2(0.5)^n$, $y[-1] = 0$ and find its zero-state response. Then use superposition and time invariance as required. For the input $(0.4)^n$ of part (b), assume $y_F[n] = Cn(0.4)^n$. For the input $n(0.5)^n$ of part (c), assume $y_F[n] = (A + Bn)(0.5)^n$.]

3.17. (System Response) Find the impulse response of the following filters.

(a) $y[n] = x[n] - x[n-1]$ (differencing operation)
(b) $y[n] = 0.5x[n] + 0.5x[n-1]$ (averaging operation)
(c) $y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k],\ N = 3$ (moving average)
(d) $y[n] = \frac{2}{N(N+1)}\sum_{k=0}^{N-1}(N-k)x[n-k],\ N = 3$ (weighted moving average)
(e) $y[n] - \alpha y[n-1] = (1-\alpha)x[n],\ N = 3,\ \alpha = \frac{N-1}{N+1}$ (exponential averaging)

3.18. (System Response) It is known that the response of the system $y[n]+\alpha y[n-1]=x[n]$, $\alpha \neq 0$, is given by $y[n]=[5+3(0.5)^n]u[n]$.

(a) Identify the natural response and forced response.
(b) Identify the values of α and $y[-1]$.
(c) Identify the zero-input response and zero-state response.
(d) Identify the input $x[n]$.

3.19. (System Response) It is known that the response of the system $y[n]+0.5y[n-1]=x[n]$ is described by $y[n]=[5(0.5)^n+3(-0.5)^n)]u[n]$.

(a) Identify the zero-input response and zero-state response.
(b) What is the zero-input response of the system $y[n]+0.5y[n-1]=x[n]$ if $y[-1]=10$?
(c) What is the response of the relaxed system $y[n]+0.5y[n-1]=x[n-2]$?
(d) What is the response of the relaxed system $y[n]+0.5y[n-1]=x[n-1]+2x[n]$?

3.20. (System Response) It is known that the response of the system $y[n]+\alpha y[n-1]=x[n]$ is described by $y[n]=(5+2n)(0.5)^n u[n]$.

(a) Identify the zero-input response and zero-state response.
(b) What is the zero-input response of the system $y[n]+\alpha y[n-1]=x[n]$ if $y[-1]=10$?
(c) What is the response of the relaxed system $y[n]+\alpha y[n-1]=x[n-1]$?
(d) What is the response of the relaxed system $y[n]+\alpha y[n-1]=2x[n-1]+x[n]$?
(e) What is the complete response of the system $y[n]+\alpha y[n-1]=x[n]+2x[n-1]$ if $y[-1]=4$?

3.21. (System Response) Find the response of the following systems.

(a) $y[n]+0.1y[n-1]-0.3y[n-2]=2u[n]$	$y[-1]=0$	$y[-2]=0$
(b) $y[n]-0.9y[n-1]+0.2y[n-2]=(0.5)^n$	$y[-1]=1$	$y[-2]=-4$
(c) $y[n]+0.7y[n-1]+0.1y[n-2]=(0.5)^n$	$y[-1]=0$	$y[-2]=3$
(d) $y[n]-0.25y[n-2]=(0.4)^n$	$y[-1]=0$	$y[-2]=3$
(e) $y[n]-0.25y[n-2]=(0.5)^n$	$y[-1]=0$	$y[-2]=0$

[**Hints and Suggestions:** For parts (b) and (e), pick $y_F[n]=Cn(0.5)^n$ because one root of the characteristic equation is 0.5.]

3.22. (System Response) Sketch a realization for each system, assuming zero initial conditions. Then evaluate the complete response from the information given. Check your answer by computing the first few values by recursion.

(a) $y[n]-0.4y[n-1]=x[n]$	$x[n]=(0.5)^n u[n]$	$y[-1]=0$
(b) $y[n]-0.4y[n-1]=2x[n]+x[n-1]$	$x[n]=(0.5)^n u[n]$	$y[-1]=0$
(c) $y[n]-0.4y[n-1]=2x[n]+x[n-1]$	$x[n]=(0.5)^n u[n]$	$y[-1]=5$
(d) $y[n]+0.5y[n-1]=x[n]-x[n-1]$	$x[n]=(0.5)^n u[n]$	$y[-1]=2$
(e) $y[n]+0.5y[n-1]=x[n]-x[n-1]$	$x[n]=(-0.5)^n u[n]$	$y[-1]=0$

[**Hints and Suggestions:** For parts (b) through (c), use the results of part (a) plus linearity (superposition) and time invariance.]

