Recap: kinematic decoupling

- **Appropriate for systems that have an arm a wrist**
  - **Such that the wrist joint axes are aligned at a point**
- **For such systems, we can split the inverse kinematics problem into two parts:**
  1. **Inverse position kinematics: position of the wrist center**
  2. **Inverse orientation kinematics: orientation of the wrist**
- **First, assume 6DOF, the last three intersecting at o_c**

$$R_6^0(q_1,\ldots,q_6) = R$$
$$o_6^0(q_1,\ldots,q_6) = o$$

- **Use the position of the wrist center to determine the first three joint angles…**

Recap: kinematic decoupling

- **Now, origin of tool frame, o_6, is a distance d_6 translated along z_5 (since z_5 and z_6 are collinear)**
  - **Thus, the third column of R is the direction of z_6 (w/ respect to the base frame) and we can write:**

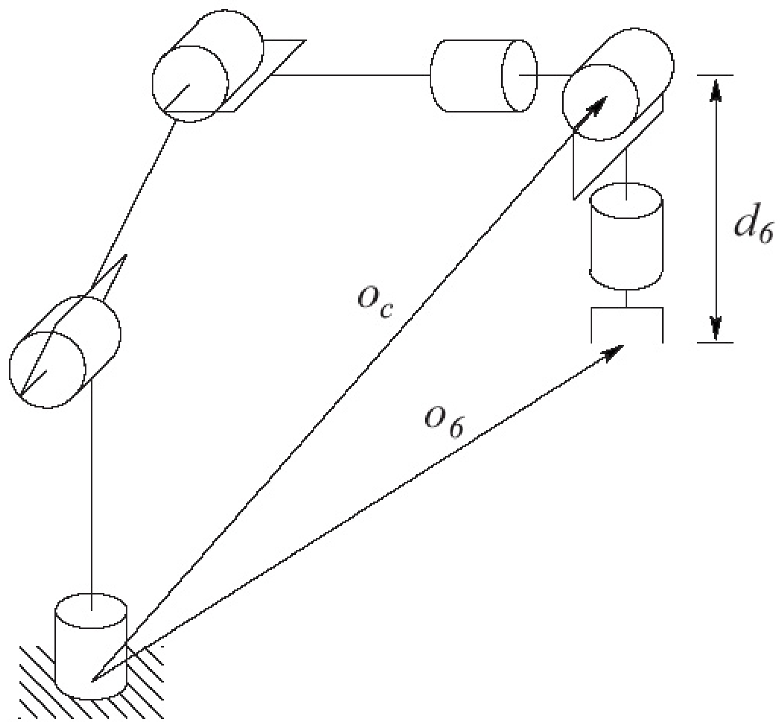
$$o = o_6^0 = o_c^o + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - **Rearranging:**

$$o_c^o = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - **Calling $o = [o_x\ o_y\ o_z]^T$, $o_c^0 = [x_c\ y_c\ z_c]^T$**

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$



Recap: kinematic decoupling

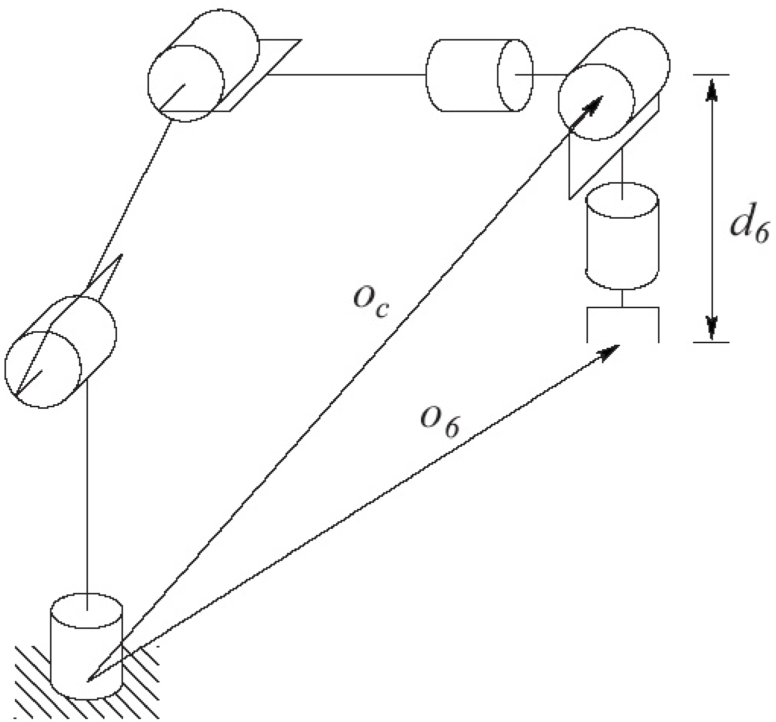
- **Since $[x_c\ y_c\ z_c]^T$ are determined from the first three joint angles, our forward kinematics expression now allows us to solve for the first three joint angles decoupled from the final three.**
  - **Thus we now have R_3^0**
  - **Note that:**

