Suggested solution INF3480 exam 2014

June 13, 2014

Exercise 1 (20 %)

a) The algorithm has the following steps.

   (a) Initialize random population
   (b) Evaluate individuals
   (c) Check if termination criterion is reached
   (d) Create new population from good individuals

b) Serial manipulators consist of one open chain, while parallel manipulators have a closed loop with ground.

c) Any of the following will give a full score

   - To solve differential equations.
   - To analyze the stability of the system
   - To find the steady-state error using the final value theorem
   - To analyze the frequency response of the system (e.g. bode plot)

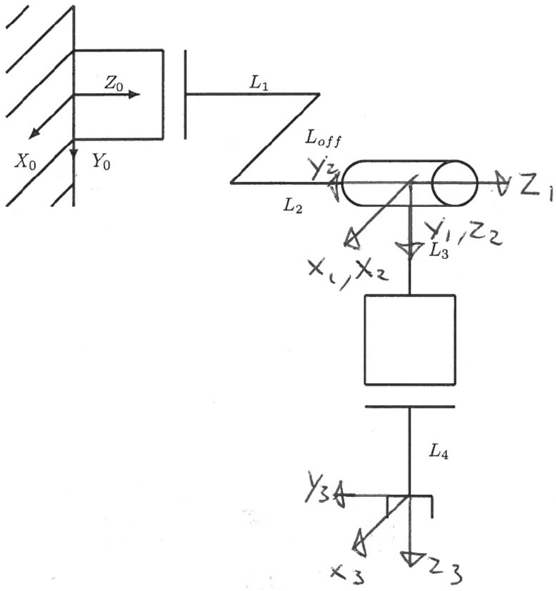
d) Typically mobile robots need exteroceptive sensors to sense the environment, while with manipulators it is often sufficient to use only proprioceptive sensors.

Exercise 2 (45 %)

a) It is a PRP robot.

b) The DH parameters are

	a_i	d_i	α_i	θ_i
1	L_{off}	$L_1 + L_2 + d_1^*$	0	0
2	0	0	$-90°$	θ_2^*
3	0	$L_3 + L_4 + d_3^*$	0	0



In the rest of the exercise we will use

$$\tilde{d}_1^* = L_1 + L_2 + d_1^* \tag{1}$$
$$\tilde{d}_3^* = L_3 + L_4 + d_3^* \tag{2}$$

c) The transformations between the joints based on the DH parameters are (see (3.10) on page 77 in the course textbook)

$$\boldsymbol{T}_1^0 = \begin{bmatrix} 1 & 0 & 0 & L_{off} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{d}_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{T}_2^1 = \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$\boldsymbol{T}_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tilde{d}_3^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Then we multiply these matrices together

$$\boldsymbol{T}_1^0 \boldsymbol{T}_2^1 = \boldsymbol{T}_2^0 = \begin{bmatrix} c_2 & 0 & -s_2 & L_{off} \\ s_2 & 0 & c_2 & 0 \\ 0 & -1 & 0 & \tilde{d}_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\boldsymbol{T}_2^0 \boldsymbol{T}_3^2 = \boldsymbol{T}_3^0 = \begin{bmatrix} c_2 & 0 & -s_2 & -s_2\tilde{d}_3^* + L_{off} \\ s_2 & 0 & c_2 & c_2\tilde{d}_3^* \\ 0 & -1 & 0 & \tilde{d}_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

d) The equations for the jacobian is given on page 133 in the textbook. It is a PRP manipulator, therefore the jacobian is:

$$\boldsymbol{J} = \begin{bmatrix} z_0 & z_1 \times (o_3 - o_1) & z_2 \\ 0 & z_1 & 0 \end{bmatrix} \tag{7}$$

The calculating the elements in the jacobian.

$$o_3 - o_1 = \begin{bmatrix} -s_2\tilde{d}_3^* + L_{off} \\ c_2\tilde{d}_3^* \\ \tilde{d}_1^* \end{bmatrix} - \begin{bmatrix} L_{off} \\ 0 \\ \tilde{d}_1^* \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} -s_2\tilde{d}_3^* \\ c_2\tilde{d}_3^* \\ 0 \end{bmatrix} \tag{9}$$

$$z_1 \times (o_3 - o_1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -s_2\tilde{d}_3^* & c_2\tilde{d}_3^* & 0 \end{vmatrix} \tag{10}$$

$$= \begin{bmatrix} -c_2\tilde{d}_3^* \\ -s_2\tilde{d}_3^* \\ 0 \end{bmatrix} \tag{11}$$