3.23. (System Response) For each system, evaluate the natural, forced, and total response. Assume that $y[-1]=0$, $y[-2]=1$. Check your answer for the total response by computing its first few values by recursion.

(a) $y[n]+4y[n-1]+3y[n-2]=u[n]$
(b) $\{1-0.5z^{-1}\}y[n]=(0.5)^n\cos(0.5n\pi)u[n]$
(c) $y[n]+4y[n-1]+8y[n-2]=\cos(n\pi)u[n]$
(d) $\{(1+2z^{-1})^2\}y[n]=n(2)^n u[n]$
(e) $\{1+\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\}y[n]=(\frac{1}{3})^n u[n]$
(f) $\{1+0.5z^{-1}+0.25z^{-2}\}y[n]=\cos(0.5n\pi)u[n]$

[**Hints and Suggestions:** For part (b), pick $y_F[n]=(0.5)^n[A\cos(0.5n\pi)+B\sin(0.5n\pi)]$, expand terms like $\cos[0.5(n-1)\pi]$ using trigonometric identities, and compare the coefficients of $\cos(0.5n\pi)$ and $\sin(0.5n\pi)$ to generate two equations to solve for A and B. For part (d), pick $y_F[n]=(C+Dn)(2)^n$ and compare like powers of n to solve for C and D.]

3.24. (System Response) For each system, evaluate the zero-state, zero-input, and total response. Assume that $y[-1]=0$, $y[-2]=1$.

(a) $y[n]+4y[n-1]+4y[n-2]=2^n u[n]$
(b) $\{z^2+4z+4\}y[n]=2^n u[n]$

[**Hints and Suggestions:** In part (b), $y[n+2]+4y[n+1]+4y[n]=(2)^n u[n]$. By time invariance, $y[n]+4y[n-1]+4y[n-2]=(2)^{n-2}u[n-2]$ and we shift the zero-state response of part (a) by two units ($n\to n-2$) and add to the zero-input response to get the result.]

3.25. (System Response) For each system, set up a difference equation and compute the zero-state, zero-input, and total response, assuming $x[n]=u[n]$ and $y[-1]=y[-2]=1$.

(a) $\{1-z^{-1}-2z^{-2}\}y[n]=x[n]$
(b) $\{z^2-z-2\}y[n]=x[n]$
(c) $\{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\}y[n]=x[n]$
(d) $\{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\}y[n]=\{1+z^{-1}\}x[n]$
(e) $\{1-0.25z^{-2}\}y[n]=x[n]$
(f) $\{z^2-0.25\}y[n]=\{2z^2+1\}x[n]$

[**Hints and Suggestions:** For part (b), use the result of part (a) and time-invariance to get the answer as $y_{zs}[n-2]+y_{zi}[n]$. For part (d), use the result of part (c) to get the answer as $y_{zi}[n]+y_{zs}[n]+y_{zs}[n-1]$. The answer for part (f) may be similarly obtained from part (e).]

3.26. (Impulse Response by Recursion) Find the impulse response $h[n]$ by recursion up to $n=4$ for each of the following systems.

(a) $y[n]-y[n-1]=2x[n]$
(b) $y[n]-3y[n-1]+6y[n-2]=x[n-1]$
(c) $y[n]-2y[n-3]=x[n-1]$
(d) $y[n]-y[n-1]+6y[n-2]=nx[n-1]+2x[n-3]$

[**Hints and Suggestions:** For the impulse response, $x[n]=1,\ n=0$ and $x[n]=0,\ n\neq 0$.]

3.27. (Analytical Form for Impulse Response) Classify each filter as recursive or FIR (nonrecursive) and causal or noncausal, and find an expression for its impulse response $h[n]$.

