Deriving J_ω

- **Now J_ω can simply be written as follows:**

$$J_\omega = \begin{bmatrix} \rho_1 z_0^0 & \rho_2 z_1^0 & \cdots & \rho_n z_{n-1}^0 \end{bmatrix}$$

  - **There are *n* columns, each is 3x1, thus J_ω is 3x*n***

Deriving J_v

- **Linear velocity of the end effector:**

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i$$

- **Therefore we can simply write the i^{th} column of J_v as:**

$$J_{v_i} = \frac{\partial o_n^0}{\partial q_i}$$

- **However, the linear velocity of the end effector can be due to the motion of revolute and/or prismatic joints**
- **Thus the end-effector velocity is a linear combination of the velocity due to the motion of each joint**
  - **w/o L.O.G. we can assume all joint velocities are zero other than the i^{th} joint**
  - **This allows us to examine the end-effector velocity due to the motion of either a revolute or prismatic joint**

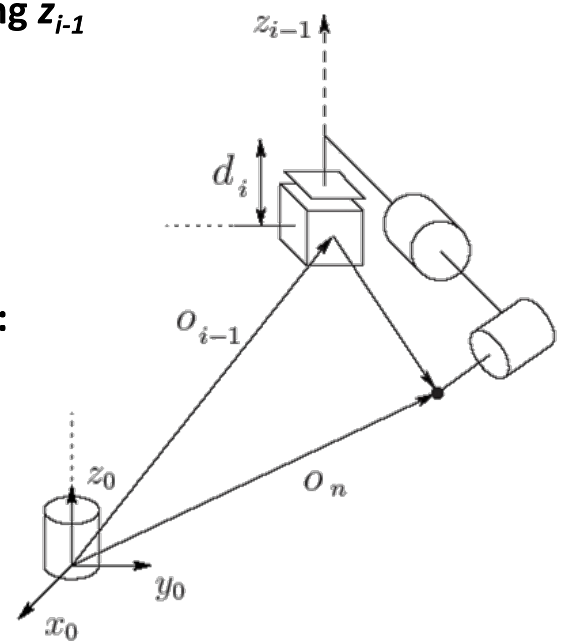
Deriving J_v

- **End-effector velocity due to prismatic joints**
  - **Assume all joints are fixed other than the prismatic joint d_i**
  - **The motion of the end-effector is pure translation along z_{i-1}**

$$\dot{o}_n^0 = \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{d}_i z_{i-1}^0$$

  - **Therefore, we can write the i^{th} column of the Jacobian:**

$$J_{v_i} = z_{i-1}^0$$



Deriving J_v

- End-effector velocity due to revolute joints
  - Assume all joints are fixed other than the revolute joint θ_i
  - The motion of the end-effector is given by:

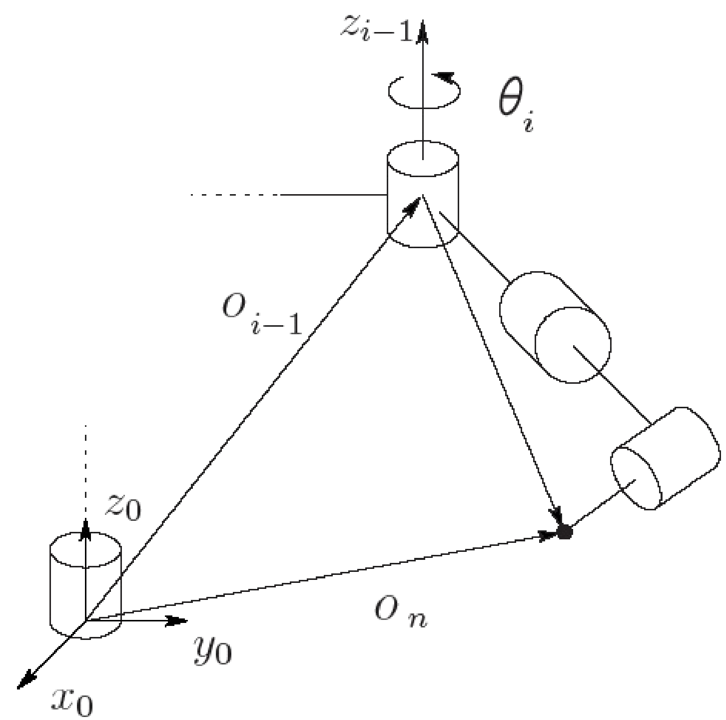
$$\dot{o}_n^0 = \omega_{i-1,i}^0 \times r = \dot{\theta}_i z_{i-1}^0 \times r$$

