UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF3470/4470 — Digital signal processing
Day of examination: January 25th, 2019
Examination hours: 09:00 – 13:00
This problem set consists of 9 pages.
Appendices: None
Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note 1: All numbers and figure axes should have units.
Note 2: Read through the whole exercise set before you start!

Problem 1 Z-transform (14 p.)

a) A causal, linear, time-invariant system is given by $y[n] = y[n-1] + y[n-2] + x[n-1]$.

- Determine the system function H(z). Justify your answer. 1 p.
- Determine the poles and the zeros of the system, and indicate their location in a pole-zero plot. Justify your answer. 2 p.
- What is the region of convergence (ROC)? Indicate the ROC in the pole-zero plot. Justify your answer. 1 p.

b) Show that the z-transform of $x[n] = a^n u[n]$ is $X(z) = 1/(1 - az^{-1})$. What is the ROC of $X(z)$? Justify your answer. 2 p.

c) Find the impulse response $h[n]$ of a causal system with

$$H(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

2 p.

d) We have the signal $x[n] = 12 \cdot 0.5^n (u[n+1] - u[n-3])$.

- Sketch $x[n]$. Have clear x- and y-axis values. 1 p.
- Sketch both $x[-n]$ and $x[n+2]$. Have clear axis values. 1 p.
- Sketch $x[-n+2]$. Have clear axis values. 1 p.

e) Consider a system with impulse response $h[n] = \{\underline{2}, -4, 2\}$. What is the output $y[n]$ if we send the signal $x[n] = \{\underline{3}, 0, 4, 1\}$ into this system?

(Continued on page 2.)

- Find $y[n]$ by staying in the time domain. Show full calculation. 1 p.
- Calculate $y[n]$ by converting to the z-transform domain. 2 p.

Problem 2 Mixed problems (9 p.)

a) Petter is bored while waiting at a bus stop. He decides to count the number of Hertz rental cars he sees passing by him. While waiting twenty minutes for his bus to come, he spots six Hertz-cars. What frequency does this correspond to in Hz? 1 p.

b) Petter is playing around with an acoustical transducer that transmits a short sound wave x_1 with sampling frequency F_s. This sound wave x_1 is sent from the transducer towards an object. It hits the object and is reflected back towards the transducer again. Petter registers the received signal x_2. It looks like a delayed and slightly noisy version of x_1. Explain how Petter can use the following equation to find the distance to the object based on the output signal $x_1[n]$, the received signal $x_2[n]$, and the wave speed v. 2 p.

$$r_{x_1x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n-l] = x_1[l] \star x_2[-l], \quad -\infty < l < \infty$$

c) Petter continues to play around with his acoustical transducer and transmits the sound signal $x[n]$ seen in Figure 1. It has a ten second duration and increases linearly in frequency up to 10 Hz. Petter then sends this signal through a filter with the magnitude response shown in Figure 2. Sketch the resulting signal $y(nT)$ after the signal $x(nT)$ has passed through the filter. Have the same x- and y-axes as in Figure 1. 1 p.

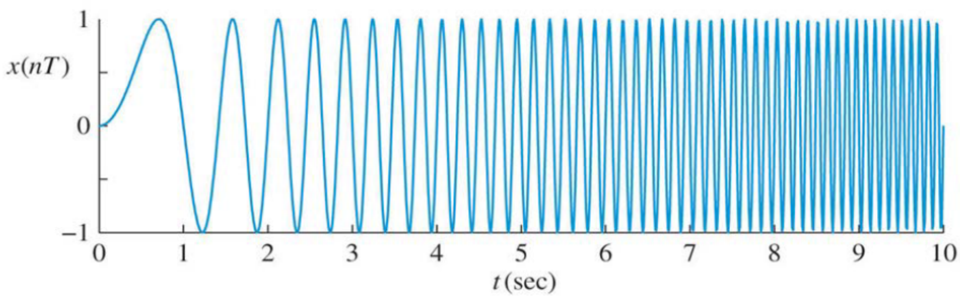

Figure 1: Signal $x(nT)$ used in c). T is the time interval between samples.

d) Is $y[n] = nx[n]$ time invariant? Justify your answer. 1 p.

e) We have a continuous signal $x(t) = 10\sin(2\pi F_1 t) + \sin(2\pi F_2 t)$, where $F_1 = 1$ Hz and $F_2 = 6$ Hz. It is sampled for eight seconds using a 20 Hz sampling frequency. The resulting $x[n]$ is normalized and shown in Figure 3. This signal is then sent through a filter, and $y[n]$ is the

