Ch. 6 Single Variable Control

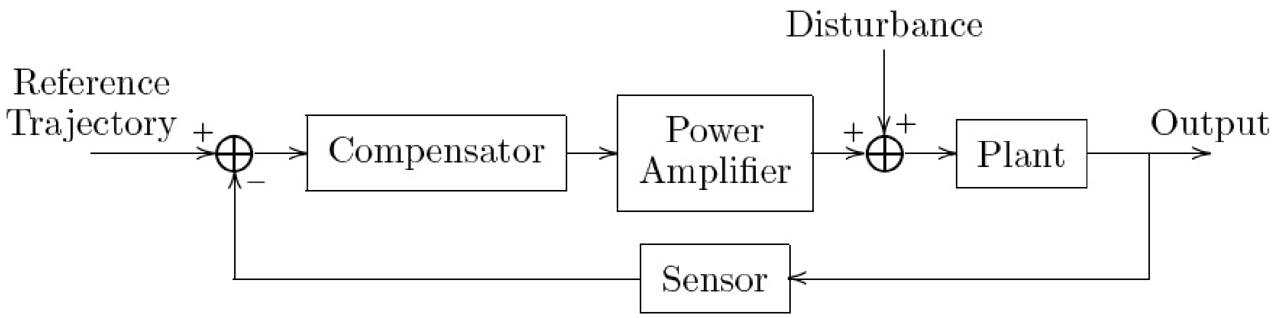
ES159/259

Single variable control

- How do we determine the motor/actuator inputs so as to command the end effector in a desired motion?
- In general, the input voltage/current does not create instantaneous motion to a desired configuration
  - Due to dynamics (inertia, etc)
  - Nonlinear effects
    - Backlash
    - Friction
  - Linear effects
    - Compliance
- Thus, we need three basic pieces of information:
  1. Desired joint trajectory
  2. Description of the system (ODE = Ordinary Differential Equation)
  3. Measurement of actual trajectory

SISO overview

- Typical single input, single output (SISO) system:



- We want the robot *tracks* the desired trajectory and *rejects* external disturbances
- We already have the desired trajectory, and we assume that we can measure the actual trajectories
- Thus the first thing we need is a system description

SISO overview

- Need a convenient input-output description of a SISO system
- Two typical representations for the plant:
  - Transfer function
  - State-space
- Transfer functions represent the system dynamics in terms of the Laplace transform of the ODEs that represent the system dynamics
- For example, if we have a 1DOF system described by:

$$\tau(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$

- We want the representation in the Laplace domain:

$$\begin{aligned}\tau(s) &= s^2 J\theta(s) + sB\theta(s) \\ &= s(sJ + B)\theta(s)\end{aligned}$$

- Therefore, we give the *transfer function* as:

$$P(s) \equiv \frac{\theta(s)}{\tau(s)} = \frac{1}{s(sJ+B)} = \frac{1/J}{s(s+B/J)}$$

Review of the Laplace transform

- Laplace transform creates algebraic equations from differential equations
- The Laplace transform is defined as follows:

$$x(s) = \int_0^\infty e^{-st} x(t) dt$$

- For example, Laplace transform of a derivative:

$$L\{\dot{x}(t)\} = L\left\{\frac{dx(t)}{dt}\right\} = \int_0^\infty e^{-st} \frac{dx(t)}{dt} dt$$

  - Integrating by parts:

$$L\left\{\frac{dx(t)}{dt}\right\} = e^{-st} x(t)\Big|_0^\infty + s\int_0^\infty e^{-st} x(t) dt$$

$$= sx(s) - x(0)$$

Review of the Laplace transform

- Similarly, Laplace transform of a second derivative:

$$L\{\ddot{x}(t)\} = L\left\{\frac{d^2x(t)}{dt^2}\right\} = \int_0^\infty e^{-st}\frac{d^2x(t)}{dt^2}dt = s^2x(s) - sx(0) - \dot{x}(0)$$

- Thus, if we have a generic 2nd order system described by the following ODE:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$$