(a) $y[n]=x[n]+x[n-1]+x[n-2]$
(b) $y[n]=x[n+1]+x[n]+x[n-1]$
(c) $y[n]+2y[n-1]=x[n]$
(d) $y[n]+2y[n-1]=x[n-1]$
(e) $y[n]+2y[n-1]=2x[n]+6x[n-1]$
(f) $y[n]+2y[n-1]=x[n+1]+4x[n]+6x[n-1]$
(g) $\{1+4z^{-1}+3z^{-2}\}y[n]=\{z^{-2}\}x[n]$
(h) $\{z^2+4z+4\}y[n]=\{z+3\}x[n]$
(i) $\{z^2+4z+8\}y[n]=x[n]$
(j) $y[n]+4y[n-1]+4y[n-2]=x[n]-x[n+2]$

[**Hints and Suggestions:** To find the impulse response for the recursive filters, assume $y[0] = 1$ and (if required) $y[-1] = y[-2] = \cdots = 0$. If the right-hand side of the recursive filter equation is anything but $x[n]$, start with the single input $x[n]$ and then use superposition and time-invariance to get the result for the required input. The results for (d) through (f) can be found from the results of (c) in this way.]

3.28. **(Stability)** Investigate the causality and stability of the following right-sided systems.

(a) $y[n] = x[n-1] + x[n] + x[n+1]$
(b) $y[n] = x[n] + x[n-1] + x[n-2]$
(c) $y[n] - 2y[n-1] = x[n]$
(d) $y[n] - 0.2y[n-1] = x[n] - 2x[n+2]$
(e) $y[n] + y[n-1] + 0.5y[n-2] = x[n]$
(f) $y[n] - y[n-1] + y[n-2] = x[n] - x[n+1]$
(g) $y[n] - 2y[n-1] + y[n-2] = x[n] - x[n-3]$
(h) $y[n] - 3y[n-1] + 2y[n-2] = 2x[n+3]$

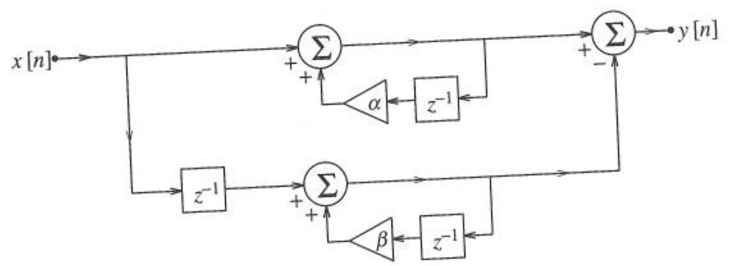
[**Hints and Suggestions:** Remember that FIR filters are always stable and for right-sided systems, every root of the caracteristic equation must have a magnitude (absolute value) less than 1.]

3.29. **(System Interconnections)** Two systems are said to be in cascade if the output of the first system acts as the input to the second. Find the response of the following cascaded systems if the input is a unit step and the systems are described as follows. In which instances does the response differ when the order of cascading is reversed? Can you use this result to justify that the order in which the systems are cascaded does not matter in finding the overall response if both systems are LTI?

(a) System 1: $y[n] = x[n] - x[n-1]$ System 2: $y[n] = 0.5y[n-1] + x[n]$
(b) System 1: $y[n] = 0.5y[n-1] + x[n]$ System 2: $y[n] = x[n] - x[n-1]$
(c) System 1: $y[n] = x^2[n]$ System 2: $y[n] = 0.5y[n-1] + x[n]$
(d) System 1: $y[n] = 0.5y[n-1] + x[n]$ System 2: $y[n] = x^2[n]$

3.30. **(Systems in Cascade and Parallel)** Consider the realization of Figure P3.30.

P.3.30 realization blem 3.30



(a) Find its impulse response if $\alpha \neq \beta$. Is the overall system FIR or IIR?
(b) Find its difference equation and impulse response if $\alpha = \beta$. Is the overall system FIR or IIR?
(c) Find its difference equation and impulse response if $\alpha = \beta = 1$. What is the function of the overall system?

3.31. **(Difference Equations from Impulse Response)** Find the difference equations describing the following systems.

(a) $h[n] = \delta[n] + 2\delta[n-1]$ **(b)** $h[n] = \{2, \overset{\Downarrow}{3}, -1\}$
(c) $h[n] = (0.3)^n u[n]$ **(d)** $h[n] = (0.5)^n u[n] - (-0.5)^n u[n]$

[**Hints and Suggestions:** For part (c), the left-hand side of the difference equation is $y[n] - 0.3y[n-1]$. So, $h[n] - 0.3h[n-1]$ simplified to get impulses leads to the right-hand side. For part (d), start with the left-hand side as $y[n] - 0.25y[n-2]$.]