$$R = R_3^0 R_6^3$$

  - **To solve for the final three joint angles:**

$$R_6^3 = \left(R_3^0\right)^{-1} R = \left(R_3^0\right)^T R$$

  - **Since the last three joints for a spherical wrist, we can use a set of Euler angles to solve for them**



Recap: Inverse position kinematics

- **Now that we have $[x_c\ y_c\ z_c]^T$ we need to find q_1, q_2, q_3**
  - **Solve for q_i by projecting onto the x_{i-1}, y_{i-1} plane, solve trig problem**
  - **Two examples**
    - **elbow (RRR) manipulator: 4 solutions (left-arm elbow-up, left-arm elbow-down, right-arm elbow-up, right-arm elbow-down)**
    - **spherical (RRP) manipulator: 2 solutions (left-arm, right-arm)**

Inverse orientation kinematics

- **Now that we can solve for the position of the wrist center (given kinematic decoupling), we can use the desired orientation of the end effector to solve for the last three joint angles**
  - **Finding a set of Euler angles corresponding to a desired rotation matrix R**
  - **We want the final three joint angles that give the orientation of the tool frame with respect to o_3 (i.e. R_6^3)**

Inverse Kinematics: general procedure

1. **Find q_1, q_2, q_3 such that the position of the wrist center is:**

$$o_c^o = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2. **Using q_1, q_2, q_3, determine R_3^0**
3. **Find Euler angles corresponding to the rotation matrix:**

$$R_6^3 = \left(R_3^0\right)^{-1} R = \left(R_3^0\right)^T R$$

Steps 1–2: inverse position kinematics

Step 3: inverse orientation kinematics

Velocity Kinematics

- **Now we know how to relate the end-effector position and orientation to the joint variables**
- **Now we want to relate end-effector linear and angular velocities with the joint velocities**
- **First we will discuss angular velocities about a fixed axis**
- **Second we discuss angular velocities about arbitrary (moving) axes**
- **We will then introduce the Jacobian**
  - **Instantaneous transformation between a vector in R^n representing joint velocities to a vector in R^6 representing the linear and angular velocities of the end-effector**
- **Finally, we use the Jacobian to discuss numerous aspects of manipulators:**
  - **Singular configurations**
  - **Dynamics**
  - **Joint/end-effector forces and torques**

Angular velocity: fixed axis

- **When a rigid body rotates about a fixed axis, every point moves in a circle**
  - **Let *k* represent the fixed axis of rotation, then the angular velocity is:**

$$\omega = \dot{\theta}\hat{k}$$

  - **The velocity of any point on a rigid body due to this angular velocity is:**

$$v = \omega \times r$$

  - **Where *r* is the vector from the axis of rotation to the point**

- **When a rigid body translates, all points attached to the body have the same velocity**

The complete Jacobian

- **The *i*th column of J_v is given by:**

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for } i \text{ revolute} \\ z_{i-1} & \text{for } i \text{ prismatic} \end{cases}$$

- **The *i*th column of J_ω is given by:**

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for } i \text{ revolute} \\ 0 & \text{for } i \text{ prismatic} \end{cases}$$