This yields the jacobian

$$\boldsymbol{J} = \begin{bmatrix} 0 & -c_2\tilde{d}_3^* & -s_2 \\ 0 & -s_2\tilde{d}_3^* & c_2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{12}$$

e) The inverse kinematics can be solved analytically and geometrically. Solving analytically we start with the tree equations

$$x = -s_2\tilde{d}_3^* + L_{off} \tag{13}$$

$$y = c_2\tilde{d}_3^* \tag{14}$$

$$z = \tilde{d}_1^* \tag{15}$$

It is already solved for $\tilde{d}_1^*$. Written with d_1^* yields

$$d_1^* = z - L_1 - L_2 \tag{16}$$

Now aquaring the to remaining equations yields

$$(x - L_{off})^2 = s_2^2\left(\tilde{d}_3^*\right)^2 \tag{17}$$

$$y^2 = c_2^2\left(\tilde{d}_3^*\right)^2 \tag{18}$$

Taking the sum of the two equations yields

$$(x - L_{off})^2 + y^2 = \left(\tilde{d}_3^*\right)^2 \tag{19}$$

The solving for d_3^*

$$d_3^* = \pm\sqrt{(x - L_{off})^2 + y^2} - L_3 - L_4 \tag{20}$$

Now only θ_2^* is remaining. Rearranging the equations for x and y yields

$$s_2 = -\frac{x - L_{off}}{\tilde{d}_3^*} \qquad\qquad c_2 = \frac{y}{\tilde{d}_3^*} \tag{21}$$

This can be solved using the *atan2* function

$$\theta_3^* = \mathrm{atan2}(y, -x + L_{off}) \tag{22}$$

Since θ_2^* does not depend on $\tilde{d}_3^*$ an alternative solution for $\tilde{d}_3^*$ can be found

$$y = c_2 \tilde{d}_3^* \tag{23}$$

$$\tilde{d}_3^* = \frac{y}{c_2} \tag{24}$$

Inserting for $\tilde{d}_3^*$ yields

$$d_3^* = \frac{y}{c_2} - L_3 - L_4 \tag{25}$$

Exercise 3 (15 %)

a) Position of the mass is (can be found directly or by simplifying the forward kinematics in exercise 2)

$$\boldsymbol{p} = \begin{bmatrix} -L_2 \sin\theta_2 \\ L_2 \cos\theta_2 \\ d_1 + L_1 \end{bmatrix} \tag{26}$$

The finding the derivative (alternatively by simplifying the jacobian in exercise 2)

$$\boldsymbol{v} = \begin{bmatrix} -L_2 c_2 \dot{\theta}_2 \\ -L_2 s_2 \dot{\theta}_2 \\ \dot{d}_1 \end{bmatrix} \tag{27}$$

The kinetic energy is

$$\mathcal{K} = \frac{1}{2} m \boldsymbol{v}^2 \tag{28}$$

$$= \frac{1}{2} m \left(\dot{d}_1^2 + L_2^2 \dot{\theta}_2^2 \right) \tag{29}$$

Potential energy is (gravity in y-direction)

$$\mathcal{P} = mgh \tag{30}$$

$$= mgL_2 \cos\theta_2 \tag{31}$$

Combining these two to the Lagrangian yields

$$\mathcal{L} = \mathcal{K} - \mathcal{P} \tag{32}$$

$$= \frac{1}{2} m \left(\dot{d}_1^2 + L_2^2 \dot{\theta}_2^2 \right) - mgL_2 \cos\theta_2 \tag{33}$$

b) Using (7.5) on page 241 in the textbook, and calculating for the individual parts

$$\frac{\partial \mathcal{L}}{\partial \dot{d}_1} = m\dot{d}_1 \qquad \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = mL_2^2 \dot{\theta}_2 \tag{34}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{d}_1} = m\ddot{d}_1 \qquad \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = mL_2^2 \ddot{\theta}_2 \tag{35}$$

