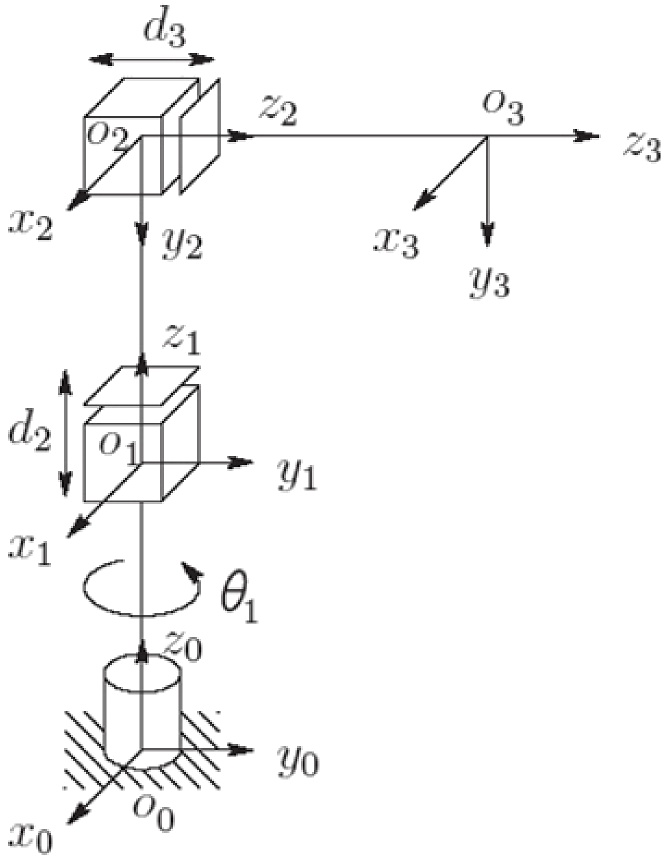
Ch. 7: Dynamics

Example: three link cylindrical robot

- **Up to this point, we have developed a systematic method to determine the forward and inverse kinematics and the Jacobian for any arbitrary serial manipulator**
  - **Forward kinematics: mapping from joint variables to position and orientation of the end effector**
  - **Inverse kinematics: finding joint variables that satisfy a given position and orientation of the end effector**
  - **Jacobian: mapping from the joint velocities to the end effector linear and angular velocities**
- **Example: three link cylindrical robot**



Dynamics Overview

- While the kinematic equations describe the motion of the robot without consideration of the forces that produces the motion.
- The dynamics explicitly describe the relationship between force and motion
- The dynamic of the robot is necessary to consider in the design of robots, simulation and animation, and in the design of control algoritms
- We want to come up with equations of motion for any *n*DOF system
  - In general, this will consist of *n* coupled second order differential equations
- These systems may be:
  - Linear or nonlinear
  - Conservative or nonconservative
- We want to develop an expression of the form: $\dot{q} = f(q,t)$
- Once we have this, we can use it to choose an appropriate controller that will put our dynamical system in a desired state (configuration)
- The dynamics is important

Euler-Lagrange Equations

- **We can derive the equations of motion for any *n*DOF system by using energy methods**
  - **All we need to know are the conservative (kinetic and potential) and non-conservative (dissipative) terms**
- **This is a shortcut to describing the motion of each particle in a rigid body along with the constraints that form rigid motions**
- **For this, we need to first use virtual displacements subject to holonomic constraints, then use the principle of virtual work, then finally use D'Alembert's Principle to derive the Euler-Lagrange equations of motion**
- **But first, an example…**

Ex: 1DOF system

- To illustrate, we derive the equations of motion for a 1DOF system
  - Consider a particle of mass *m*
  - Using Newton's second law:

$$m\ddot{y} = f - mg$$

  - Now define the kinetic and potential energies:

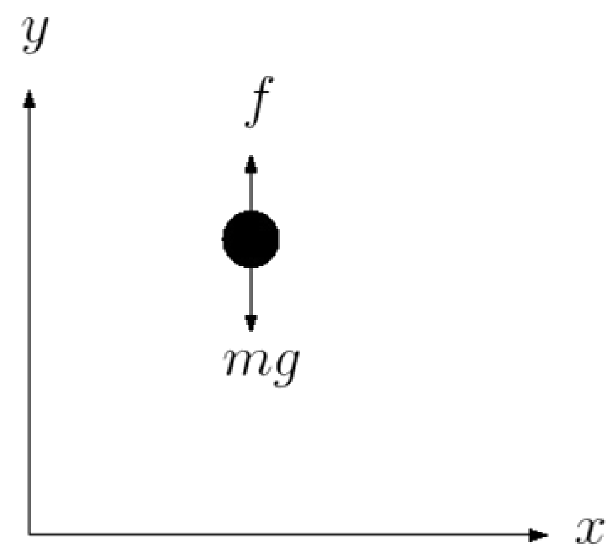
$$K = \frac{1}{2} m\dot{y}^2 \qquad P = mgy$$

  - Rewrite the above differential equation
    - Left side:

$$m\ddot{y} = \frac{d}{dt}\left(m\dot{y}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2\right) = \frac{d}{dt}\frac{\partial K}{\partial \dot{y}}$$

    - Right side:

$$mg = \frac{\partial}{\partial y}\left(mgy\right) = \frac{\partial P}{\partial y}$$



Ex: 1DOF system

- **Thus we can rewrite the initial equation:**

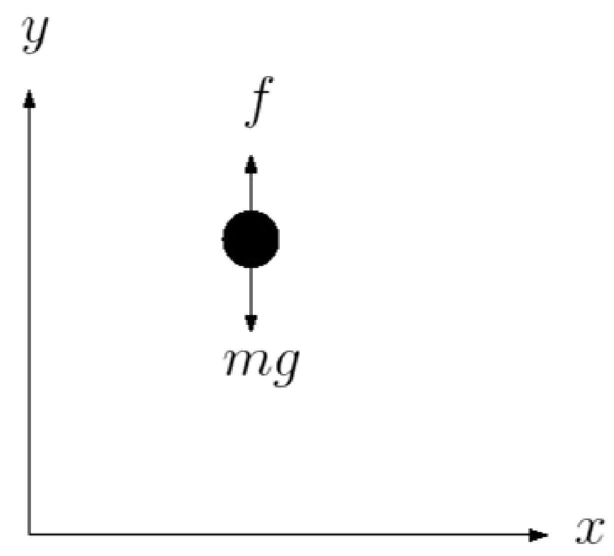
$$\frac{d}{dt}\frac{\partial K}{\partial \dot{y}} = f - \frac{\partial P}{\partial y}$$