Example: two-link planar manipulator

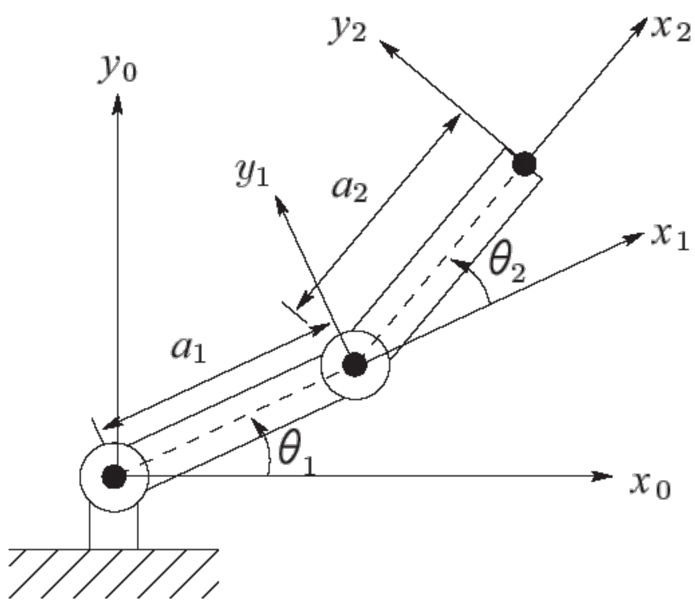
- **Calculate *J* for the following manipulator:**
  - **Two joint angles, thus *J* is 6x2**

$$J(q)=\begin{bmatrix} z_0^0\times(o_2-o_0) & z_1^0\times(o_2-o_1) \\ z_0^0 & z_1^0 \end{bmatrix}$$

  - **Where:**

$$o_0=\begin{bmatrix}0\\0\\0\end{bmatrix}, o_1=\begin{bmatrix}a_1c_1\\a_1s_1\\0\end{bmatrix}, o_2=\begin{bmatrix}a_1c_1+a_2c_{12}\\a_1s_1+a_2s_{12}\\0\end{bmatrix} \qquad z_0^0=z_1^0=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$J(q)=\begin{bmatrix} -a_1s_1-a_2s_{12} & -a_2s_{12} \\ a_1c_1+a_2c_{12} & a_2c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



Singularities

- **We can now derive the Jacobian as a mapping given by the following:**

$$\xi = J(q)\dot{q}$$

- **This means that the columns of *J* form a basis for the space of possible end effector velocities, meaning that all possible end-effector velocities are linear combinations of the columns of the Jacobean matrix:**

$$\xi = J_1\dot{q}_1 + J_2\dot{q}_2 + J_3\dot{q}_3 + J_4\dot{q}_4$$

- **Thus, for the end effector to be able to achieve any arbitrary body velocity ξ, *J* must have rank 6, which is the number of linearly independent columns**
- **We know that *J* is 6x*n* and that:**

$$\text{for } A \in \mathbf{R}^{m\times n},\ \operatorname{rank}(A) \le \min(m,n)$$

- **Thus,** $\operatorname{rank}(J) \le \min(6,n)$
- **For example, for the two link planar manipulator,** $\operatorname{rank}(J) \le 2$
- **For example, for the Stanford manipulator,** $\operatorname{rank}(J) \le 6$
- **Note that the columns the Jacobian of a kinematically redundant manipulator are never linearly independent**

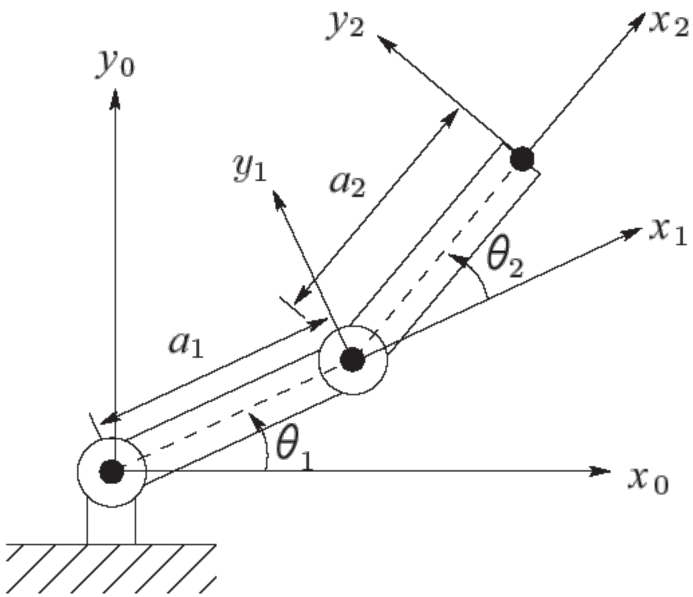
Singularities

- **But the rank of the Jacobian is not necessarily constant… it will depend upon the configuration, q**
- **Definition: we say that any configuration in which the rank of *J* is less than its maximum is a singular configuration**
  - **i.e. any configuration that causes *J* to lose rank is a singular configuration**
- **Characteristics of singularities:**
  - **At a singularity, motion in some directions will not be possible**
  - **At and near singularities, bounded end effector velocities would require unbounded joint velocities**
  - **At and near singularities, bounded joint torques may produce unbounded end effector forces and torques**
  - **Singularities often occur along the workspace boundary (i.e. when the arm is fully extended)**