- And we want to get a transfer function representation of the system, take the Laplace transform of both sides:

$$mL\{\ddot{x}(t)\} + bL\{\dot{x}(t)\} + kL\{x(t)\} = L\{F(t)\}$$
$$m\left(s^2x(s) - sx(0) - \dot{x}(0)\right) + b\left(sx(s) - x(0)\right) + kx(s) = F(s)$$

Review of the Laplace transform

- Continuing:

$$\left(ms^2 + bs + k\right)x(s) = F(s) + m\dot{x}(0) + (ms + c)x(0)$$

- The *transient response* is the solution of the above ODE if the *forcing function* $F(t) = 0$
- Ignoring the transient response, we can rearrange:

$$\frac{x(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

- This is the input-output transfer function and the denominator is called the *characteristic equation*

Review of the Laplace transform

- Properties of the Laplace transform
  - Takes an ODE to a algebraic equation
  - Differentiation in the time domain is multiplication by *s* in the Laplace domain
  - Integration in the time domain is multiplication by 1/*s* in the Laplace domain
  - Considers initial conditions
    - i.e. transient and steady-state response
  - The Laplace transform is a linear operator

Review of the Laplace transform

- for this class, we will rely on a table of Laplace transform pairs for convenience

Time domain	Laplace domain
$x(t)$	$x(s) = L\{x(t)\} = \int_0^\infty e^{-st} x(t) dt$
$\dot{x}(t)$	$sx(s) - x(0)$
$\ddot{x}(t)$	$s^2 x(s) - sx(0) - \dot{x}(0)$
Ct	$\frac{C}{s^2}$
step	$\frac{1}{s}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$

Review of the Laplace transform

Time domain	Laplace domain
$x(t-\alpha)H(t-\alpha)$	$e^{-\alpha s}x(s)$
$e^{-at}x(t)$	$x(s+a)$
$x(at)$	$\frac{1}{a}x\left(\frac{s}{a}\right)$
$C\delta(t)$	C

SISO overview

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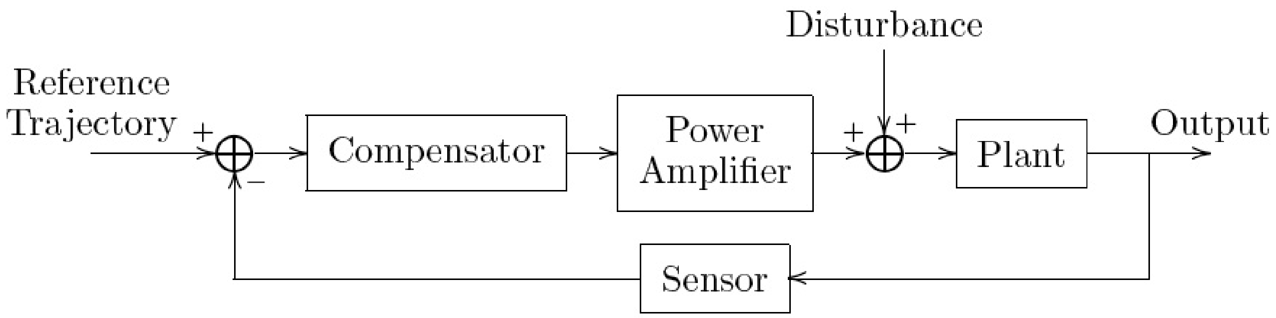

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- Therefore, we give the *transfer function* as:

$$P(s) \equiv \frac{\theta(s)}{\tau(s)} = \frac{1}{s(sJ+B)} = \frac{1/J}{s(s+B/J)}$$

System descriptions

- A generic 2nd order system can be described by the following ODE:

$$m\ddot{x}(t)+b\dot{x}(t)+kx(t)=F(t)$$

- And we want to get a transfer function representation of the system, take the Laplace transform of both sides:

$$mL\{\ddot{x}(t)\}+bL\{\dot{x}(t)\}+kL\{x(t)\}=L\{F(t)\}$$

$$m\left(s^2x(s)-sx(0)-\dot{x}(0)\right)+b\left(sx(s)-x(0)\right)+kx(s)=F(s)$$

$$\left(ms^2+bs+k\right)x(s)=F(s)+m\dot{x}(0)+\left(ms+c\right)x(0)$$

- Ignoring the transient response, we can rearrange:

$$\frac{x(s)}{F(s)}=\frac{1}{ms^2+bs+k}$$

- This is the input-output transfer function and the denominator is called the *characteristic equation*