- **Now we make the following definition:**

$$L = K - P$$

- ***L* is called the *Lagrangian***
  - **We can rewrite our equation of motion again:**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = f$$



- **Thus, to define the equation of motion for this system, all we need is a description of the potential and kinetic energies**

Euler-Lagrange Equations

- **If we represent the variables of the system as generalized coordinates, then we can write the equations of motion for an *n*DOF system as:**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i$$

- **We will come back to this, but it is important to recognize the form of the above equation:**
  - **The left side contains the conservative terms**
  - **The right side contains the non-conservative terms**
- **This formulation leads to a set of *n* coupled 2nd order differential equations**

Ex: 1DOF system

- **Single link, single motor coupled by a drive shaft**
  - **θ_m and θ_l are the angular displacements of the shaft and the link respectively, related by a gear ratio, *r*:**

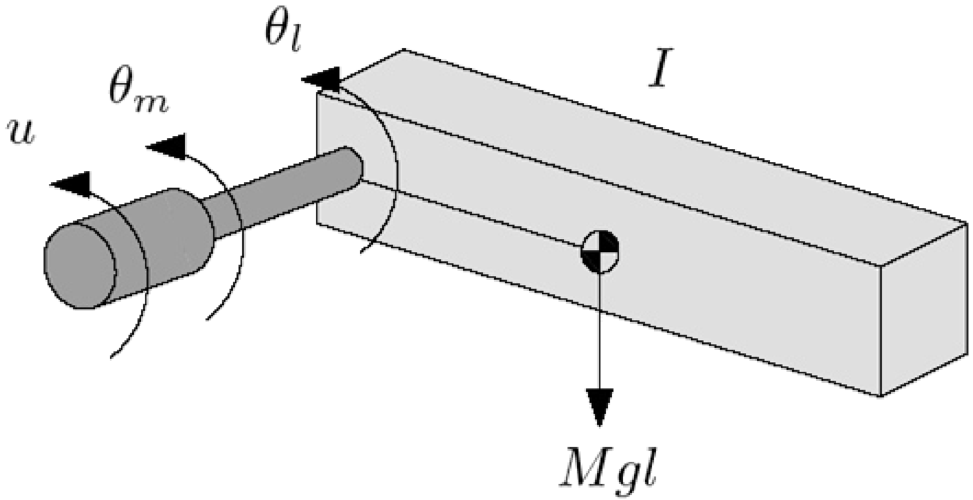
$$\theta_m = r\theta_l$$

  - **Start by determining the kinetic and potential energies:**

$$K = \frac{1}{2} J_m \dot{\theta}_m^{\ 2} + \frac{1}{2} J_l \dot{\theta}_l^{\ 2}$$

$$= \frac{1}{2}\left(r^2 J_m + J_l\right)\dot{\theta}_l^{\ 2}$$

$$P = \frac{MgL}{2}\left(1 - \cos\theta_l\right)$$



  - **J_m and J_l are the motor/shaft and link inertias respectively and *M* and *L* are the mass and length of the link respectively**

Ex: 1DOF system

- **Let the total inertia, *J*, be defined by:**

$$J = r^2 J_m + J_l$$

- **Now write the Lagrangian:**

$$L = \frac{1}{2} J \dot{\theta}_l^{\,2} - \frac{MgL}{2}\left(1 - \cos\theta_l\right)$$

- **Thus we can write the equation of motion for this 1DOF system as:**

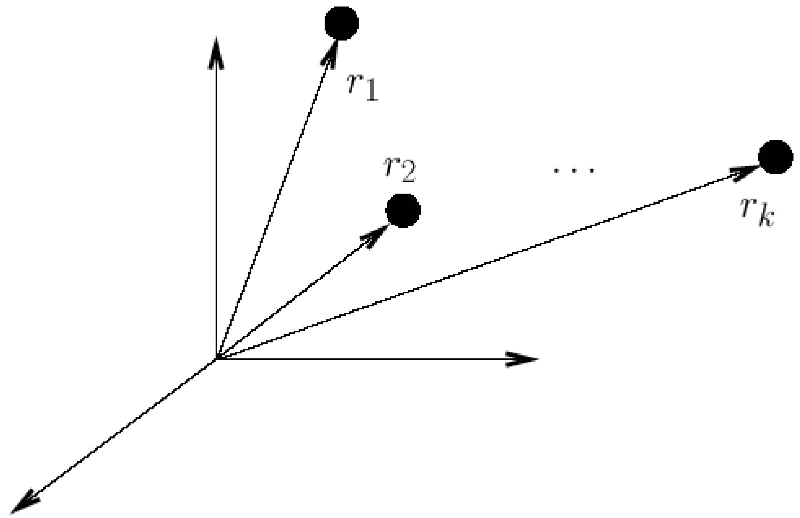
$$J\ddot{\theta}_l + \frac{MgL}{2}\sin\theta_l = \tau_l$$

- **The right side contains the non-conservative terms such as:**
  - **The input motor torque:** $u = r\tau_m$
  - **Damping torques:** $B = rB_m + B_l$
- **Therefore we can rewrite the equation of motion:**

$$J\ddot{\theta}_l + B\dot{\theta}_l + \frac{MgL}{2}\sin\theta_l = u$$

Holonomic constraints

- **Motivation: reduce the number of DOFs of the system**
  - **i.e. reduce the number of coordinates necessary to fully describe a system of particles**
- **Consider a set of *k* independent particles $r_1, r_2, \ldots, r_k$:**
  - **Requires 3*k* generalized coordinates to fully describe the set**



- **If we add constraints, the number of DOFs (and hence generalized coordinates) is reduced**

Holonomic constraints

- **Without constraints, we can simply write:** $m\ddot{r}_i = f_i$
  - **i.e. *k* uncoupled differential equations**
- **If there are constraints on the motion of the particles, we must also consider constraint forces**
- **Example, consider a system of two particles with positions r_1, r_2**
  - **If there are no constraints, we need 6 parameters to fully describe this system**
  - **Consider the following constraint: the distance between the particles is constant**
  - **Now we can write the equations by finding all the forces that will ensure the constraints are met**

$$\left\| r_1 - r_2 \right\| = \sqrt{\left(r_1 - r_2\right)^T \left(r_1 - r_2\right)} = L$$