Singularities

- **How do we determine singularities?**
  - **Simple: construct the Jacobian and observe when it will lose rank**
- **EX: two link manipulator**
  - **Previously, we found *J* to be:**

$$J(q) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



  - **This loses rank if we can find some α such that:**

$$J_1 = \alpha J_2 \quad \text{for } \alpha \in \mathbf{R}$$

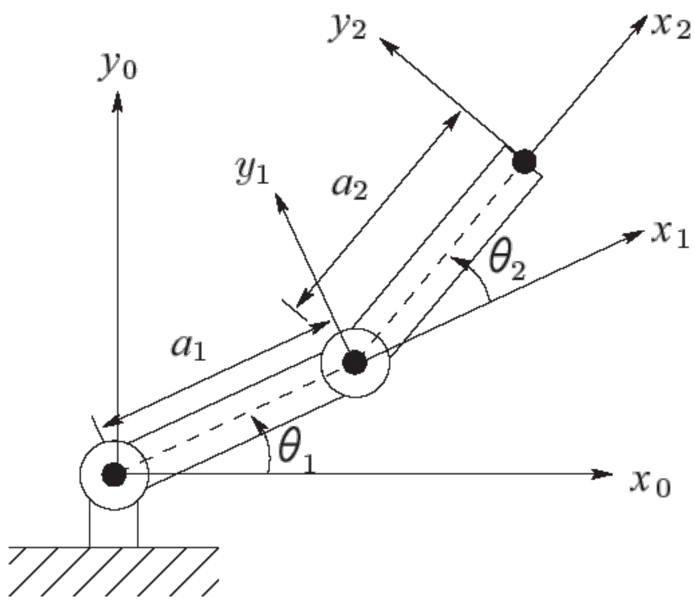
Singularities

- **This is equivalent to the following:**

$$a_1 s_1 + a_2 s_{12} = \alpha(a_2 s_{12})$$
$$a_1 c_1 + a_2 c_{12} = \alpha(a_2 c_{12})$$

- **Thus if $s_{12} = s_1$, we can always find an α that will reduce the rank of J**
- **Thus $\theta_2 = 0, \pi$ are two singularities**

$$\alpha = \frac{a_1 + a_2}{a_2} \qquad \alpha = \frac{a_2 - a_1}{a_2}$$



Determining Singular Configurations

- **In general, all we need to do is observe how the rank of the Jacobian changes as the configuration changes**
- **But it is not always as easy as the last example to observe how the rank changes**
- **There are some shortcuts for common manipulators: decoupling singularities**
  - **Analogous to kinematic decoupling**
  - **Assume that we have a 6DOF manipulator and that we can break the Jacobian into a block form**
  - **Then we can separate singularities into *arm singularities* and *wrist singularities***

Decoupling of Singularities

- Assume that we have a 6DOF manipulator that has a 3-axis arm and a spherical wrist
  - thus the Jacobian is 6x6 and the maximum rank J can have is 6
  - Now we can say that the manipulator is in a singular configuration if det(J(q)) = 0
- For the case of a kinematically decoupled manipulator, we can break up the Jacobian as follows:
  - Where J_p and J_o are represent the position and orientation portions of the Jacobian
  - J_o is given by the following:

$$J = \begin{bmatrix} J_p & J_o \end{bmatrix}$$

$$J_o = \begin{bmatrix} z_3 \times (o_6 - o_3) & z_4 \times (o_6 - o_4) & z_5 \times (o_6 - o_5) \\ z_3 & z_4 & z_5 \end{bmatrix}$$