  - Where the term r is the distance from the tool frame o_n to the frame o_{i-1}

$$\dot{o}_n^0 = \dot{\theta}_i z_{i-1}^0 \times (o_n - o_{i-1})$$

  - Thus we can write the i^{th} column of J_v:

$$J_{v_i} = z_{i-1}^0 \times (o_n - o_{i-1})$$



The complete Jacobian

- **The *i*th column of J_v is given by:**

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for } i \text{ revolute} \\ z_{i-1} & \text{for } i \text{ prismatic} \end{cases}$$

- **The *i*th column of J_ω is given by:**

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for } i \text{ revolute} \\ 0 & \text{for } i \text{ prismatic} \end{cases}$$

Example: two-link planar manipulator

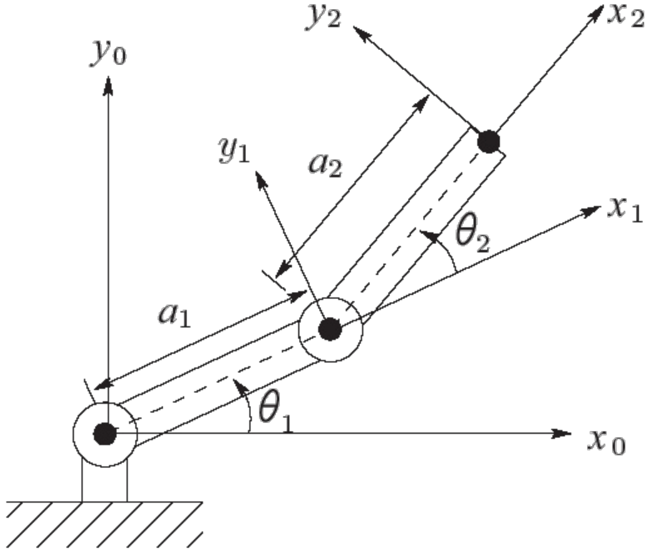
- **Calculate *J* for the following manipulator:**
  - **Two joint angles, thus *J* is 6x2**

$$J(q) = \begin{bmatrix} z_0^0 \times (o_2 - o_0) & z_1^0 \times (o_2 - o_1) \\ z_0^0 & z_1^0 \end{bmatrix}$$

  - **Where:**

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix}, o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \qquad z_0^0 = z_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J(q) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