3.32. **(Difference Equations from Impulse Response)** A system is described by the impulse response $h[n] = (-1)^n u[n]$. Find the difference equation of this system. Then find the difference equation of the inverse system. Does the inverse system describe an FIR filter or IIR filter? What function does it perform?

3.33. **(Difference Equations)** For the filter realization shown in Figure P3.33, find the difference equation relating $y[n]$ and $x[n]$ if the impulse response of the filter is given by

(a) $h[n] = \delta[n] - \delta[n-1]$ **(b)** $h[n] = 0.5\delta[n] + 0.5\delta[n-1]$

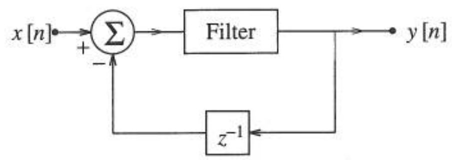


FIGURE P.3.33 Filter realization for Problem 3.33

3.34. **(Difference Equations from Differential Equations)** This problem assumes some familiarity with analog theory. Consider an analog system described by

$$y''(t) + 3y'(t) + 2y(t) = 2u(t)$$

(a) Confirm that this describes a stable analog system.
(b) Convert this to a difference equation using the backward Euler algorithm and check the stability of the resulting digital filter.
(c) Convert this to a difference equation using the forward Euler algorithm and check the stability of the resulting digital filter.
(d) Which algorithm is better in terms of preserving stability? Can the results be generalized to any arbitrary analog system?

3.35. **(Inverse Systems)** Are the following systems invertible? If not, explain why; if invertible, find the inverse system.

(a) $y[n] = x[n] - x[n-1]$ (differencing operation)
(b) $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$ (moving average operation)
(c) $y[n] = 0.5x[n] + x[n-1] + 0.5x[n-2]$ (weighted moving average operation)
(d) $y[n] - \alpha y[n-1] = (1-\alpha)x[n]$, $0 < \alpha < 1$ (exponential averaging operation)
(e) $y[n] = \cos(n\pi)x[n]$ (modulation)
(f) $y[n] = \cos(x[n])$
(g) $y[n] = e^{x[n]}$

[**Hints and Suggestions:** The inverse system is found by switching input and output and rearranging. Only one of these systems is not invertible.]

3.36. (An Echo System and Its Inverse) An echo system is described by

$$y[n] = x[n] + 0.5x[n - N]$$

Assume that the echo arrives after 1 ms and the sampling rate is 2 kHz.

(a) What is the value of N? Sketch a realization of this echo system.
(b) What is the impulse response and step response of this echo system?
(c) Find the difference equation of the inverse system. Then, sketch its realization and find its impulse response and step response.

3.37. (Reverb) A reverb filter is described by $y[n] = x[n] + 0.25y[n - N]$. Assume that the echoes arrive every millisecond and the sampling rate is 2 kHz.

(a) What is the value of N? Sketch a realization of this reverb filter.
(b) What is the impulse response and step response of this reverb filter?
(c) Find the difference equation of the inverse system. Then, sketch its realization and find its impulse response and step response.

3.38. (Periodic Signal Generators) Find the difference equation of a filter whose impulse response is a periodic sequence with first period $x[n] = \{\overset{\Downarrow}{1}, 2, 3, 4, 6, 7, 8\}$. Sketch a realization for this filter.

3.39. (Recursive and IIR Filters) The terms *recursive* and *IIR* are not always synonymous. A recursive filter could in fact have a finite impulse response. Use recursion to find the the impulse response $h[n]$ for each of the following recursive filters. Which filters (if any) describe IIR filters?

(a) $y[n] - y[n-1] = x[n] - x[n-2]$
(b) $y[n] - y[n-1] = x[n] - x[n-1] - 2x[n-2] + 2x[n-3]$

3.40. (Recursive Forms of FIR Filters) An FIR filter may always be recast in recursive form by the simple expedient of including identical factors on the left-hand and right-hand side of its difference equation in operational form. For example, the filter $y[n] = (1 - z^{-1})x[n]$ is FIR, but the identical filter $(1 + z^{-1})y[n] = (1 + z^{-1})(1 - z^{-1})x[n]$ has the difference equation $y[n] + y[n-1] = x[n] - x[n-2]$ and can be implemented recursively. Find two different recursive difference equations (with different orders) for each of the following filters.