$$\frac{\partial \mathcal{L}}{\partial d_1} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \theta_2} = mgL_2 \sin\theta_2 \tag{36}$$

$$\tag{37}$$

This yields the two dynamic equations

$$m\ddot{d}_1 = f_1 \tag{38}$$

$$mL_2^2 \ddot{\theta}_2 - mgL_2 \sin\theta_2 = \tau_2 \tag{39}$$

In matrix form

$$\begin{bmatrix} m & 0 \\ 0 & mL_2^2 \end{bmatrix} \ddot{\boldsymbol{q}} + \begin{bmatrix} 0 \\ -mgL_2 \sin\theta_2 \end{bmatrix} = \boldsymbol{\tau} \tag{40}$$

Exercise 4 (20 %)

a) Proportional controller (P-controller)

b) The system without and with controller

$$Js^2\theta = -D - K\theta \qquad\qquad Js^2\theta = -D - K\theta + C(\theta^d - \theta) \tag{41}$$

The rearranging the equations yield

$$Js^2\theta + K\theta + D = 0 \qquad\qquad Js^2\theta + (K + C)\theta + D = C\theta^d \tag{42}$$

Then tranforming the equations into the time domain yields

$$J\frac{d^2\theta}{dt^2} + K\theta + D = 0 \qquad\qquad J\frac{d^2\theta}{dt^2} + (K + C)\theta + D = C\theta^d \tag{43}$$

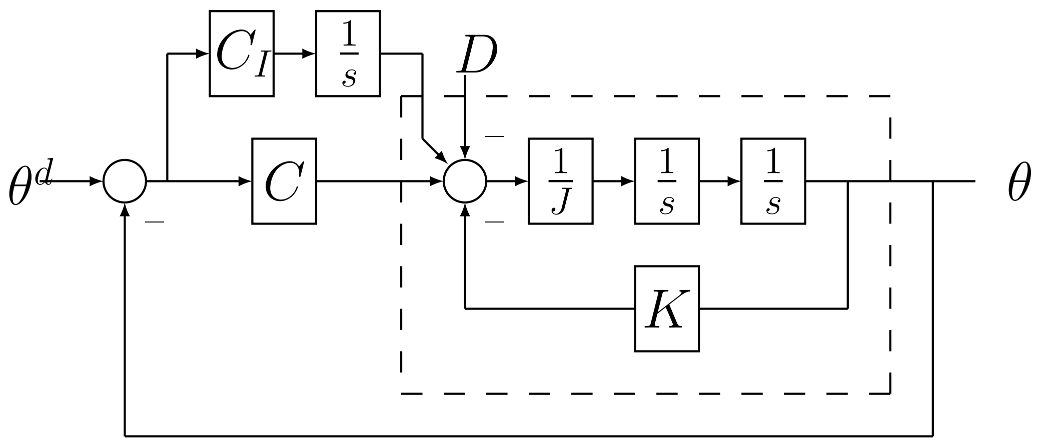
c) Rearranging the equation (system with controller) from b) yields

$$\theta = \frac{C\theta^d - D}{Js^2 + K + C} \tag{44}$$

Inserting θ into the final value theorem

$$\begin{aligned}
\lim_{t\to\infty} \theta(t) &= \lim_{s\to 0} s\theta(s) && (45)\\
&= \lim_{s\to 0} \cancel{s}\frac{C\frac{1}{\cancel{s}}T - \frac{1}{\cancel{s}}D_c}{Js^2 + K + C} && (46)\\
&= \lim_{s\to 0} \frac{CT - D_c}{\cancelto{0}{Js^2} + K + C} && (47)\\
&= \frac{C}{K + C}T - \frac{1}{K + C}D_c && (48)
\end{aligned}$$

d) Integral controller (I-controller / PI-controller)



Exercise 5 (X %) (Master only)