Example: motor dynamics

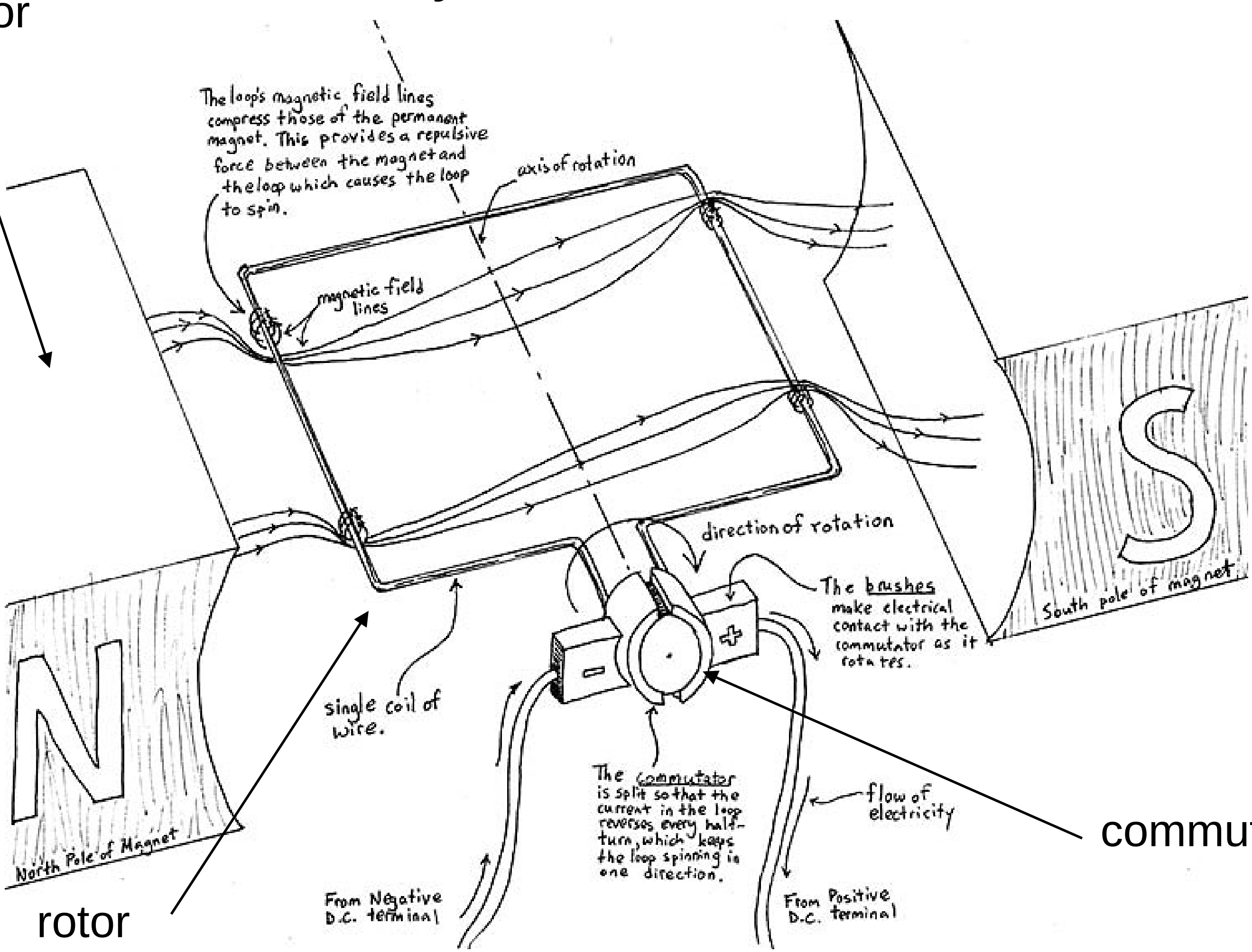
- DC motors are ubiquitous in robotics applications
- Here, we develop a transfer function that describes the relationship between the input voltage and the output angular displacement
- First, a physical description of the most common motor: permanent magnet…