(a) $y[n] = x[n] - x[n-2]$ (b) $h[n] = \{1, \overset{\Downarrow}{2}, 1\}$

3.41. (Nonrecursive Forms of IIR Filters) An FIR filter may always be represented exactly in recursive form, but we can also approximate an IIR filter as an FIR filter by truncating its impulse response to N terms. The larger the truncation index N, the better is the approximation. Consider the IIR filter described by $y[n] - 0.8y[n-1] = x[n]$. Find its impulse response $h[n]$ and truncate it to three terms to obtain $h_3[n]$, the impulse response of the approximate FIR equivalent. Would you expect the greatest mismatch in the response of the two filters to identical inputs to occur for lower or higher values of n? Compare the step response of the two filters up to $n = 6$ to justify your expectations.

3.42. (Nonlinear Systems) One way to solve nonlinear difference equations is by recursion. Consider the nonlinear difference equation $y[n]y[n-1] - 0.5y^2[n-1] = 0.5Au[n]$.

(a) What makes this system nonlinear?
(b) Using $y[-1] = 2$, recursively obtain $y[0]$, $y[1]$, and $y[2]$.

(c) Use $A = 2$, $A = 4$, and $A = 9$ in the results of part (b) to confirm that this system finds the square root of A.
(d) Repeat parts (b) and (c) with $y[-1] = 1$ to check whether the choice of the initial condition affects system operation.

3.43. (LTI Concepts and Stability) Argue that neither of the following describes an LTI system. Then, explain how you might check for their stability and determine which of the systems are stable.

(a) $y[n] + 2y[n-1] = x[n] + x^2[n]$ (b) $y[n] - 0.5y[n-1] = nx[n] + x^2[n]$

3.44. (Response of Causal and Noncausal Systems) A difference equation may describe a causal or noncausal system depending on how the initial conditions are prescribed. Consider a first-order system governed by $y[n] + \alpha y[n-1] = x[n]$.

(a) With $y[n] = 0, n < 0$, this describes a causal system. Assume $y[-1] = 0$ and find the first few terms $y[0], y[1], \ldots$ of the impulse response and step response, using recursion, and establish the general form for $y[n]$.
(b) With $y[n] = 0, n > 0$, we have a noncausal system. Assume $y[0] = 0$ and rewrite the difference equation as $y[n-1] = \{-y[n] + x[n]\}/\alpha$ to find the first few terms $y[0], y[-1], y[-2], \ldots$ of the impulse response and step response, using recursion, and establish the general form for $y[n]$.

3.45. (Time Reversal) For each signal $x[n]$, sketch $g[k] = x[3-k]$ versus k and $h[k] = x[2+k]$ versus k.

(a) $x[n] = \{\overset{\Downarrow}{1}, 2, 3, 4\}$ (b) $x[n] = \{3, 3, \overset{\Downarrow}{3}, 2, 2, 2\}$

[**Hints and Suggestions:** Note that $g[k]$ and $h[k]$ will be plotted against the index k.]

3.46. (Closed-Form Convolution) Find the convolution $y[n] = x[n] * h[n]$ for the following:

(a) $x[n] = u[n]$ $h[n] = u[n]$
(b) $x[n] = (0.8)^n u[n]$ $h[n] = (0.4)^n u[n]$
(c) $x[n] = (0.5)^n u[n]$ $h[n] = (0.5)^n\{u[n+3] - u[n-4]\}$
(d) $x[n] = \alpha^n u[n]$ $h[n] = \alpha^n u[n]$
(e) $x[n] = \alpha^n u[n]$ $h[n] = \beta^n u[n]$
(f) $x[n] = \alpha^n u[n]$ $h[n] = \text{rect}(n/2N)$

[**Hints and Suggestions:** The summations will be over the index k and functions of n should be pulled out before evaluating them using tables. For (a), (b), (d) and (e), summations will be from $k = 0$ to $k = n$. For part (c) and (f), use superposition. For (a) and (d), the sum $\Sigma(1)^k = \Sigma(1)$ from $k = 0$ to $k = n$ equals $n + 1$.]

3.47. (Convolution with Impulses) Find the convolution $y[n] = x[n] * h[n]$ of the following signals.