Virtual displacements

- **Definition: for a system of *k* particles with *m* holonomic constraints, the set of virtual displacements is the set:** $\delta r_1, \ldots, \delta r_k$
  - **These displacements are also consistent with the holonomic constraints**
- **Ex: again, consider a system of two particles that has a fixed distance**

$$\left(r_1 - r_2\right)^T \left(r_1 - r_2\right) = L^2$$

  - **Now supposed that both r_1 and r_2 are perturbed by the virtual displacements δr_1 and δr_2**
  - **Thus, the same constraint has to be satisfied:**

$$\left(r_1 + \delta r_1 - r_2 - \delta r_2\right)^T \left(r_1 + \delta r_1 - r_2 - \delta r_2\right) = L^2$$

  - **Rearranging this gives:**

$$\left(r_1 - r_2 + \delta r_1 - \delta r_2\right)^T \left(r_1 - r_2 + \delta r_1 - \delta r_2\right) = L^2$$

$$\Rightarrow \left(r_1 - r_2\right)^T \left(r_1 - r_2\right) + 2\left(r_1 - r_2\right)^T \left(\delta r_1 - \delta r_2\right) + \left(\delta r_1 - \delta r_2\right)^T \left(\delta r_1 - \delta r_2\right) = L^2$$

Virtual displacements

- **If we ignore second order terms in the virtual displacements:**

$$(r_1 - r_2)^T (r_1 - r_2) + 2(r_1 - r_2)^T (\delta r_1 - \delta r_2) = L^2$$

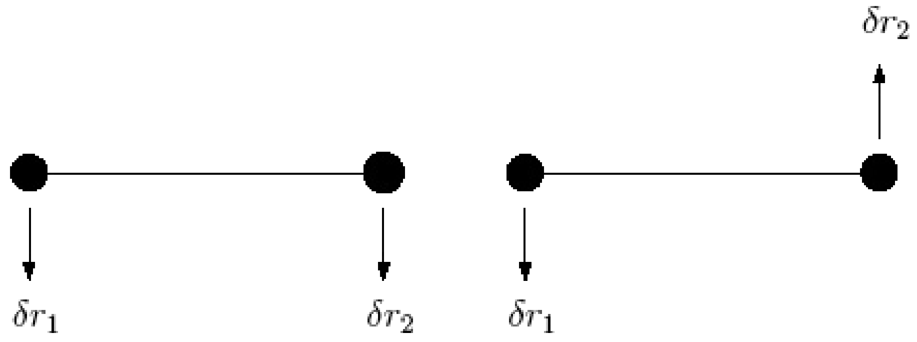
- **Now our initial constraint is:**

$$(r_1 - r_2)^T (r_1 - r_2) = L^2$$

- **Therefore:**

$$2(r_1 - r_2)^T (\delta r_1 - \delta r_2) = 0$$

- **This says that in order for the virtual displacements to satisfy the holonomic constraints, they must be orthogonal to the constraint forces**
  - **For example:**



Principle of virtual work

- So, if, due to the holonomic constraints on a system of particles, the coordinates of the particles can be fully defined by a set of generalized coordinates, then we can write the set of virtual displacements as:

$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \,, \quad i = 1,\ldots,k$$

  – There are no constraints on the generalized coordinates, so there are no constraints on the virtual displacements of the generalized coordinates

- Now suppose each particle is in equilibrium
  – Thus the net force on each particle is zero
  – This implies that the work done by each set of virtual displacements is zero

$$\sum_{i=1}^{k} F_i^T \delta r_i = 0$$

    - F_i is the total force acting on particle *i*

Principle of virtual work

- **At the beginning of this discussion, we said that a system with holonomic constraints has two sets of forces acting on it:**
  - **External forces, f_i**
  - **Forces that impose the constraints, f_i^a**
- **If the work done by the constraint force is zero, we can write:**

$$\sum_{i=1}^{k} f_i^{a^T} \delta r_i = 0$$

  - **This is true whenever the constraint force between particles is directed along the vector connecting the two particles**
    - **i.e. the virtual displacements are orthogonal to the constraint forces**
- **Thus when the work done by the constraint forces is zero, we can also say that:**

$$\sum_{i=1}^{k} f_i^{T} \delta r_i = 0$$

Principle of virtual work

- **This is called the principle of virtual work:**
  - **The work done by external forces corresponding to any set of virtual displacements is zero!**
  - **Only true if the work done by the constraint forces is zero**
    - **Which is only true if the constraint forces are directed along the vector connecting two particles**
    - **In this case the virtual displacement vector will be orthogonal to the constraint force vector**
      - **This is by definition of the virtual displacements and the fact that they also must satisfy the holonomic constraints**
- **Ex: again, consider two particles that are constrained by:** $(r_1 - r_2)^T (r_1 - r_2) = L^2$
  - **The constraint force must be along the vector connecting the two**
  - **Assume that there is a rigid wire connecting the two**
  - **Thus the constraint force acting on the first particle is:** $f_1^a = c(r_1 - r_2)$
  - **And clearly the force on the second particle is:** $f_2^a = -c(r_1 - r_2)$

Principle of virtual work

- **Thus we can write the work done by the constraint forces as:**

$$f_1^{a^T}\delta r_1 + f_2^{a^T}\delta r_2 = c(r_1 - r_2)^T\delta r_1 - c(r_1 - r_2)^T\delta r_2$$
$$= c(r_1 - r_2)^T(\delta r_1 - \delta r_2)$$

- **But we already showed that this is zero since the virtual displacements are also consistent with the holonomic constraints**
  - **Thus the work done by the constraint forces is zero**
- **Thus what we have shown is that for a system of points in which the points are subjected to holonomic constraints, the principle of virtual work says that the work done by the forces that maintain the holonomic constraints is zero**
- **We can represent a rigid body as an infinite set of particles with an infinite set of holonomic constraints**
  - **Thus the principle of virtual work also holds in the case of rigid motion**

D'Alembert's principle

- **The previous discussion assumed that the system of particles was in equilibrium**
- **To make this more general, we assume that the system is not in equilibrium**
  - **Thus adding an additional force will put the more general system in equilibrium (D'Alembert's Principle)**
  - **This additional force is:** $-\dot{p}_i$
    - **Where p_i is the momentum of the i^{th} particle**
  - **And we can include this as follows:**

$$\sum_{i=1}^{k} \left(F_i - \dot{p}_i\right)^T \delta r_i = 0$$

  - **We can again use the principle of virtual work to remove the constraint forces from F_i and this becomes:**

$$\sum_{i=1}^{k} f_i^T \delta r_i - \sum_{i=1}^{k} \dot{p}_i \delta r_i = 0$$

D'Alembert's principle

- **Note that the virtual displacements are not independent**
  - **But the generalized coordinates are**
- **Thus we want to express this relationship in terms of the generalized coordinates instead of the virtual displacements**
  - **First, remember that:**