torque on the rotor:

$$\tau_m = K_1 \phi i_a$$

Physical instantiation

stator



commutator

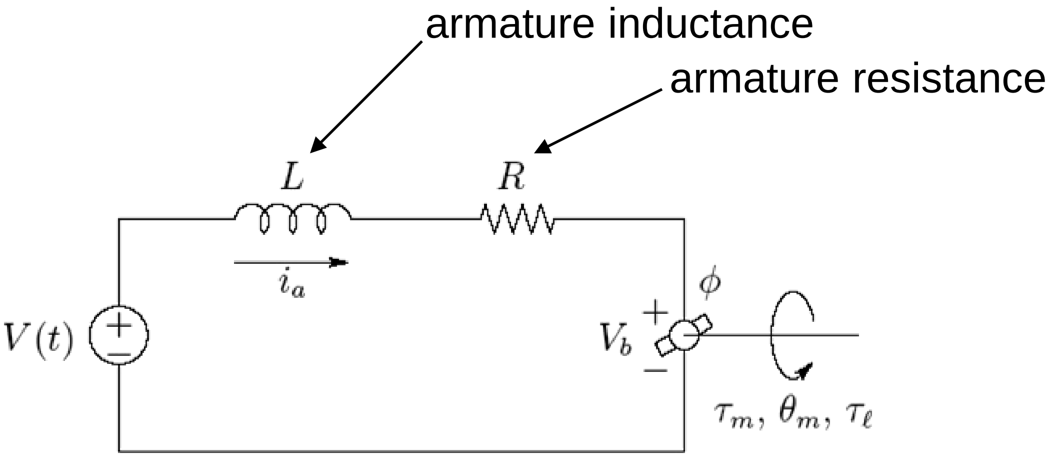
rotor
(armature)

Motor dynamics

- When a conductor moves in a magnetic field, a voltage is generated
  - Called *back EMF*:

$$V_b = K_2 \phi \omega_m$$

  - Where ω_m is the rotor angular velocity



$$L\frac{di_a}{dt} + Ri_a = V - V_b$$

Motor dynamics

- Since this is a permanent magnet motor, the magnetic flux is constant, we can write:

$$\tau_m = K_1 \phi i_a = K_m i_a$$

torque constant

- Similarly:

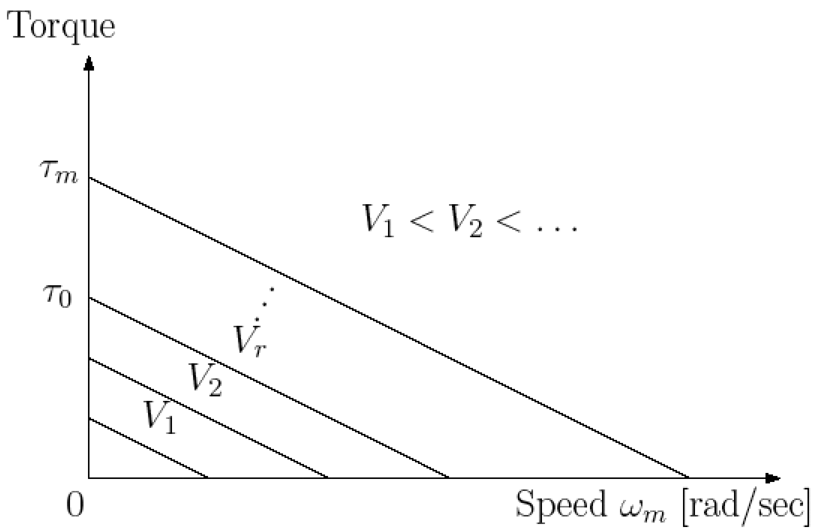
$$V_b = K_2 \phi \omega_m = K_b \frac{d\theta_m}{dt}$$

back EMF constant

- K_m and K_b are numerically equivalent, thus there is one constant needed to characterize a motor