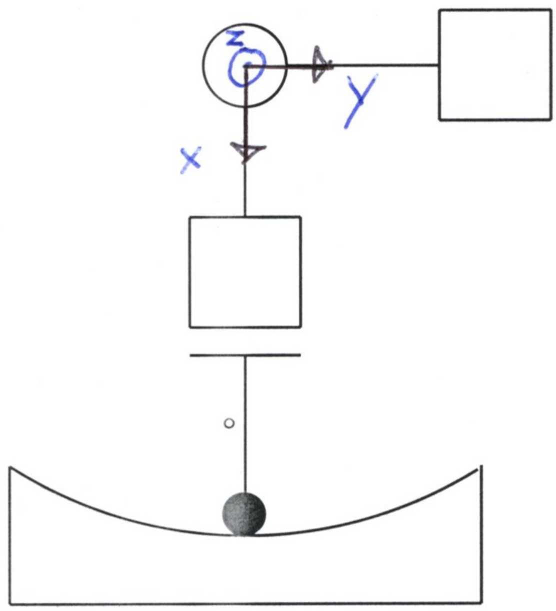
a) Using (4.105) on page 149 in the textbook

$$\boldsymbol{\tau} = \boldsymbol{J}^T\boldsymbol{F} \tag{49}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{50}$$

$$= \begin{bmatrix} 1 \\ -\sqrt{3} - \frac{3}{2} \\ \frac{3}{2}\sqrt{3} - 1 \end{bmatrix} \tag{51}$$

b) The compliance frame is placed in the center of joint 2 with the x-axis pointing towards the tool:



First we define the natural constraints. The robot cannot move in the x-direction as it would collide with the surface. There are no forces that can interact in the y- and z-direction. The robot cannot rotate around the x and y axis, thus the velocities are zero. The is no external force when rotating around the z-axis. This is summarized in the table below.

For the artificial constraints the force in the x-direction should be constant. The robot should move to paint the entire surface, which means that the robot must have a variable velocity in v_x and ω_z. The rest of the artificial constraints should be zero. The reciprocity condition is met with these constraints.

Natural	Artificial
$v_x = 0$	$f_x = f_c$
$f_y = 0$	$v_x = 0$
$f_z = 0$	$v_x = variable$
$\omega_x = 0$	$\tau_x = 0$
$\omega_y = 0$	$\tau_y = 0$
$\tau_z = 0$	$\omega_z = variable$

c) Using the two equations provided in the exercise one get

$$M\ddot{x} - G + F_e = Ma_x - G + a_f \tag{52}$$
$$M\ddot{x} = Ma_x + a_f - F_e \tag{53}$$
$$\ddot{x} = a_x + M^{-1}(a_f - F_e) \tag{54}$$
$$\tag{55}$$

d) Using a_x and a_f as they are defined in the textbook

$$a_x = \ddot{x}^d - M_d^{-1}(B_d\dot{\tilde{x}} + K_d\tilde{x} + F_e) \tag{56}$$
$$a_f = F_e \tag{57}$$

Then insert this into the equation in c)

$$\begin{aligned}
\ddot{x} &= a_x + M^{-1}(a_f - F_e) && (58)\\
\ddot{x} &= \ddot{x}^d - M_d^{-1}(B_d\dot{\tilde{x}} + K_d\tilde{x} + F_e) + M^{-1}(F_e - F_e) && (59)\\
\ddot{x} - \ddot{x}^d &= -M_d^{-1}(B_d\dot{\tilde{x}} + K_d\tilde{x} + F_e) && (60)\\
M_d\ddot{\tilde{x}} &= -(B_d\dot{\tilde{x}} + K_d\tilde{x} + F_e) && (61)\\
M_d\ddot{\tilde{x}} + B_d\dot{\tilde{x}} + K_d\tilde{x} &= -F_e && (62)
\end{aligned}$$

We have now shown that the chosen a_x and a_f gives the desired system.

e) Using the equation for the controller

$$\begin{aligned}
u &= Ma_x - Gx + a_f && (63)\\
&= M\ddot{x}^d - M_d^{-1}(B_d\dot{\tilde{x}} + K_d\tilde{x} + F_e) - Gx + F_e && (64)
\end{aligned}$$

Then inserting the numbers

$$\begin{aligned}
u &= 2 \cdot 0 - 1(0.5 \cdot 0 + 0.25 \cdot 0.1 + 2) - 20 + 2 && (65)\\
&= -(0.25 \cdot 0.1 + 2) - 20 + 2 && (66)\\
&= -(0.25 \cdot 0.1 + 2) - 20 + 2 && (67)\\
&= 20.025 && (68)
\end{aligned}$$