$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \ , \quad i = 1, \ldots, k$$

  - **Substituting this in gives (for the first term):**

$$\sum_{i=1}^{k} f_i^T \delta r_i = \sum_{i=1}^{k} \sum_{j=1}^{n} f_i^T \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \psi_j \delta q_j$$

  - **Where the j^{th} generalized force ψ_j is:**

$$\psi_j = \sum_{i=1}^{k} f_i^T \frac{\partial r_i}{\partial q_j}$$

D'Alembert's principle

- **The momentum of particle *i* is:** $p_i = m_i \dot{r}_i$
- **Therefore we can write the second term as:**

$$\sum_{i=1}^{k} \dot{p}_i^T \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i^T \delta r_i = \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j$$

- **Now note that:**

$$\frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] = m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right]$$

- **Rearranging gives:**

$$m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right]$$

- **Remember that the holonomic constraints allows us to write the coordinates of the particles as:** $r_i = r_i(q_1, \ldots, q_n), \quad i = 1, \ldots, k$

D'Alembert's principle

- **Differentiating this gives:**

$$v_i \equiv \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j$$

- **Note that this implies:**

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

- **Now the last term that we need to identify is:**

$$\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] = \sum_{u=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_u} \dot{q}_u = \frac{\partial}{\partial q_j}\left(\sum_{u=1}^{n} \frac{\partial r_i}{\partial q_u} \dot{q}_u\right) = \frac{\partial v_i}{\partial q_j}$$

- **Recall the following**

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right] \right\}$$

$$v_i$$

D'Alembert's principle

- **Now that we know each term, we can rewrite this as follows:**

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j} \right] - m_i v_i^T \frac{\partial v_i}{\partial q_j} \right\}$$

- **Now define the kinetic energy of a system of particles:**

$$K = \sum_{i=1}^{k} \frac{1}{2} m_i v_i^T v_i$$

- **Take the partial derivative of the kinetic energy with respect to the generalized coordinates and their derivatives:**

$$\frac{\partial K}{\partial q_j} = \sum_{i=1}^{k} m_i v_i^T \frac{\partial v_i}{\partial q_j} \qquad \frac{\partial K}{\partial \dot{q}_j} = \sum_{i=1}^{k} m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j}$$

- **Therefore we can rewrite once more:**

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}$$

D’Alembert’s principle

- **Putting this pack into our first equation for the work from the fictitious force:** $$\sum_{i=1}^{k} p_i^{\ T} \delta r_i = \sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \right\} \delta q_j$$
- **Finally, combining this with the external forces:**

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j \right\} \delta q_j = 0$$

- **Since the virtual displacements of the generalized coordinates are independent, we can say that each coefficient of δq_j is zero:** $$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \psi_j \ , \quad j = 1, \ldots, n$$

Euler-Lagrange Equations

- **If we assume that the generalized force ψ_j is the sum of an external force and a force due to potential energy, we can simplify further**
  - **Assume that there exists a potential energy function *P*(*q*) and external forces τ_j:**

$$\psi_j = -\frac{\partial P}{\partial q_j} + \tau_j$$

  - **Then remembering the definition of the Lagrangian: *L* = *K* – *P*:**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \tau_j$$

  - **This is the Euler-Lagrange equation and it allows us to derive the equations of motion for an *n*DOF system subject to holonomic constraints**

Dynamics Overview

- **We want to come up with equations of motion for any *n*DOF system**
  - **In general, this will consist of *n* coupled second order ODEs**
- **These systems may be:**
  - **Linear or nonlinear**
  - **Conservative or nonconservative**
- **We want to develop an expression of the form:**

$$\dot{q} = f(q,t)$$

- **Once we have this, we can use it to choose an appropriate controller that will put our dynamical system in a desired state (configuration)**

Euler-Lagrange Equations

- **We can derive the equations of motion for any *n*DOF system by using energy methods**
  - **All we need to know are the conservative (kinetic and potential) and non-conservative (dissipative) terms**
- **Ideally, the terms in the Euler-Lagrange equation are functions of the generalized coordinates**
- **This is a shortcut to describing the motion of each particle in a rigid body along with the constraints that form rigid motions**
- **For this, we need to first use virtual displacements subject to holonomic constraints, then use the principle of virtual work, then finally use D'Alembert's Principle to derive the Euler-Lagrange equations of motion**

Euler-Lagrange Equations

- **If we represent the variables of the system as generalized coordinates, then we can write the equations of motion for an *n*DOF system as:**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i$$

- **It is important to recognize the form of the above equation:**
  - **The left side contains the conservative terms**
  - **The right side contains the non-conservative terms**
- **This formulation leads to a set of *n* coupled 2nd order differential equations**

Kinetic Energy

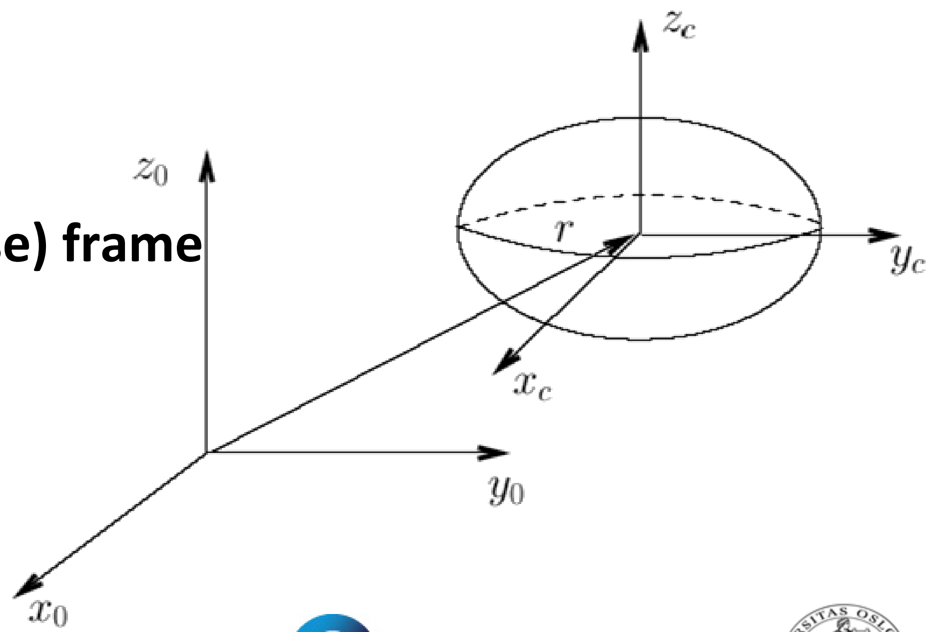
- Need to define the kinetic energy for an *n*-link manipulator
- DH joint variables are the generalized coordinates
- The kinetic energy is a function that takes a vector of joint velocities to a scalar
- The kinetic energy is the sum of two terms:
  - A translational term equivalent to concentrating all the mass at the center of mass
  - A rotational term due to rotation about the center of mass