Motor dynamics

- This constant is determined from torque-speed curves
  - Remember, torque and displacement are work conjugates

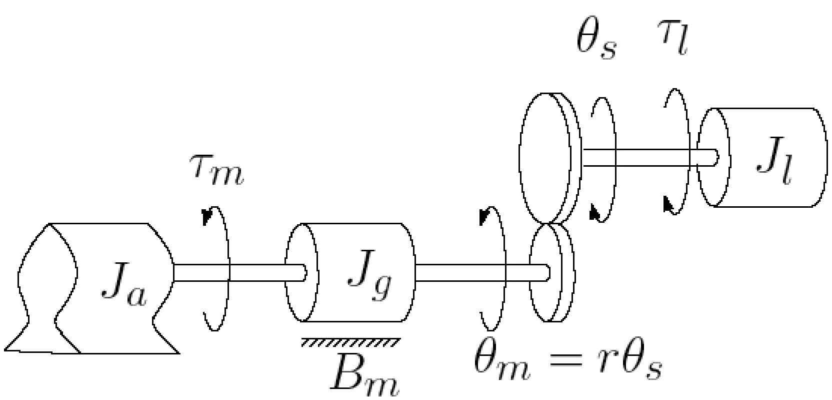


  - τ_0 is the *blocked torque*

Single link/joint dynamics

- Now, lets take our motor and connect it to a link
- Between the motor and link there is a gear such that: $\theta_m = r\theta_L$
- Lump the actuator and gear inertias: $J_m = J_a + J_g$
- Now we can write the dynamics of this mechanical system:

$$J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} = \tau_m - \frac{\tau_L}{r} = K_m i_a - \frac{\tau_L}{r}$$



Motor dynamics

- Now we have the ODEs describing this system in both the electrical and mechanical domains:

$$L\frac{di_a}{dt}+Ri_a=V-K_b\frac{d\theta_m}{dt}$$

$$J_m\frac{d^2\theta_m}{dt^2}+B_m\frac{d\theta_m}{dt}=K_m i_a-\frac{\tau_L}{r}$$

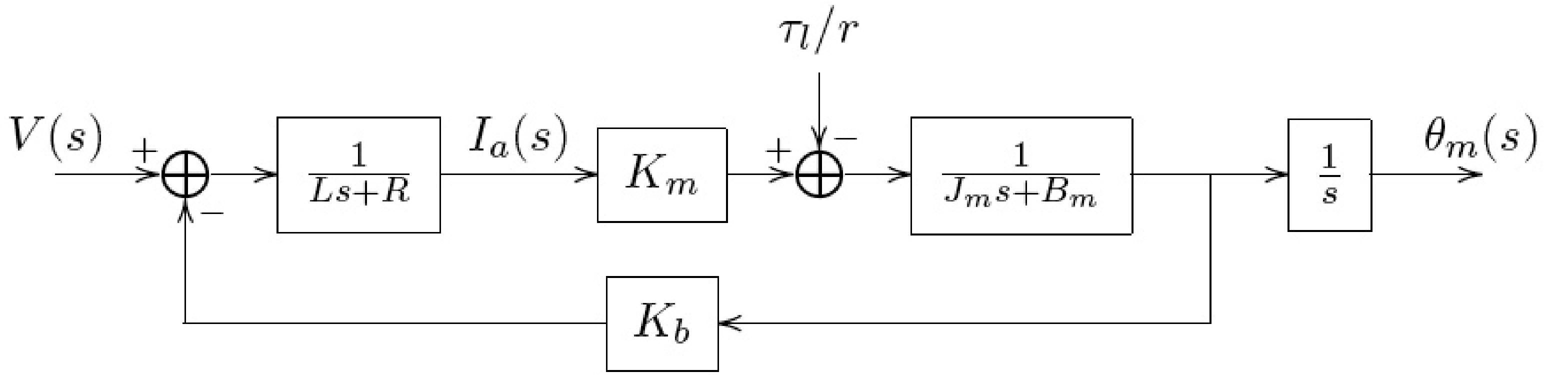
- In the Laplace domain:

$$(Ls+R)I_a(s)=V(s)-K_b s\Theta_m(s)$$

$$(J_m s^2+B_m s)\Theta_m(s)=K_m I_a(s)-\frac{\tau_L(s)}{r}$$

Motor dynamics

- These two can be combined to define, for example, the input-output relationship for the input voltage, load torque, and output displacement:



Motor dynamics

- Remember, we want to express the system as a transfer function from the input to the output angular displacement
  - But we have two potential inputs: the load torque and the armature voltage
  - First, assume τ_L = 0 and solve for $\Theta_m(s)$:

$$\frac{\left(J_m s^2 + B_m s\right)\Theta_m(s)}{K_m} = I_a(s) \longrightarrow \frac{(Ls + R)\left(J_m s^2 + B_m s\right)}{K_m}\Theta_m(s) = V(s) - K_b s \Theta_m(s)$$

$$\longrightarrow \frac{\Theta_m(s)}{V(s)} = \frac{K_m}{s\left[(Ls + R)(J_m s + B_m) + K_b K_m\right]}$$

Motor dynamics

- Now consider that $V(s) = 0$ and solve for $\Theta_m(s)$:

$$I_a(s) = \frac{-K_b s \Theta_m(s)}{Ls + R} \longrightarrow \left(J_m s^2 + B_m s\right)\Theta_m(s) = \frac{-K_m K_b s \Theta_m(s)}{Ls + R} - \frac{\tau_L(s)}{r}$$

$$\longrightarrow \frac{\Theta_m(s)}{\tau_L(s)} = \frac{-(Ls + R)/r}{s\left[(Ls + R)(J_m s + B_m) + K_b K_m\right]}$$

- Note that this is a function of the gear ratio
  - The larger the gear ratio, the less effect external torques have on the angular displacement

Motor dynamics

- In this system there are two 'time constants'
  - Electrical: L/R
  - Mechanical: J_m/B_m
- For intuitively obvious reasons, the electrical time constant is assumed to be small compared to the mechanical time constant
  - Thus, ignoring electrical time constant will lead to a simpler version of the previous equations:

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m/R}{s\left[J_m s + B_m + K_b K_m/R\right]}$$

$$\frac{\Theta_m(s)}{\tau_L(s)} = \frac{-1/r}{s\left[J_m s + B_m + K_b K_m/R\right]}$$

Motor dynamics

- Rewriting these in the time domain gives:

$$\frac{\Theta_m(s)}{V(s)} = \frac{K_m / R}{s[J_m s + B_m + K_b K_m / R]} \longrightarrow J_m \ddot{\theta}_m(t) + (B_m + K_b K_m / R)\dot{\theta}_m(t) = (K_m / R)V(t)$$

$$\frac{\Theta_m(s)}{\tau_L(s)} = \frac{-1/r}{s[J_m s + B_m + K_b K_m / R]} \longrightarrow J_m \ddot{\theta}_m(t) + (B_m + K_b K_m / R)\dot{\theta}_m(t) = -(1/R)\tau_L(t)$$

- By superposition of the solutions of these two linear 2nd order ODEs:

$$\underbrace{J_m}_{J} \ddot{\theta}_m(t) + \underbrace{(B_m + K_b K_m / R)}_{B}\dot{\theta}_m(t) = \underbrace{(K_m / R)V(t)}_{u(t)} - \underbrace{(1/R)\tau_L(t)}_{d(t)}$$

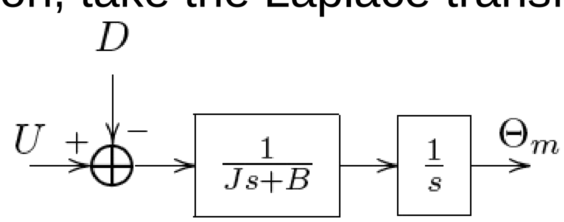
Motor dynamics

- Therefore, we can write the dynamics of a DC motor attached to a load as:

$$J\ddot{\theta}(t) + B\dot{\theta}(t) = u(t) - d(t)$$

  - Note that *u*(*t*) is the input and *d*(*t*) is the disturbance (e.g. the dynamic coupling from motion of other links)
- To represent this as a transfer function, take the Laplace transform:

$$(Js^2 + Bs)\Theta(s) = U(s) - D(s)$$



Setpoint controllers

- We will first discuss three initial controllers: P, PD and PID
  - Both attempt to drive the error (between a desired trajectory and the actual trajectory) to zero
- The system can have any dynamics, but we will concentrate on the previously derived system