Decoupling of Singularities

- **Now, one further assumption: $o_3 = o_4 = o_5 = o_6 = o$**
  - **This allows us to note the form of Jo:**

$$J_o = \begin{bmatrix} 0 & 0 & 0 \\ z_3 & z_4 & z_5 \end{bmatrix}$$

  - **And we can split the total manipulator Jacobian as follows:**

$$J = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

  - **Thus we can say:**

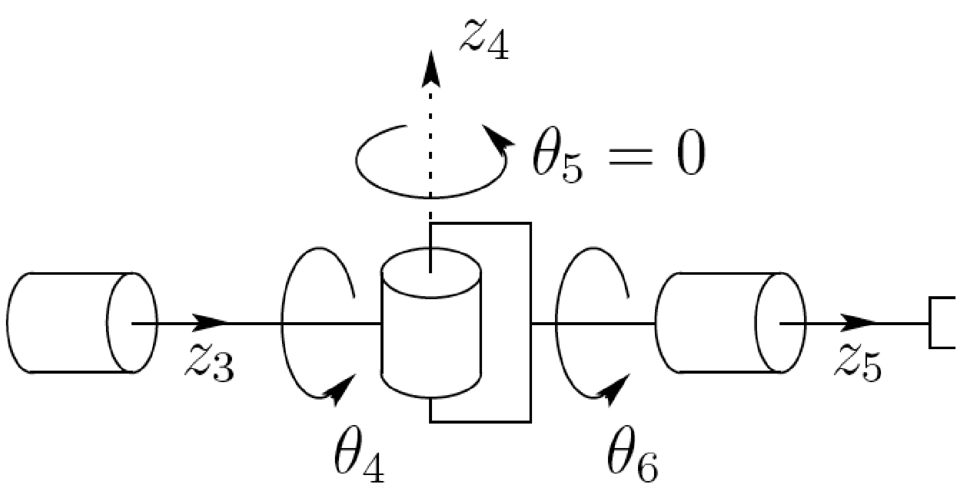
$$\det(J) = \det(J_{11})\det(J_{22})$$

Wrist singularities

- To determine the wrist singularities, we observe the determinant of J_{22}

$$J_{22} = [z_3 \quad z_4 \quad z_5]$$

- Thus the J_{22} has rank 3 when the three axes are linearly independent
  - This is always true, except when two of the axes are collinear
  - i.e. $\theta_5 = 0, \pi$ are the singularities for a spherical wrist



Arm singularities

- To determine the arm singularities, we observe the determinant of J_{11}
  - First, if the i^{th} joint is revolute, the i^{th} column is J_{11} is given as follows:

$$J_{11,i} = \left[z_{i-1} \times \left(o - o_{i-1}\right)\right]$$

  - First, if the i^{th} joint is prismatic, the i^{th} column is J_{11} is given as follows:

$$J_{11,i} = \left[z_{i-1}\right]$$

- We will now give examples for the common configurations we have been using: elbow, spherical, and SCARA manipulators

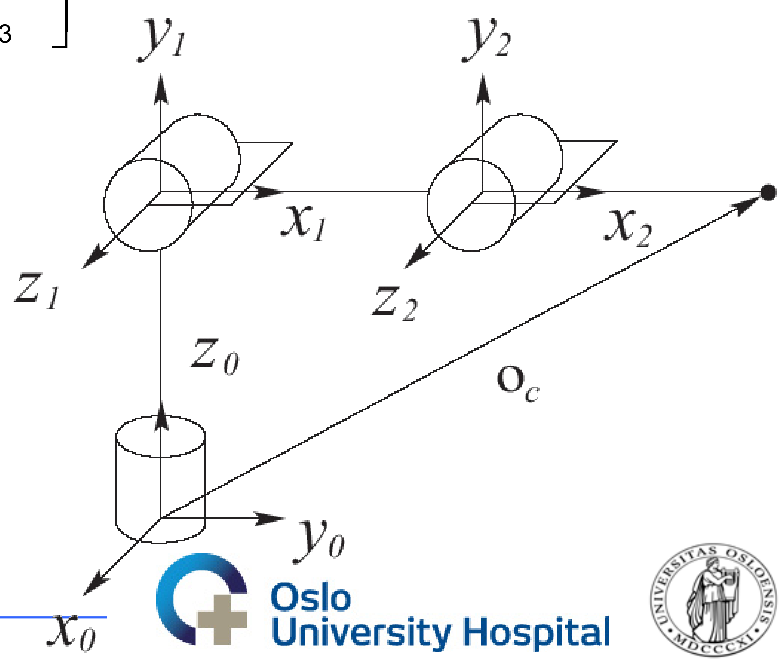
Ex: elbow manipulator

- **To determine the arm singularities, we observe the determinant of J_{11}**
  - **First, J_{11} is given as follows:**