(a) $x[n] = \delta[n-1]$ $h[n] = \delta[n-1]$
(b) $x[n] = \cos(0.25n\pi)$ $h[n] = \delta[n] - \delta[n-1]$
(c) $x[n] = \cos(0.25n\pi)$ $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$
(d) $x[n] = (-1)^n$ $h[n] = \delta[n] + \delta[n-1]$

[**Hints and Suggestions:** Start with $\delta[n] * g[n] = g[n]$ and use linearity and time invariance.]

3.48. (Convolution) Find the convolution $y[n] = x[n] * h[n]$ for each pair of signals.

(a) $x[n] = (0.4)^{-n}u[n]$ $\quad h[n] = (0.5)^{-n}u[n]$
(b) $x[n] = \alpha^{-n}u[n]$ $\quad h[n] = \beta^{-n}u[n]$
(c) $x[n] = \alpha^{n}u[-n]$ $\quad h[n] = \beta^{n}u[-n]$
(d) $x[n] = \alpha^{-n}u[-n]$ $\quad h[n] = \beta^{-n}u[-n]$

[**Hints and Suggestions:** For parts (a) and (b) write the exponentials in the form r^n. For parts (c) and (d) find the convolution of $x[-n]$ and $h[-n]$ and flip the result to get $y[n]$.]

3.49. (Convolution of Finite Sequences) Find the convolution $y[n] = x[n] * h[n]$ for each of the following signal pairs. Use a marker to indicate the origin $n = 0$.

(a) $x[n] = \{\overset{\Downarrow}{1}, 2, 0, 1\}$ $\quad h[n] = \{\overset{\Downarrow}{2}, 2, 3\}$
(b) $x[n] = \{\overset{\Downarrow}{0}, 2, 4, 6\}$ $\quad h[n] = \{\overset{\Downarrow}{6}, 4, 2, 0\}$
(c) $x[n] = \{-3, -2, -\overset{\Downarrow}{1}, 0, 1\}$ $\quad h[n] = \{\overset{\Downarrow}{4}, 3, 2\}$
(d) $x[n] = \{3, 2, \overset{\Downarrow}{1}, 1, 2\}$ $\quad h[n] = \{4, \overset{\Downarrow}{2}, 3, 2\}$
(e) $x[n] = \{3, 0, 2, 0, \overset{\Downarrow}{1}, 0, 1, 0, 2\}$ $\quad h[n] = \{4, 0, \overset{\Downarrow}{2}, 0, 3, 0, 2\}$
(f) $x[n] = \{\overset{\Downarrow}{0}, 0, 0, 3, 1, 2\}$ $\quad h[n] = \{4, \overset{\Downarrow}{2}, 3, 2\}$

[**Hints and Suggestions:** Since the starting index of the convolution equals the sum of the starting indices of the sequences convolved, ignore markers during convolution and assign as the last step.]

3.50. (Convolution of Symmetric Sequences) The convolution of sequences that are symmetric about their midpoint is also endowed with symmetry (about its midpoint). Compute $y[n] = x[n] * h[n]$ for each pair of signals and use the results to establish the type of symmetry (about the midpoint) in the convolution if the convolved signals are both even symmetric (about their midpoint), both odd symmetric (about their midpoint), or one of each type.

(a) $x[n] = \{2,\ 1,\ 2\}$ $\quad h[n] = \{1,\ 0,\ 1\}$
(b) $x[n] = \{2,\ 1,\ 2\}$ $\quad h[n] = \{1,\ 1\}$
(c) $x[n] = \{2,\ 2\}$ $\quad h[n] = \{1,\ 1\}$
(d) $x[n] = \{2,\ 0,\ -2\}$ $\quad h[n] = \{1,\ 0,\ -1\}$
(e) $x[n] = \{2,\ 0,\ -2\}$ $\quad h[n] = \{1,\ -1\}$
(f) $x[n] = \{2,\ -2\}$ $\quad h[n] = \{1,\ -1\}$
(g) $x[n] = \{2,\ 1,\ 2\}$ $\quad h[n] = \{1,\ 0,\ -1\}$
(h) $x[n] = \{2,\ 1,\ 2\}$ $\quad h[n] = \{1,\ -1\}$
(i) $x[n] = \{2,\ 2\}$ $\quad h[n] = \{1,\ -1\}$