Kinetic Energy

- **For an arbitrary rigid body, the kinetic energy is:**

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T I \omega$$

  - ***v* is the linear velocity of the center of mass**
  - **ω is the angular velocity**
  - **I is the inertia tensor**
    - **expressed in the inertial (base) frame**



Inertia

- **First, we need to express the inertia in the body-attached frame**
  - **Note that the rotation between the inertial frame and the body attached frame is just *R***
  - **Note also the relationship between the angular velocity ω (in the inertial frame) and the rotational transformation between the body attached frame and the inertial frame:**

$$S(\omega) = \dot{R}R^T$$

  - **So the inertia in the inertial frame is related to the inertia in the body frame by a similarity transform:**

$$I = RIR^T$$

    - ***I* is the inertia in the body attached frame**
  - **The inertia in the inertial frame is dependant upon the configuration**
  - **The inertia in the body attached frame is independent of configuration**

Inertia

- **Inertia is an intrinsic property of a rigid body**
  - **In the body frame, it is a constant 3x3 matrix:**

$$I = [I_{ij}] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

  - **The diagonal elements are called the principal moments of inertia and are a representation of the mass distribution of a body with respect to an axis of rotation:**

$$I_{ii} = \int_V r^2 dm = \iiint_V r^2 \rho(x,y,z) dV = \iiint_V r^2 \rho(x,y,z) dxdydz$$

    - ***r* is the distance from the axis of rotation to the particle**

Inertia

- **The elements are defined by:**

$\rho(x,y,z)$ is the density

$$I_{xx} = \iiint (y^2 + z^2)\rho(x,y,z)dxdydz$$
$$I_{yy} = \iiint (x^2 + z^2)\rho(x,y,z)dxdydz$$
$$I_{zz} = \iiint (x^2 + y^2)\rho(x,y,z)dxdydz$$

principal moments of inertia

$$I_{xy} = I_{yx} = -\iiint xy\rho(x,y,z)dxdydz$$
$$I_{xz} = I_{zx} = -\iiint xz\rho(x,y,z)dxdydz$$
$$I_{yz} = I_{zy} = -\iiint yz\rho(x,y,z)dxdydz$$

cross products of inertia

Ex: uniform rectangular block

- **Consider a uniform (amorphous) solid block with the reference frame in the geometric center**

$$I_{xx} = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}\left(y^2+z^2\right)\rho(x,y,z)dxdydz$$

$$= \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\left(y^2+z^2\right)x\Big|_{-a/2}^{a/2}\rho dydz$$

$$= \rho a\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\left(y^2+z^2\right)dydz$$

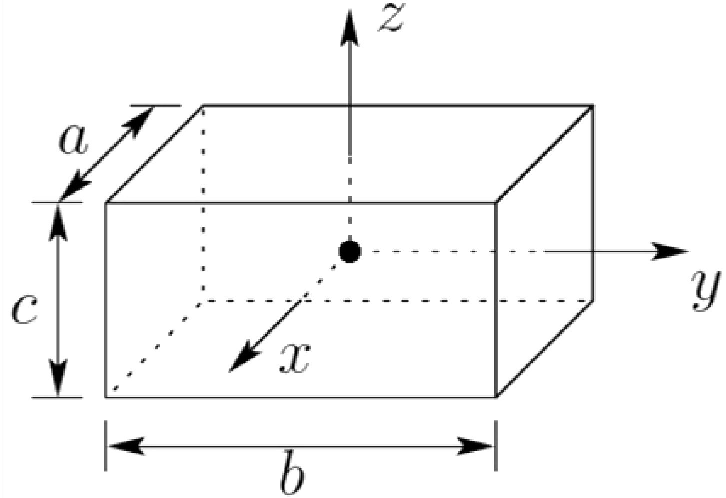
$$= \rho a\int_{-c/2}^{c/2}\left(\frac{y^3}{3}+z^2y\right)\Bigg|_{-b/2}^{b/2}dz$$

$$= \rho ab\int_{-c/2}^{c/2}\left(\frac{b^2}{12}+z^2\right)dz = \rho ab\left(\frac{b^2}{12}z+\frac{z^3}{3}\right)\Bigg|_{-c/2}^{c/2}$$

$$= \rho\frac{abc}{12}\left(b^2+c^2\right) = \frac{m}{12}\left(b^2+c^2\right)$$

$$I_{yy} = \frac{m}{12}\left(a^2+c^2\right)$$

$$I_{zz} = \frac{m}{12}\left(a^2+b^2\right)$$



$$I_{xy} = -\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}xy\rho(x,y,z)dxdydz$$

$$= -\int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\frac{yx^2}{2}\Bigg|_{-a/2}^{a/2}\rho dydz = 0$$

Ex: uniform cylinder

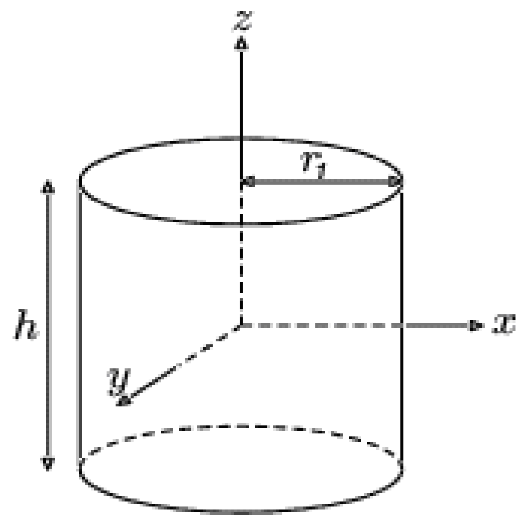
- **Consider a uniform (amorphous) solid cylinder with the reference frame in the geometric center**

$$I_{zz} = \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^{r_1} r^2 \rho(x,y,z) r dr d\theta dz$$

$$= \int_{-h/2}^{h/2}\int_0^{2\pi} \frac{r^4}{4}\bigg|_0^{r_1} \rho d\theta dz = \rho\int_{-h/2}^{h/2}\int_0^{2\pi} \frac{r_1^{\,4}}{4} d\theta dz$$

$$= \rho\int_{-h/2}^{h/2} \frac{r_1^{\,4}}{4}\theta\bigg|_0^{2\pi} dz = \frac{\rho\pi r_1^{\,4}}{2} z\Big|_{-h/2}^{h/2} = \left(\rho\pi r_1^{\,2} h\right)\frac{1}{2} r_1^{\,2}$$

$$= \frac{1}{2} m r_1^{\,2}$$



Kinetic energy of an *n*-link manipulator

- **We said that the kinetic energy of a rigid body is the sum of two terms:**

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T I \omega$$

- **Noting the relation between the inertia tensor (inertial frame) and the inertia matrix (body-attached frame), this becomes:**