Proportional Controller

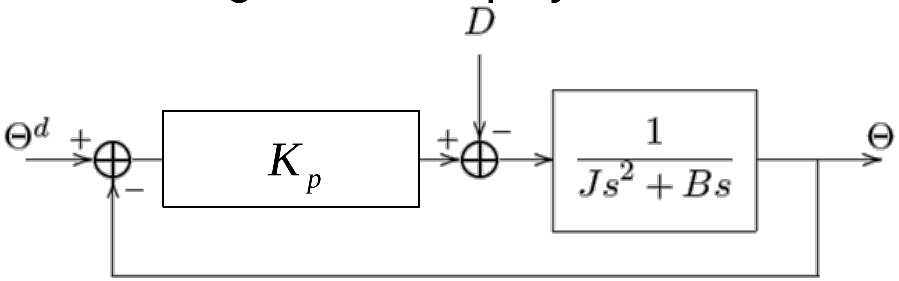
- Control law:

$$u(t) = K_p e(t)$$

  - Where $e(t) = \theta^d(t) - \theta(t)$
- in the Laplace domain:

$$U(s) = K_p E(t)$$

- This gives the following closed-loop system:



PD controller

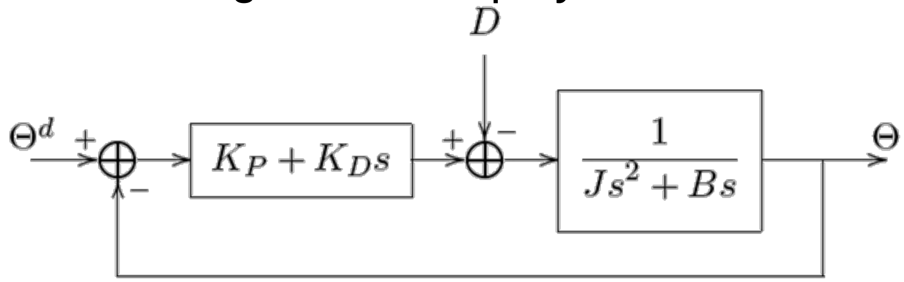
- Control law:

$$u(t) = K_p e(t) + K_d \dot{e}(t)$$

  – Where $e(t) = \theta^d(t) - \theta(t)$
- in the Laplace domain:

$$U(s) = (K_p + sK_d)E(t)$$

- This gives the following closed-loop system:



PD controller

- This system can be described by:

$$\Theta(s) = \frac{U(s) - D(s)}{Js^2 + Bs}$$

- Where, again, $U(s)$ is:

$$U(s) = (K_p + sK_d)(\Theta^d(s) - \Theta(s))$$

- Combining these gives us:

$$\Theta(s) = \frac{(K_p + sK_d)(\Theta^d(s) - \Theta(s)) - D(s)}{Js^2 + Bs}$$

- Solving for Θ gives:

$$(Js^2 + Bs)\Theta(s) + (K_p + sK_d)\Theta(s) = (K_p + sK_d)\Theta^d(s) - D(s)$$

$$\Rightarrow (Js^2 + (B + K_d)s + K_p)\Theta(s) = (K_p + sK_d)\Theta^d(s) - D(s)$$

$$\Rightarrow \Theta(s) = \frac{(K_p + sK_d)\Theta^d(s) - D(s)}{Js^2 + (B + K_d)s + K_p}$$

PD controller

- The denominator is the *characteristic polynomial*
- The roots of the characteristic polynomial determine the performance of the system

$$s^2 + \frac{(B + K_d)}{J}s + \frac{K_p}{J} = 0$$

- If we think of the closed-loop system as a damped second order system, this allows us to choose values of K_p and K_d

$$s^2 + 2\zeta\omega s + \omega^2 = 0$$

- Thus K_p and K_d are:

$$K_p = \omega^2 J$$

$$K_d = 2\varsigma\omega J - B$$

- A natural choice is $\zeta = 1$ (critically damped)