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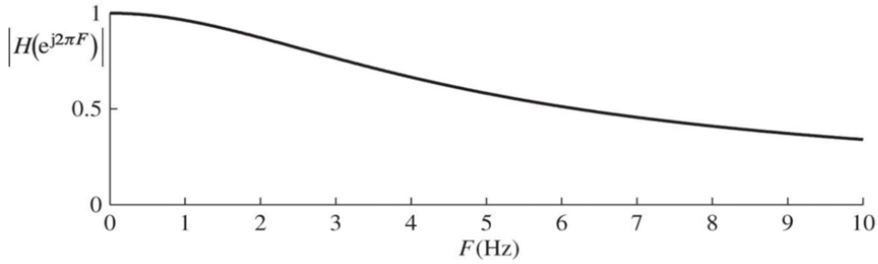

Figure 2: Magnitude response of the filter used in c)

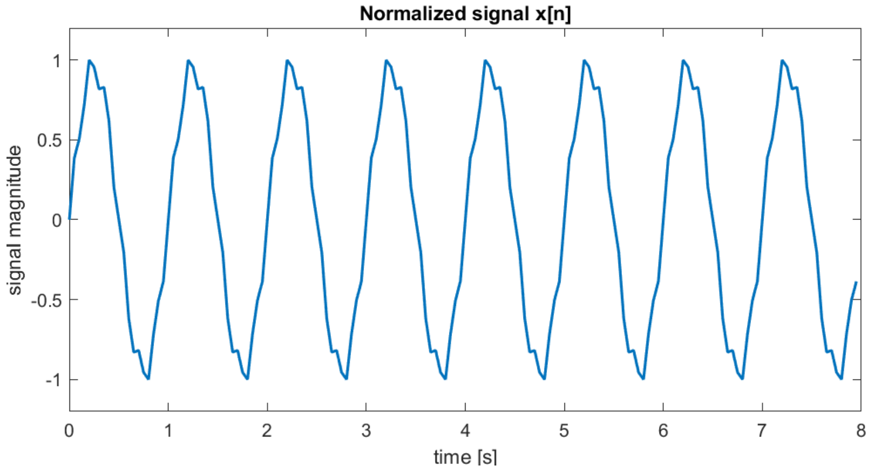

Figure 3: Signal $x[n]$ used in e)

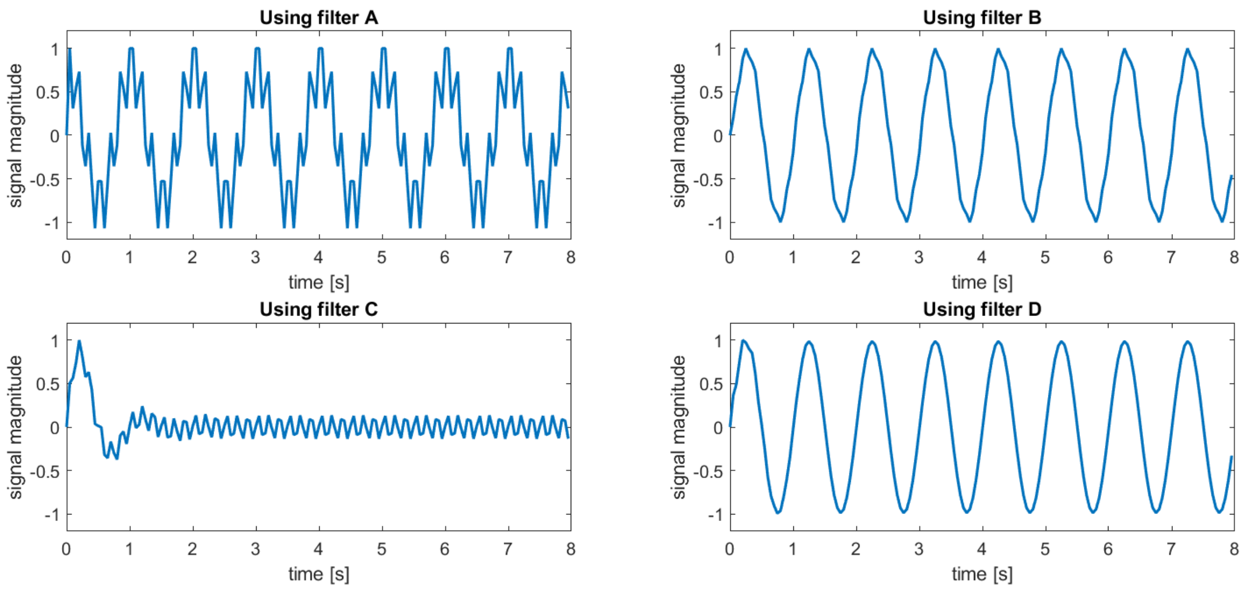

Figure 4: Filtered and normalized signals by using either filter A, B, C or D

resulting output signal. Figure 4 shows the filtered signal after $x[n]$ has been sent through either filter A, B, C or D, and then normalized. Figure 5 shows the seven possible filters. Determine which of these are filters A, B, C and D and **justify your answers**. 4 p.