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T R I R^T \omega$$

- **Similar to a set of particles, the total kinetic energy for an *n*-link manipulator is the sum of the kinetic energy of each link:**

$$K = \sum_{i=1}^{n} \left\{ \frac{1}{2} m_i v_i{}^T v_i + \frac{1}{2} \omega_i{}^T R_i I_i R_i{}^T \omega_i \right\}$$

Kinetic energy of an *n*-link manipulator

- **Note that the rotational matrices are functions of the generalized coordinates**
  - **i.e. functions of the joint variables:**

$$K=\sum_{i=1}^{n}\left\{\frac{1}{2}m_i v_i{}^T v_i+\frac{1}{2}\omega_i{}^T R_i(q) I_i R_i(q)^T \omega_i\right\}$$

- **Lastly, we need a relation between the joint velocities and the linear and angular velocity of the CM (Center of Mass) of each link**
  - **This is simply the Jacobian:**

$$v_i=J_{v_i}(q)\dot{q} \qquad \omega_i=J_{\omega_i}(q)\dot{q}$$

- **Thus we can rewrite the kinetic energy as a function of the joint variables:**

$$K=\sum_{i=1}^{n}\left\{\frac{1}{2}m_i\dot{q}^T J_{v_i}(q)^T J_{v_i}(q)\dot{q}+\frac{1}{2}\dot{q}^T J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q)\dot{q}\right\}$$

Kinetic energy of an *n*-link manipulator

- **Simplifying this gives:**

$$K = \frac{1}{2}\dot{q}^T \underbrace{\left[\sum_{i=1}^{n} \left\{ m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q) \right\} \right]}_{\text{Inertia matrix}} \dot{q} = \frac{1}{2}\dot{q}^T \underbrace{D(q)}_{\text{Inertia matrix}} \dot{q}$$

- **Properties of the inertia matrix:**
  - ***n* x *n***
  - **Symmetric**
  - **Positive-definite**

Potential energy of an *n*-link manipulator

- **For a rigid manipulator, only potential energy is from gravity**
- **The potential energy of the i^{th} link is found by assuming all mass is concentrated at the center of mass**
  - **let r_{ci} be the position of the center of mass of the i^{th} link (in the inertial frame)**
  - **let g be the gravity vector in the inertial frame**

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} m_i g^T r_{ci}$$

- **Potential energy is independent of the joint velocities**

Equations of motion

- **Now we are finally ready to use the Euler-Lagrange formulation**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \tau_j$$

- **Now we can write the Lagrangian as:**

$$L = K - P = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q) = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j - P(q)$$

- **Now take the partial derivatives of the Lagrangian:**

$$\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j\right) - \frac{\partial}{\partial \dot{q}_k}P(q)$$

$$= \sum_{j=1}^{n} d_{kj}(q)\dot{q}_j$$

Equations of motion

- **Now take the partial derivatives of the Lagrangian:**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n} \frac{d}{dt} d_{kj}\dot{q}_j$$

$$= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j$$

- **Partial with respect to the positions:**

$$\frac{\partial L}{\partial q_k} = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial P}{\partial q_k}$$

- **Combining these back into the Euler-Lagrange equation:**

$$\sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

Equations of motion

- **Lastly, we do a trick to simplify the second term**

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i}+\frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j \longrightarrow$$

Simply because of the symmetry of the inertia matrix

- **Therefore the second term can be rewritten as:**

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i}-\frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i}+\frac{\partial d_{ki}}{\partial q_j}-\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j$$

- **For simplicity, we make the following definition:**

$$c_{ijk} \equiv \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i}+\frac{\partial d_{ki}}{\partial q_j}-\frac{\partial d_{ij}}{\partial q_k}\right)$$

  - **c_{ijk} are the *Christoffel symbols***

- **Define the gravity force on the *k*th link as:**

$$g_k = \frac{\partial P}{\partial q_k}$$

Equations of motion

- **We now write the Euler-Lagrange equations for an n-link manipulator:**

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k$$

$$\underbrace{\text{if } i = j, \text{then } \dot{q}_i^{\,2}}_{\textit{centrifugal}} \qquad \underbrace{\text{if } i \neq j, \text{then } \dot{q}_i\dot{q}_j}_{\textit{Coriolis}}$$

Equations of motion

- **Commonly, this is written as:**

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

- ***D*(*q*) is the inertia matrix**
- **The *C* matrix is given as:**

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i$$

- **And the gravity vector is:**

$$g(q) = \begin{bmatrix} q_1(q) & q_2(q) & \cdots & q_n(q) \end{bmatrix}^T$$

- **This holds for any *n*-link manipulator that has kinetic energy defined by two terms and potential energy that is independent of velocity**

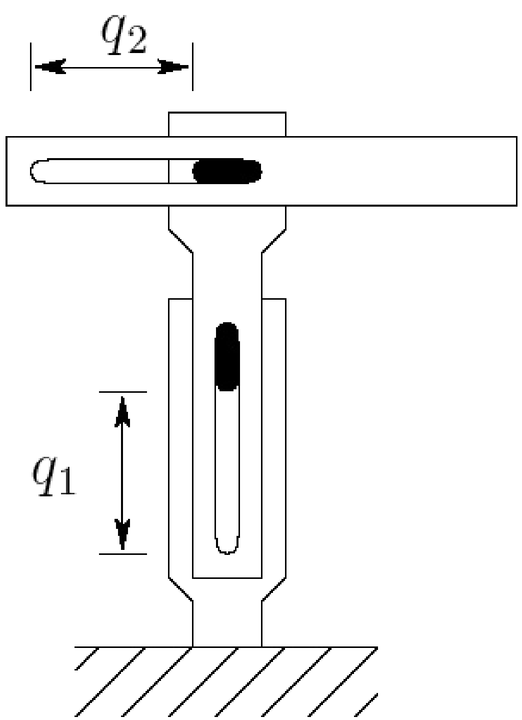
summary

1. If we know the configuration, we can calculate the Jacobian
   - If we know the Jacobian, we know the velocities of each CM
2. Once we know the mass and inertia properties of each link and the linear and angular velocities of each link, we can formulate the kinetic energy (for the whole system)
3. If we know the forward kinematics and the mass of each link, we can find the position of the CM
   - We can calculate the potential energy
4. Once we have *K* and *P*, we can easily form *L*
5. To obtain the equations of motion we can:
   - Take the partial derivatives with respect to the joint positions and velocities (and time derivatives) and plug into the Euler-Lagrange formulation
   - Or determine *D*, *C*, and *g*