PD controller

- Limitations of the PD controller:
  - for illustration, let our desired trajectory be a step input and our disturbance be a constant as well:

$$\Theta^d(s) = \frac{C}{s}, D(s) = \frac{D}{s}$$

  - Plugging this into our system description gives:

$$\Theta(s) = \frac{(K_p + sK_d)C - D}{s(Js^2 + (B + K_d)s + K_p)}$$

  - For these conditions, what is the steady-state value of the displacement?

$$\theta_{ss} = \lim_{s\to 0} \frac{s(K_p + sK_d)C - sD}{s(Js^2 + (B + K_d)s + K_p)} = \lim_{s\to 0} \frac{(K_p + sK_d)C - D}{Js^2 + (B + K_d)s + K_p} = \frac{K_p C - D}{K_p} = C - \frac{D}{K_p}$$

  - Thus the steady state error is $-D/K_p$
  - Therefore to drive the error to zero in the presence of large disturbances, we need large gains… so we turn to another controller

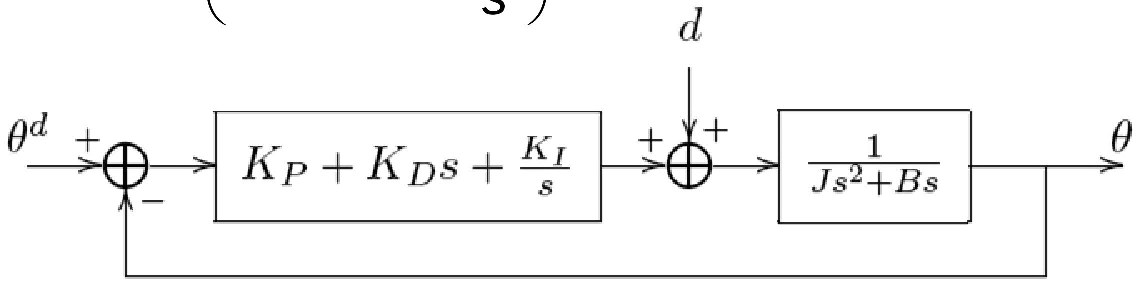
PID controller

- Control law:

$$u(t) = K_p e(t) + K_d \dot{e}(t) + K_i \int e(t)\,dt$$

- In the Laplace domain:

$$U(s) = \left(K_p + K_d s + \frac{K_i}{s}\right) E(s)$$



$$\Theta(s) = \frac{\left(K_d s^2 + K_p s + K_i\right)\Theta^d(s) - sD(s)}{Js^3 + (B + K_d)s^2 + K_p s + K_i}$$

PID controller

- The integral term eliminates the steady state error that can arise from a large disturbance
- How to determine PID gains
  1. Set $K_i = 0$ and solve for K_p and K_d
  2. Determine K_i to eliminate steady state error
     - However, we need to be careful of the stability conditions

$$K_i < \frac{(B + K_d)K_p}{J}$$

PID controller

- Stability
  - The closed-loop stability of these systems is determined by the roots of the characteristic polynomial
  - If all roots (potentially complex) are in the ‘left-half’ plane, our system is stable
    - for any bounded input and disturbance
  - A description of how the roots of the characteristic equation change (as a function of controller gains) is very valuable
    - Called the *root locus*

Summary

- Proportional
  - A pure proportional controller will have a steady-state error
  - Adding a integration term will remove the bias
  - High gain (Kp) will produce a fast system
  - High gain may cause oscillations and may make the system unstable
  - High gain reduces the steady-state error
- Integral
  - Removes steady-state error
  - Increasing Ki accelerates the controller
  - High Ki may give oscillations
  - Increasing Ki will increase the settling time
- Derivative
  - Larger K_d decreases oscillations
  - Improves stability for low values of K_d
  - May be highly sensitive to noise if one takes the derivative of a noisy error
  - High noise leads to instability