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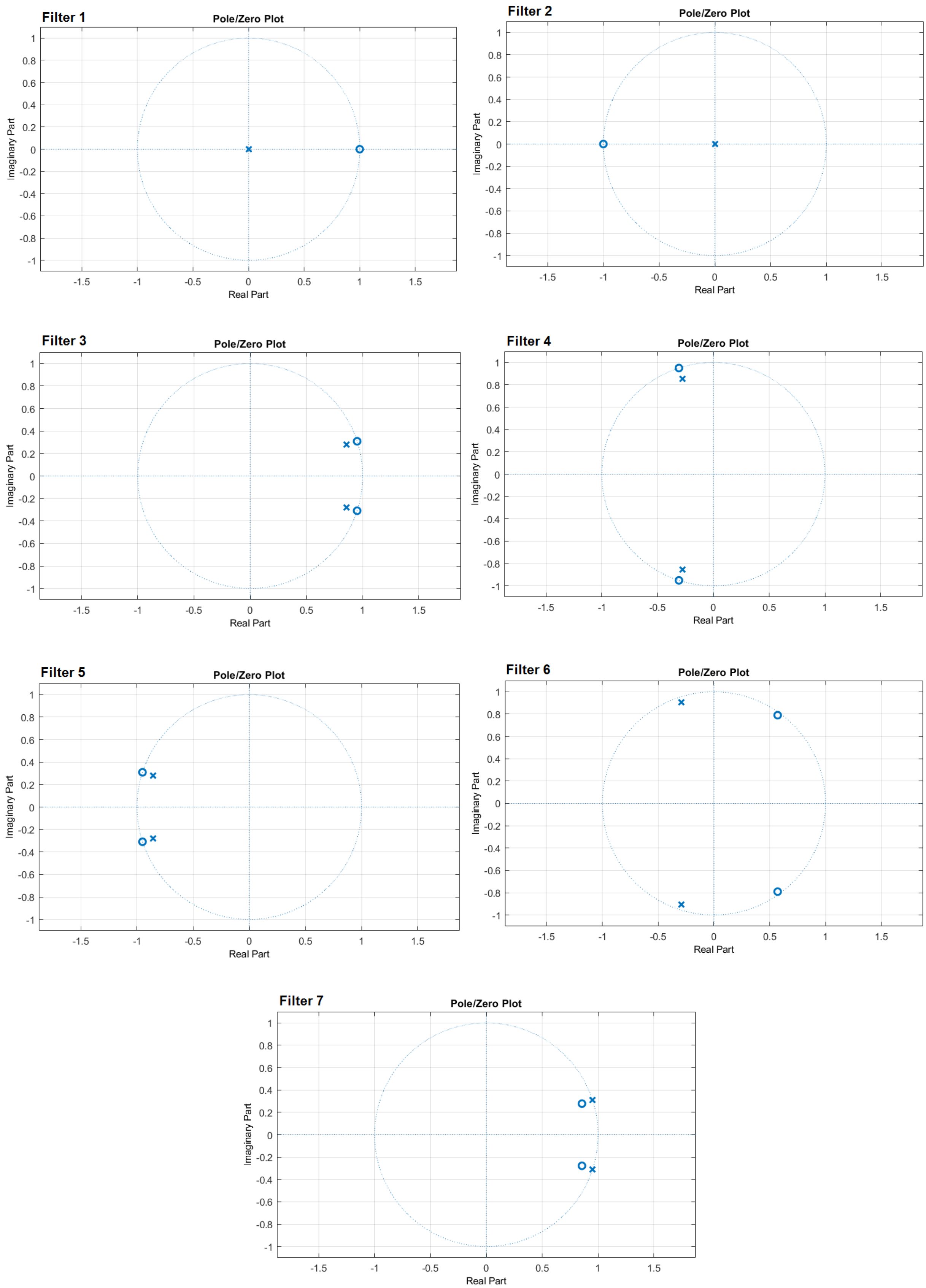

Figure 5: Seven different filters numbered 1-7. Four of them were used to make the filtered signals presented in Figure 4.