$$J_{11} = \left[z_0 \times (o_c - o_0) \quad z_1 \times (o_c - o_1) \quad z_2 \times (o_c - o_2) \right]$$
$$= \begin{bmatrix} -a_2 s_1 c_2 - a_3 s_1 c_{23} & -a_2 s_2 c_1 - a_3 s_{23} c_1 & -a_3 c_1 s_{23} \\ a_2 c_1 c_2 + a_3 c_1 c_{23} & -a_2 s_1 s_2 - a_3 s_1 s_{23} & -a_3 s_1 s_{23} \\ 0 & a_2 c_2 + a_3 c_{23} & a_3 c_{23} \end{bmatrix}$$

  - **The determinant of J_{11} is:**

$$\det(J_{11}) = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

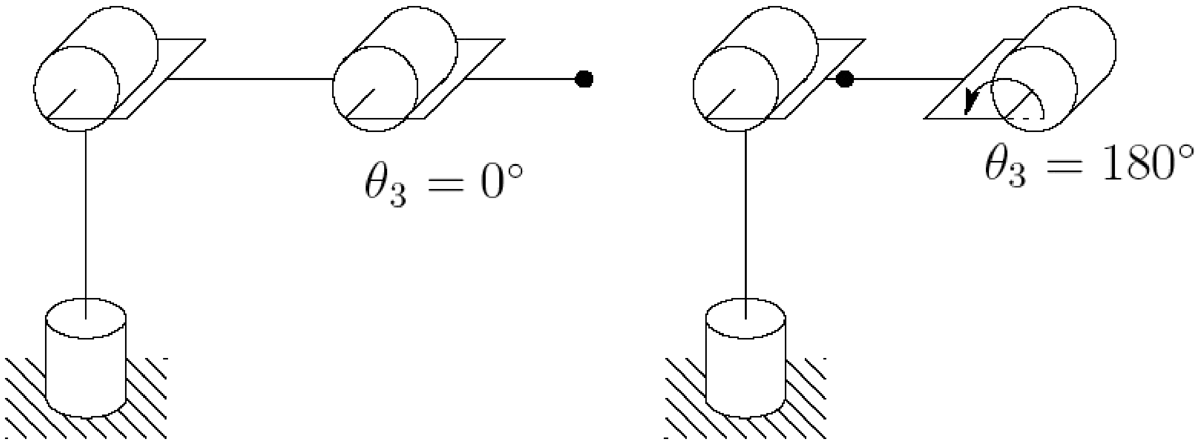


Ex: elbow manipulator

- **The determinant of J_{11} is:**

$$\det(J_{11}) = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

  - **Thus the arm is singular when $s_3 = 0$, i.e. $\theta_3 = 0, \pi$**
  - **This corresponds to the elbow being fully extended or fully retracted:**

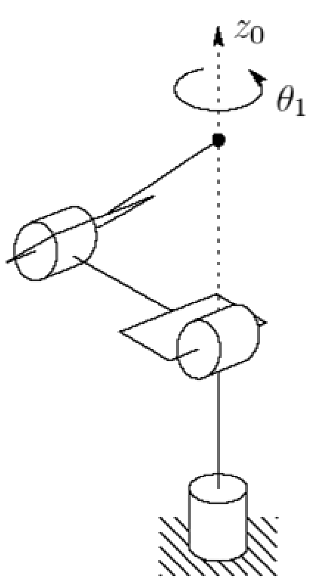


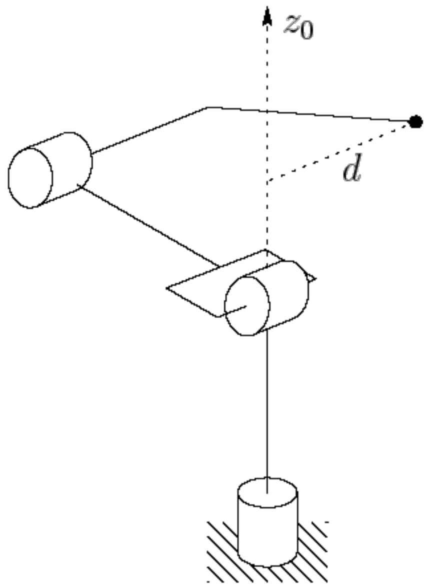
Ex: elbow manipulator

- **The determinant of J_{11} is:**

$$\det(J_{11}) = a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

  - **Thus the arm is also singular when $a_2c_2 + a_3c_{23} = 0$**
  - **This corresponds to the wrist center intersecting the z_0 axis:**