Ex: two-link cartesian manipulator

- m_1 and m_2 are the masses of the two links
- Generalized coordinates are the prismatic displacements
- By inspection:

$$v_1 = J_{v_1}\dot{q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \qquad v_2 = J_{v_2}\dot{q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$



- Thus, the kinetic energy is:

$$K = \frac{1}{2}\dot{q}^T \left\{ \sum_{i=1}^{n} m_i J_{v_i}{}^T J_{v_i} \right\} \dot{q} = \frac{1}{2}\dot{q}^T \left\{ m_1 J_{v_1}{}^T J_{v_1} + m_2 J_{v_2}{}^T J_{v_2} \right\} \dot{q}$$

- And the inertia matrix is:

$$D(q) = m_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + m_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix}$$

Ex: two-link cartesian manipulator

- **The potential energy of the system is:**

$$P(q) = \sum_{i=1}^{n} P_i = g(m_1 + m_2)q_1$$

- **Note that the inertia matrix is not a function of *q***
  - **Therefore the partial derivatives are zero**
  - **Thus the Christoffel symbols are zero**
- **We can write the equations of motions as:**

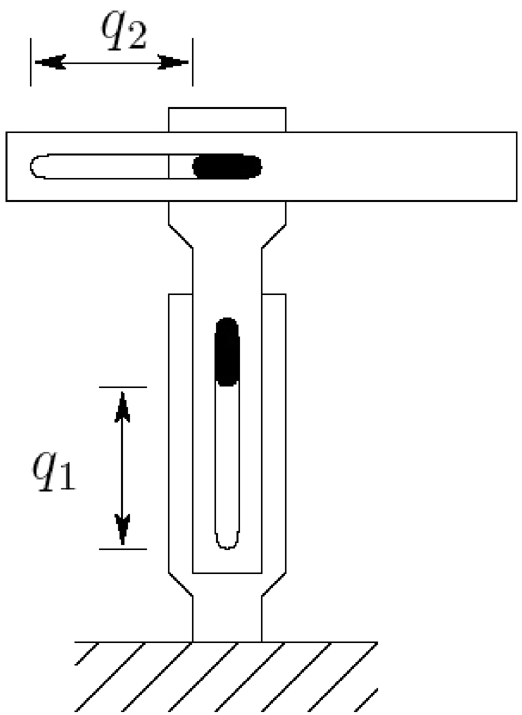
$$D\ddot{q} + g(q) = f$$

  - **Where the gravity vector is the partial of *P*:**

$$g_1 = \frac{\partial P}{\partial q_1} = g(m_1 + m_2), \; g_2 = \frac{\partial P}{\partial q_2} = 0$$

- **Therefore, the equations of motion are:**

$$(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) = f_1$$
$$(m_2)\ddot{q}_2 = f_2$$

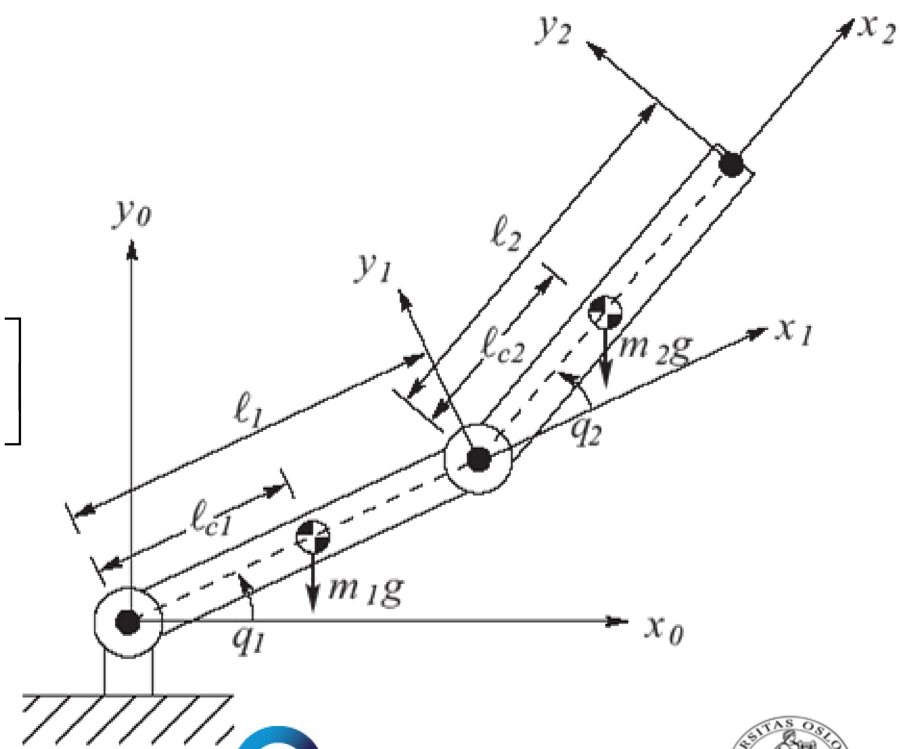


Ex: two-link elbow manipulator

- **We need the linear and angular velocity of each link:**

$$v_1 = J_{v_1}\dot{q} = \begin{bmatrix} -L_{c1}\sin q_1 & 0 \\ L_{c1}\cos q_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

$$\begin{aligned} v_2 &= J_{v_2}\dot{q} \\ &= \begin{bmatrix} -L_1\sin q_1 - L_{c2}\sin(q_1+q_2) & -L_{c2}\sin(q_1+q_2) \\ L_1\cos q_1 + L_{c2}\cos(q_1+q_2) & L_{c2}\cos(q_1+q_2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \end{aligned}$$