3.51. (Properties) Let $x[n] = h[n] = \{\overset{\Downarrow}{3}, 4, 2, 1\}$. Compute the following:

(a) $y[n] = x[n] * h[n]$ $\quad$ **(b)** $g[n] = x[-n] * h[-n]$
(c) $p[n] = x[n] * h[-n]$ $\quad$ **(d)** $f[n] = x[-n] * h[n]$
(e) $r[n] = x[n-1] * h[n+1]$ $\quad$ **(f)** $s[n] = x[n-1] * h[n+4]$

[**Hints and Suggestions:** The results for (b) and (d) can be found by flipping the results for (a) and (c) respectively. The result for (f) can be found by shifting the result for (e) (time-invariance).]

3.52. (Properties) Let $x[n] = h[n] = \{\overset{\Downarrow}{2}, 6, 0, 4\}$. Compute the following:

(a) $y[n] = x[2n] * h[2n]$
(b) Find $g[n] = x[n/2] * h[n/2]$, assuming zero interpolation.
(c) Find $p[n] = x[n/2] * h[n]$, assuming step interpolation where necessary.
(d) Find $r[n] = x[n] * h[n/2]$, assuming linear interpolation where necessary.

3.53. (Application) Consider a 2-point averaging filter whose present output equals the average of the present and previous input.

(a) Set up a difference equation for this system.
(b) What is the impulse response of this system?
(c) What is the response of this system to the sequence $\{\overset{\Downarrow}{1}, 2, 3, 4, 5\}$?
(d) Use convolution to show that the system performs the required averaging operation.

3.54. (Step Response) Given the impulse response $h[n]$, find the step response $s[n]$ of each system.

(a) $h[n] = (0.5)^n u[n]$ $\quad$ **(b)** $h[n] = (0.5)^n \cos(n\pi)u[n]$
(c) $h[n] = (0.5)^n \cos(n\pi + 0.5\pi)u[n]$ $\quad$ **(d)** $h[n] = (0.5)^n \cos(n\pi + 0.25\pi)u[n]$
(e) $h[n] = n(0.5)^n u[n]$ $\quad$ **(f)** $h[n] = n(0.5)^n \cos(n\pi)u[n]$

[**Hints and Suggestions:** Note that $s[n] = x[n] * h[n]$ where $x[n] = u[n]$. In part (b) and (f), note that $\cos(n\pi) = (-1)^n$. In part (d) expand $\cos(n\pi + 0.25\pi)$ and use the results of parts (b).]

3.55. (Convolution and System Response) Consider the system $y[n] - 0.5y[n-1] = x[n]$.

(a) What is the impulse response $h[n]$ of this system?
(b) Find its output if $x[n] = (0.5)^n u[n]$ by convolution.
(c) Find its output if $x[n] = (0.5)^n u[n]$ and $y[-1] = 0$ by solving the difference equation.
(d) Find its output if $x[n] = (0.5)^n u[n]$ and $y[-1] = 2$ by solving the difference equation.
(e) Are any of the outputs identical? Should they be? Explain.

[**Hints and Suggestions:** For part (e), remember that convolution finds the zero-state response.]

3.56. (Convolution and Interpolation) Let $x[n] = \{\overset{\Downarrow}{2}, 4, 6, 8\}$.

(a) Find the convolution $y[n] = x[n] * x[n]$.
(b) Find the convolution $y_1[n] = x[2n] * x[2n]$. Is $y_1[n]$ related to $y[n]$? Should it be? Explain.
(c) Find the convolution $y_2[n] = x[n/2] * x[n/2]$, assuming zero interpolation. Is $y_2[n]$ related to $y[n]$? Should it be? Explain.
(d) Find the convolution $y_3[n] = x[n/2] * x[n/2]$, assuming step interpolation. Is $y_3[n]$ related to $y[n]$? Should it be? Explain.
(e) Find the convolution $y_4[n] = x[n/2] * x[n/2]$, assuming linear interpolation. Is $y_4[n]$ related to $y[n]$? Should it be? Explain.

3.57. (Linear Interpolation) Consider a system that performs linear interpolation by a factor of N. One way to construct such a system (as shown) is to perform up-sampling by N (zero interpolation between signal samples) and pass the up-sampled signal through a filter with