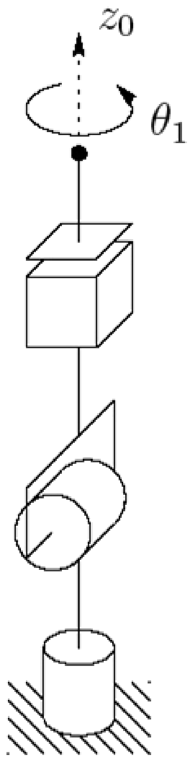




  - **But this is not possible if there is a shoulder offset:**

Ex: spherical manipulator

- **Since there is no ‘elbow’, the only singularity is when the wrist center intersects the base axis**



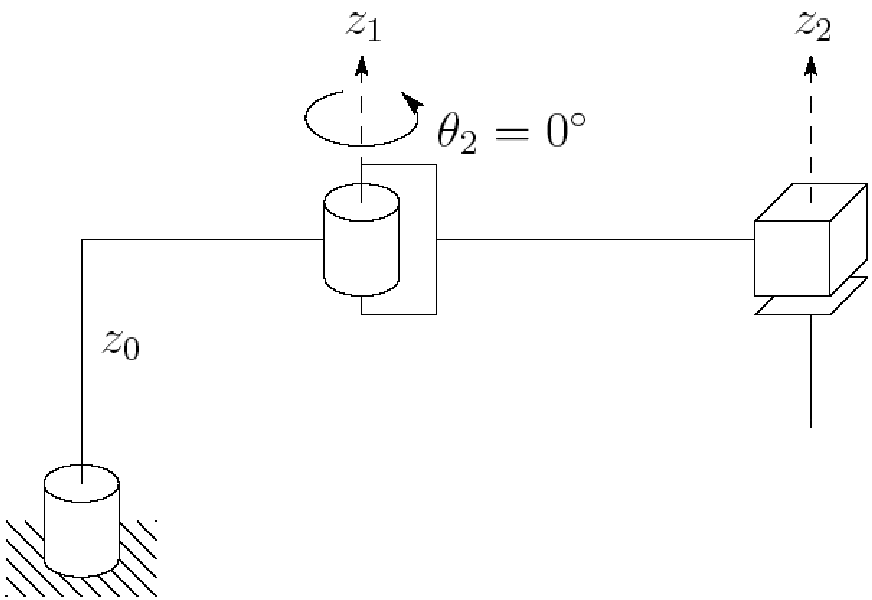
Ex: SCARA manipulator

- **First, we observe the construction of the Jacobian:**

$$J_{11} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_1 s_{12} & 0 \\ a_1 c_1 + a_2 c_{12} & a_1 c_{12} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- **The determinant is:**

$$\begin{aligned} \det(J_{11}) &= a_1^2 c_1 s_{12} - a_1^2 s_1 c_{12} \\ &= a_1^2 (c_1 s_{12} - s_1 c_{12}) \\ &= a_1^2 (c_1(s_1 c_2 + c_1 s_2) - s_1(c_1 c_2 - s_1 s_2)) \\ &= a_1^2 s_2 \end{aligned}$$