Ex: two-link elbow manipulator

- **Now the angular velocity of each link:**

$$\omega_i^T I_i \omega_i = \dot{q}^T J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q)\dot{q}$$

$$R_1(q) = \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_2(q) = \begin{bmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_1 = \begin{bmatrix} I_{xx,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz,1} \end{bmatrix} \qquad I_2 = \begin{bmatrix} I_{xx,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz,2} \end{bmatrix}$$

$$\omega_1 = J_{\omega_1}\dot{q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \qquad \omega_2 = J_{\omega_2}\dot{q} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

Ex: two-link elbow manipulator

- **Thus the rotational component of the kinetic energy is:**

$$K_{rot} = \frac{1}{2}\dot{q}^T \left\{ \sum_{i=1}^{2} J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q) \right\} \dot{q}$$

$$= \frac{1}{2}\dot{q}^T \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos q_1 & -\sin q_1 & 0 \\ \sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{xx,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz,1} \end{bmatrix} \begin{bmatrix} \cos q_1 & \sin q_1 & 0 \\ -\sin q_1 & \cos q_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \dot{q}$$

$$+ \frac{1}{2}\dot{q}^T \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(q_1+q_2) & -\sin(q_1+q_2) & 0 \\ \sin(q_1+q_2) & \cos(q_1+q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{xx,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{zz,2} \end{bmatrix} \begin{bmatrix} \cos(q_1+q_2) & \sin(q_1+q_2) & 0 \\ -\sin(q_1+q_2) & \cos(q_1+q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \right\} \dot{q}$$

$$= \frac{1}{2}\dot{q}^T \left\{ \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} \right\} \dot{q}$$

Ex: two-link elbow manipulator

- **Thus, the total kinetic energy of the system is:**

$$K = \frac{1}{2}\dot{q}^T \left\{ \sum_{i=1}^{n} m_i J_{v_i}{}^T J_{v_i} + J_{\omega_i}{}^T I_i J_{\omega_i} \right\} \dot{q}$$

$$= \frac{1}{2}\dot{q}^T \left\{ m_1 J_{v_1}{}^T J_{v_1} + m_2 J_{v_2}{}^T J_{v_2} + I_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\} \dot{q}$$

- **And the inertia matrix can be written as:**

$$D(q) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

  - **Where:**

$$d_{11} = m_1 L_{c1}{}^2 + m_2 \left(L_1{}^2 + L_{c2}{}^2 + L_1 L_{c2} \sin(q_1) \sin(q_1 + q_2) + L_1 L_{c2} \cos(q_1) \cos(q_1 + q_2) \right) + I_1 + I_2$$

$$d_{12} = m_2 \left(L_{c2}{}^2 + L_1 L_{c2} \sin(q_1) \sin(q_1 + q_2) + L_1 L_{c2} \cos(q_1) \cos(q_1 + q_2) \right) + I_2$$

$$d_{22} = m_2 L_{c2}{}^2 + I_2$$

Ex: two-link elbow manipulator

- **Now since $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$, we can rewrite the elements of the inertia matrix:**

$$d_{11} = m_1 L_{c1}^{\ 2} + m_2\left(L_1^{\ 2} + L_{c2}^{\ 2} + 2L_1 L_{c2}\cos(q_2)\right) + I_1 + I_2$$
$$d_{12} = m_2\left(L_{c2}^{\ 2} + L_1 L_{c2}\cos(q_2)\right) + I_2$$
$$d_{22} = m_2 L_{c2}^{\ 2} + I_2$$

- **We next determine the Christoffel symbols:**

$$c_{111} = \frac{1}{2}\left\{\frac{\partial d_{11}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_1} - \frac{\partial d_{11}}{\partial q_1}\right\} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0$$
$$c_{121} = \frac{1}{2}\left\{\frac{\partial d_{12}}{\partial q_1} + \frac{\partial d_{11}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_1}\right\} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 L_1 L_{c2}\sin q_2$$
$$c_{221} = \frac{1}{2}\left\{\frac{\partial d_{12}}{\partial q_2} + \frac{\partial d_{12}}{\partial q_2} - \frac{\partial d_{22}}{\partial q_1}\right\} = -m_2 L_1 L_{c2}\sin q_2$$

for $k = 1$

Ex: two-link elbow manipulator

- **We next determine the Christoffel symbols:**

$$\left.\begin{aligned}
c_{112} &= \frac{1}{2}\left\{\frac{\partial d_{21}}{\partial q_1} + \frac{\partial d_{21}}{\partial q_1} - \frac{\partial d_{11}}{\partial q_2}\right\} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = m_2 L_1 L_{c2} \sin q_2 \\
c_{122} &= \frac{1}{2}\left\{\frac{\partial d_{22}}{\partial q_1} + \frac{\partial d_{21}}{\partial q_2} - \frac{\partial d_{12}}{\partial q_2}\right\} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\
c_{222} &= \frac{1}{2}\left\{\frac{\partial d_{22}}{\partial q_2} + \frac{\partial d_{22}}{\partial q_2} - \frac{\partial d_{22}}{\partial q_2}\right\} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0
\end{aligned}\right\} \text{for } k = 2$$

- **Thus we can write the *C* matrix:**

$$\begin{aligned}
C(q,\dot{q}) &= \begin{bmatrix} c_{111} & c_{121} \\ c_{121} & c_{221} \end{bmatrix} + \begin{bmatrix} c_{112} & c_{122} \\ c_{122} & c_{222} \end{bmatrix} \\
&= \begin{bmatrix} (-m_2 L_1 L_{c2} \sin q_2)\dot{q}_2 & -m_2 L_1 L_{c2} \sin q_2(\dot{q}_1 + \dot{q}_2) \\ (m_2 L_1 L_{c2} \sin q_2)\dot{q}_1 & 0 \end{bmatrix}
\end{aligned}$$

Ex: two-link elbow manipulator

- **We next determine the potential energy:**

$$P_1 = m_1 g L_{c1} \sin q_1$$
$$P_2 = m_2 g\left(L_1 \sin q_1 + L_{c2} \sin(q_1 + q_2)\right)$$

- **And the gravity vector:**

$$g_1 = \frac{\partial P}{\partial q_1} = (m_1 L_{c1} + m_2 L_1) g \cos q_1 + m_2 L_{c2} g \cos(q_1 + q_2)$$
$$g_2 = \frac{\partial P}{\partial q_2} = m_2 L_{c2} g \cos(q_1 + q_2)$$

- **Combining each of these terms gives us the equations of motion:**

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_1\dot{q}_2 + c_{221}\dot{q}_2^{\,2} + g_1 = \tau_1$$
$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^{\,2} + g_2 = \tau_2$$