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Problem 3 Structures for discrete-time systems (8 p.)

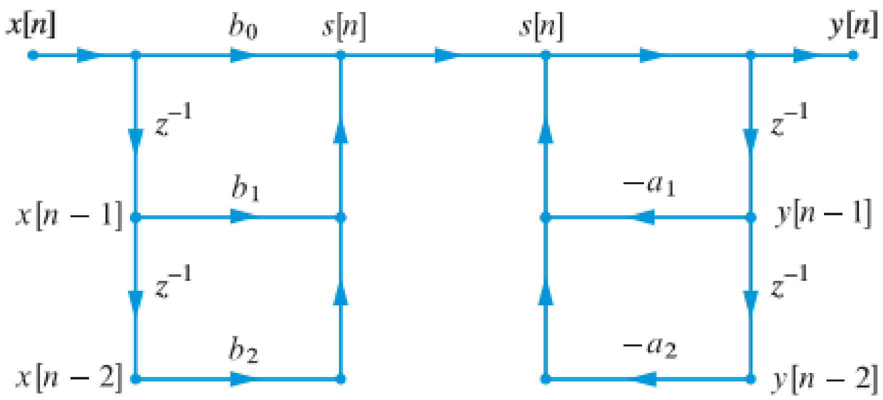

Figure 6: Direct form I structure for second order IIR system.

a) Given a second order system

$$H(z) = \frac{1 + z^{-1} + 0.75z^{-2}}{1 - 2.5z^{-1} + 5z^{-2}}$$

and its corresponding Direct form I structure of figure 6, find the values of b_0, b_1, b_2, a_1, a_2 2 p.

b) Sketch the transposed direct form I of the system. 2 p.

c) Direct form I structures are called "zeros first realizations". Direct form II structures have poles first. Sketch a direct form II realization of the system. Comment on the advantage(s) compared to Direct form I. 2 p.

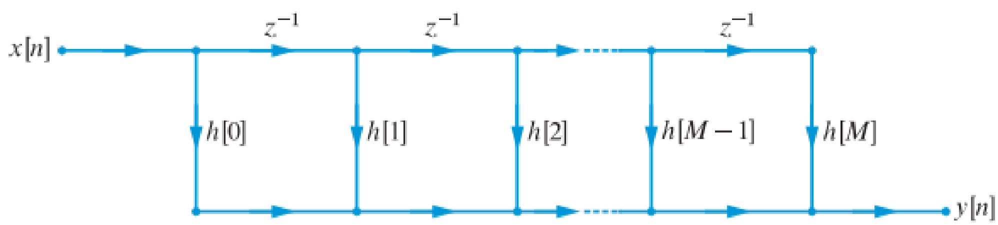

Figure 7: Direct form structure for a FIR system.

d) A general FIR filter can be realized using a FIR direct form, as in figure 7. A hardware implementation of a second order FIR system using FIR direct form, requires 3 multiplication blocks and 2 addition blocks. Now assume that the second order FIR filter has linear phase. Sketch a FIR linear phase realization requiring only 2 multiplication blocks and 2 addition blocks. 2 p.

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Problem 4 Filter design (12 p.)

You have been given the task to design a lowpass filter to remove noise with frequency higher than 4 MHz from a digital signal sampled at $F_s = 10$MHz. The original signal, without noise, is a bandlimited signal in the range $1 - 2$ MHz. It is of utmost importance that you do not distort the original signal, and in addition, it is preferable with a short filter. The transition band should not exceed 1 MHz. The ripple is maximally $\delta_p = 0.02$ in the passband and maximally $\delta_s = 0.01$ in the stopband.

a)
- Calculate the stopband edge frequency in normalized frequency, ω_s. 1 p.
- Calculate the passband edge frequency in normalized frequency, ω_p. 1 p.

b) Sketch the low-pass filter using normalized frequency axes and indicate δ_p, δ_s, ω_s and ω_p on the figure. Make sure to add all axis labels. 2 p.

c) You are now going to design a FIR filter using fixed windows. Using table 1, which windows can you use for your design? Explain your answer. 1 p.