- **Thus, the SCARA is singular**

for s_2 = 0, i.e. θ_2 = 0, π

Force/torque relationships

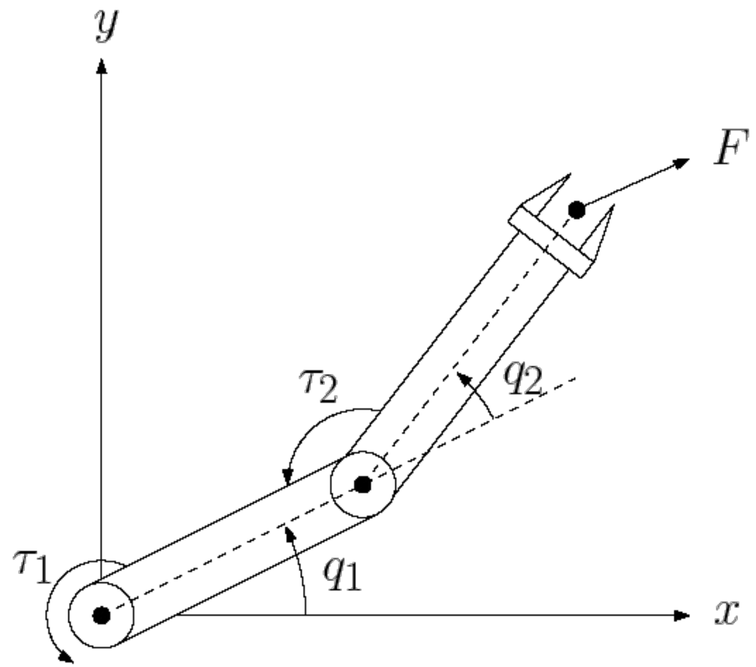
- **Similar to the relationship between the joint velocities and the end effector velocities, we are interested in expressing the relationship between the joint torques and the forces and moments at the end effector**
  - **Important for dynamics, force control, etc**
- **Let the vector of forces and moments at the end effector be represented as:** $F = \begin{bmatrix} F_x & F_y & F_z & n_x & n_y & n_z \end{bmatrix}^T$
- **Then we can express the joint torques, τ, as:**

$$\tau = J^T(q)F$$

Force/torque relationships

- **Example: for a force *F* applied to the end of a planar two-link manipulator, what are the resulting joint torques?**
  - **First, remember that the Jacobian is:**

$$J(q) = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

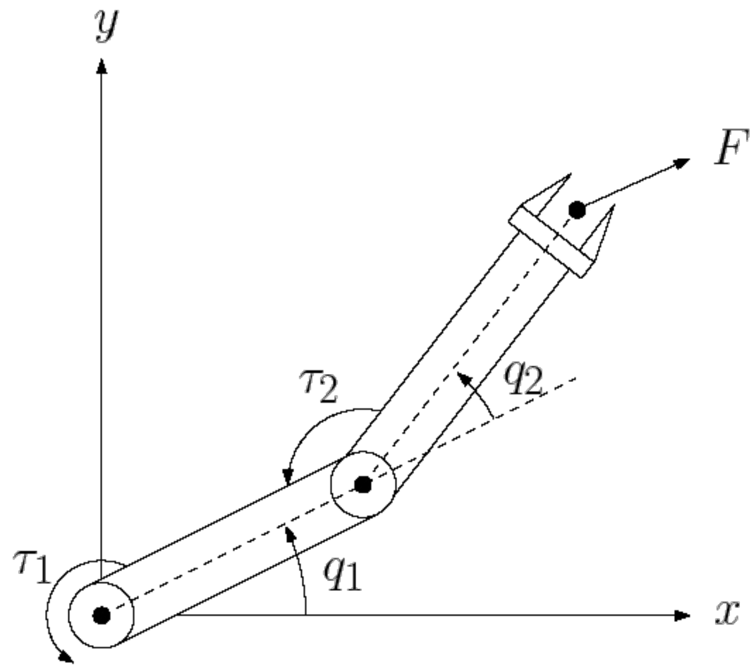


Force/torque relationships

- **Example: for a force *F* applied to the end of a planar two-link manipulator, what are the resulting joint torques?**
  - **Thus the joint torques are:**

$$\tau = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & a_1 c_1 + a_2 c_{12} & 0 & 0 & 0 & 1 \\ -a_2 s_{12} & a_2 c_{12} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \\ n_x \\ n_y \\ n_z \end{bmatrix}$$

$$= \begin{bmatrix} F_x(-a_1 s_1 - a_2 s_{12}) + F_y(a_1 c_1 + a_2 c_{12}) \\ F_x(-a_2 s_{12}) + F_y(a_2 c_{12}) \end{bmatrix}$$



Inverse velocity

- **We have developed the Jacobian as a mapping from joint velocities to end effector velocities:**

$$\xi = J\dot{q}$$

- **Now we want the inverse: what are the joint velocities for a specified end effector velocity?**
- **Simple case: if the Jacobian is square and nonsingular,**

$$\dot{q} = J^{-1}\xi$$

- **In all other cases, we need another method**
- **For systems that do not have exactly 6DOF, we cannot directly invert the Jacobian**

Next class...

- **Introduction to dynamics**