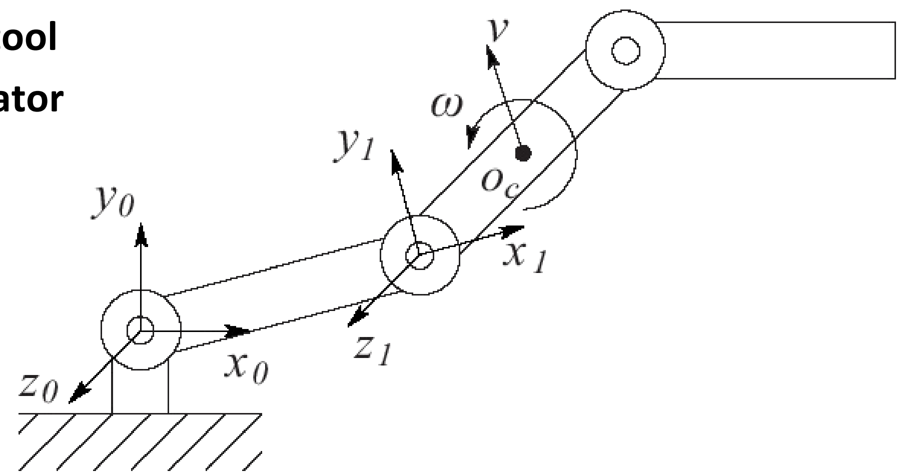


Example: velocity of an arbitrary point

- **We can also use the Jacobian to calculate the velocity of any arbitrary point on the manipulator**

$$J(q)=\begin{bmatrix} z_0^0 \times (o_c - o_0) & z_1^0 \times (o_c - o_1) & 0 \\ z_0^0 & z_1^0 & 0 \end{bmatrix}$$

  - **This is identical to placing the tool frame at any point of the manipulator**



Example: Stanford manipulator

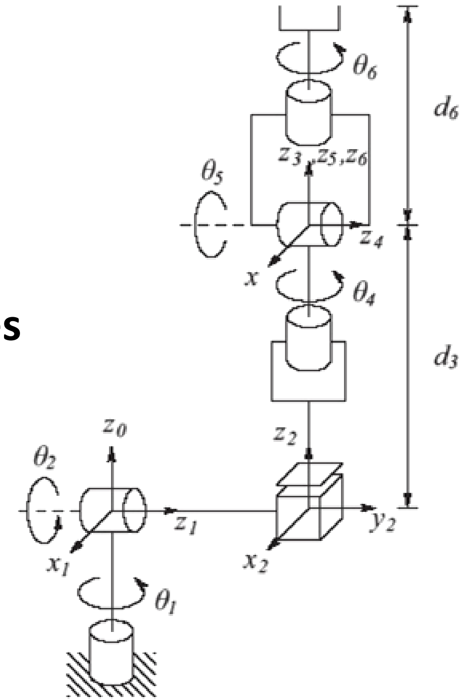
- **The configuration of the Stanford manipulator allows us to make the following simplifications:**

$$J_i = \begin{bmatrix} z_{i-1} \times (o_6 - o_{i-1}) \\ z_{i-1} \end{bmatrix}, \; i = 1,2$$

$$J_3 = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}$$

$$J_i = \begin{bmatrix} z_{i-1} \times (o_6 - o) \\ z_{i-1} \end{bmatrix}, \; i = 4,5,6$$

- **Where *o* is the common origin of the o_3, o_4, and o_5 frames**



Example: Stanford manipulator

- **From the forward kinematics of the Stanford manipulator, we calculated the homogeneous transformations for each joint:**

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Stanford manipulator

- To complete the derivation of the Jacobian, we need the following quantities: z_0, z_1, … , z_5, o_0, o_1, o_3, o_6
  - o_3 is o and $o_0 = [0\ 0\ 0]^T$
- We determine these quantities by noting the construction of the T matrices
  - o_j is the first three elements of the last column of T_j^0
  - z_j is $R_j^0 k$, or equivalently, the first three elements of the third column of T_j^0
- Thus we can calculate the Jacobian by first determining the T_j^0
  - Thus the z_i terms are given as follows:

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, z_1 = \begin{bmatrix} -S_1 \\ C_1 \\ 0 \end{bmatrix}, z_2 = \begin{bmatrix} C_1 S_2 \\ S_1 S_2 \\ C_2 \end{bmatrix}, z_3 = \begin{bmatrix} C_1 S_2 \\ S_1 S_2 \\ C_2 \end{bmatrix}, z_4 = \begin{bmatrix} -C_1 C_2 S_4 - S_1 C_4 \\ -S_1 C_2 S_4 + C_1 C_4 \\ S_2 S_4 \end{bmatrix}, z_5 = \begin{bmatrix} C_1 C_2 C_4 S_5 - S_1 S_4 S_5 + C_1 S_2 C_5 \\ S_1 C_2 C_4 S_5 + C_1 S_4 S_5 + S_1 S_2 C_5 \\ -S_2 C_4 S_5 + C_2 C_5 \end{bmatrix}$$