Window name	Side lobe level [dB]	Mainlobe width	Transition band width ($\Delta\omega$)	Ripple level ($\delta_p \approx \delta_s$)	A_p [dB]	A_s [dB]
Rectangular	-13	$4\pi/L$	$1.8\pi/L$	0.09	1.57	21
Bartlett	-25	$8\pi/L$	$6.1\pi/L$	0.05	0.87	26
Hann	-31	$8\pi/L$	$6.2\pi/L$	0.0063	0.11	44
Hamming	-41	$8\pi/L$	$6.6\pi/L$	0.0022	0.038	53
Blackman	-57	$12\pi/L$	$11\pi/L$	0.0002	0.0035	74

Table 1: Characteristics of fixed windows used for FIR filter design. L represents the length of the filter's impulse response.

d) What is the smallest filter order, M, you can achieve? 1 p.

e) If the filter requirement is changed to have a narrower transition band of just 500 kHz, what effect will this have on the filter length? 1 p.

f) Will a filter designed by this procedure always have a linear phase? Explain your answer. 1 p.

(Continued on page 7.)

g) You are considering to use an IIR filter instead. What are the main advantages and disadvantages of using an IIR filter 2 p.

h) The Butterworth IIR lowpass filter is based on the expression

$$|H_B(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}, N = 1, 2, \ldots$$

Use this equation to find an expression for Ω_{-3dB} (the frequency for which $H_B(j\Omega)$ has dropped 3dB). Assume that $10^{3/10} \approx 2$. 2 p.

Problem 5 DFT and Convolution (6 p.)

Assume the following discrete signals with corresponding DFTs:

$$\begin{aligned} x_1[n] &= \{\underline{0}, 0, 1, 0, 1, 0\}, & x_1[n] \overset{DFT}{\underset{N}{\leftrightarrow}} X_1[k] \\ x_2[n] &= \{\underline{0}, 0, 0, 0, 0, 1\}, & x_2[n] \overset{DFT}{\underset{N}{\leftrightarrow}} X_2[k] \end{aligned}$$

a) Find $y[n]$ when $y[n] \overset{DFT}{\underset{N}{\leftrightarrow}} Y[k]$ and $Y[k] = X_1[k]X_2[k]$. You may assume $N = 6$. 2 p.

b) Assume the following signals: $x_3[n] = \{\underline{1}, -3, 0, 2\}$ and $x_4[n] = \{\underline{0}, 1, 2, -1, 0, 1\}$.

- Calculate the linear convolution $x_3[n] * x_4[n]$. 1 p.
- Calculate the 6 point circular convolution $x_3[n]ⓖx_4[n]$. 1 p.

c) Explain the reasons for the different results in b) and c). How long must the circular convolution be in order for the circular and linear convolution to provide the same result. 1 p.

d) Explain why we are sometimes applying zeropadding to a signal before DFT (keywords: filtering, convolution, frequency sampling, visualization, new information). 1 p.

(Continued on page 8.)

Formula sheet

Basic relations:

$$
\begin{aligned}
\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
\sin\alpha + \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha - \sin\beta &= 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha + \cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha - \cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos^2\alpha + \sin^2\alpha &= 1 \\
\cos\alpha &= \frac{1}{2}(e^{j\alpha} + e^{-j\alpha}) \\
\sin\alpha &= \frac{1}{2j}(e^{j\alpha} - e^{-j\alpha}) \\
\sum_{n=0}^{N-1} a^n &= \begin{cases} N & \text{for } a = 1 \\ \frac{1-a^N}{1-a} & \text{otherwise} \end{cases} \\
ax^2 + bx + c = 0 \quad &\Leftrightarrow \quad x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

Linear convolution:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k]h[k] = h[n] * x[n]$$

Circular convolution:

$$y[n] = x[n] \circledN h[n] = \sum_{k=0}^{N-1} x[k]h[< n-k >_N] = \sum_{k=0}^{N-1} x[< n-k >_N]h[k] = h[n] \circledN x[n]$$

Continuous Time Fourier Transform (CTFT):

$$
\begin{aligned}
\text{Analysis:} \qquad X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\
\text{Synthesis:} \qquad x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega
\end{aligned}
$$

Continuous Time Fourier Series (CTFS):

$$
\begin{aligned}
\text{Analysis:} \qquad c_k &= \frac{1}{T_o}\int_{T_0} x(t)e^{-jk\Omega_0 t}dt \\
\text{Synthesis:} \qquad x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}
\end{aligned}
$$

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Discrete Time Fourier Series (DTFS):

$$\text{Analysis:} \qquad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}$$

$$\text{Synthesis:} \qquad x[n] = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi}{N}kn}$$

Discrete Time Fourier Transform (DTFT):

$$\text{Analysis:} \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$$\text{Synthesis:} \qquad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Discrete Fourier Transform (DFT):

$$\text{Analysis:} \qquad X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}, \quad 0 \le k \le N-1$$

$$\text{Synthesis:} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \quad 0 \le n \le N-1$$

z-transform:

$$\text{Analysis:} \qquad X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$