Example: Stanford manipulator

- **And the *o*$_i$ terms are given as:**

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, o_1 = \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix}, o_3 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}, o_6 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6 \left(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5 \right) \\ s_1 s_2 d_3 - c_1 d_2 + d_6 \left(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2 \right) \\ c_2 d_3 + d_6 \left(c_2 c_5 - c_4 s_2 s_5 \right) \end{bmatrix}$$

- **Finally, the Jacobian can be assembled as follows:**

$$J_1 = \begin{bmatrix} -d_y \\ d_x \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, J_2 = \begin{bmatrix} c_1 d_z \\ s_1 d_z \\ -s_1 d_y - c_1 d_x \\ -s_1 \\ c_1 \\ 0 \end{bmatrix}, J_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, J_4 = \begin{bmatrix} s_1 s_2 \left(d_z - o_{3,z} \right) + c_2 \left(d_y - o_{3,y} \right) \\ -c_1 s_1 \left(d_z - o_{3,z} \right) + c_2 \left(d_x - o_{3,x} \right) \\ -c_1 c_2 s_4 - s_1 c_4 \\ s_2 s_4 \\ 0 \\ 0 \end{bmatrix}$$

Example: Stanford manipulator

- **Finally, the Jacobian can be assembled as follows:**

$$J_5 = \begin{bmatrix} (-s_1c_2s_4 + c_1c_4)(d_z - o_{3,z}) - s_2s_4(d_y - o_{3,y}) \\ (-c_1c_2s_4 + s_1c_4)(d_z - o_{3,z}) + s_2s_4(d_x - o_{3,x}) \\ (-c_1c_2s_4 - s_1c_4)(d_y - o_{3,y}) + (s_1c_2s_4 - c_1c_4)(d_x - o_{3,x}) \\ -c_1c_2c_4 - s_1c_4 \\ s_2s_4 \\ 0 \end{bmatrix}$$

$$J_6 = \begin{bmatrix} (s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5)(d_y - o_{3,y}) + (s_2c_4s_5 - c_2c_5)(d_y - o_{3,y}) \\ -(c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5)(d_z - o_{3,z}) + (s_2c_4s_5 - c_2c_5)(d_x - o_{3,x}) \\ c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2c_5 \\ s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c \\ -s_2c_4s_5 + c_2c_5 \\ 0 \end{bmatrix}$$

Example: SCARA manipulator

- **Jacobian will be a 6x4 matrix**

$$J = \begin{bmatrix} z_0^0 \times (o_4 - o_0) & z_1^0 \times (o_4 - o_1) & z_2^0 & z_3^0 \times (o_4 - o_3) \\ z_0^0 & z_1^0 & 0 & z_3^0 \end{bmatrix}$$
$$= \begin{bmatrix} z_0^0 \times (o_4 - o_0) & z_1^0 \times (o_4 - o_1) & z_2^0 & 0 \\ z_0^0 & z_1^0 & 0 & z_3^0 \end{bmatrix}$$

- **Thus we will need to determine the following quantities:** $z_0, z_1, \ldots, z_3, o_0, o_1, o_2, o_4$
  - **Since all the joint axes are parallel, we can see the following:**

$$z_0^0 = z_1^0 = \hat{k},\ z_2^0 = z_3^0 = -\hat{k}$$

  - **From the homogeneous transformation matrices we can determine the origins of the coordinate frames**

Example: SCARA manipulator

- **Thus *o_0, o_1, o_2, o_4* are given by:**

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix}, o_4 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ d_3 - d_4 \end{bmatrix}$$

- **Finally, we can assemble the Jacobian:**

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

Next class…

- **Formal definition of singularities**
- **Tool velocity**
- **manipulability**