INF3580 – Semantic Technologies – Spring 2011

Lecture 5: Mathematical Foundations

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22nd February 2011

Department of Informatics

University of Oslo

Today's Plan

1. Basic Set Algebra
2. Pairs and Relations
3. Propositional Logic

Outline

Motivation

- The great thing about Semantic Technologies is. . .

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- ...Semantics!

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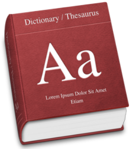

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- There are some problems with this, but it's good enough for us!

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- We use $\in$ to say that something is element of a set:

$$1 \in \{\text{'a'}, 1, \triangle\}$$

$$\text{'b'} \notin \{\text{'a'}, 1, \triangle\}$$

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- $\{1,2,3\}$ has 3 elements, what about $\{a,b,c\}$?

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- $x \notin \emptyset$, whatever x is!

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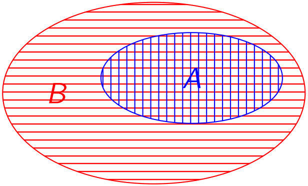

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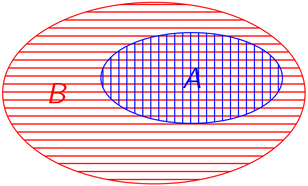

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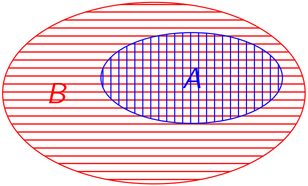

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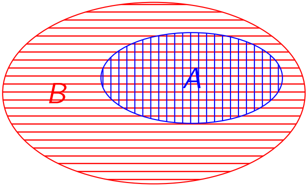

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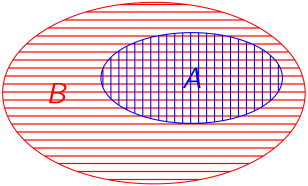

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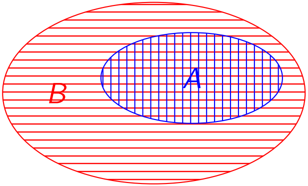

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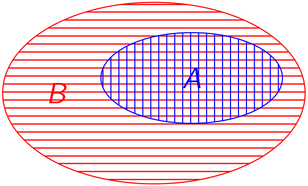

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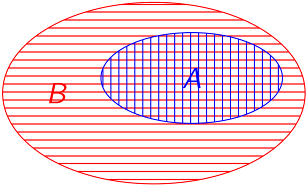

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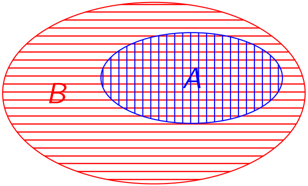

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- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$

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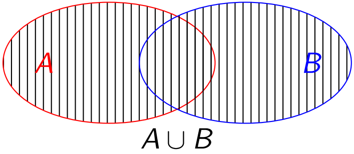

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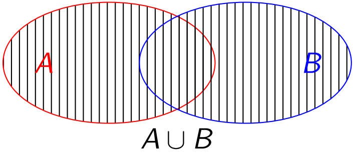

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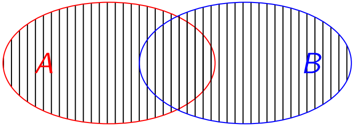

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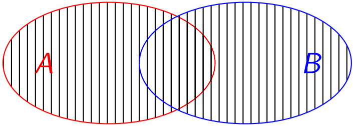

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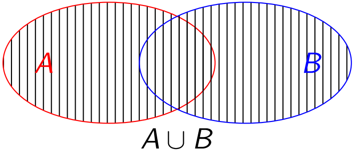

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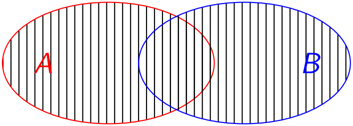

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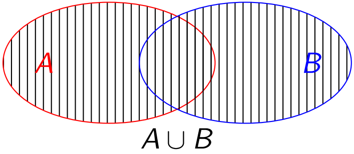

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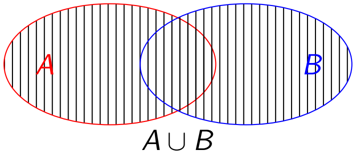

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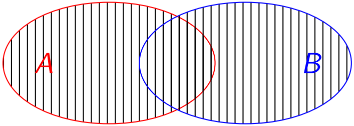

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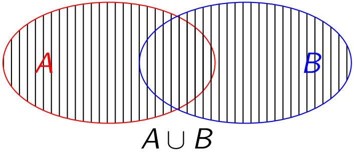

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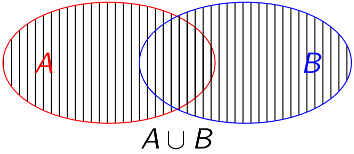

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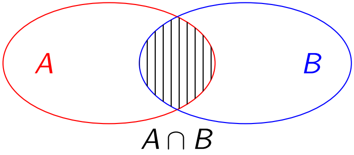

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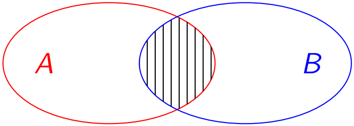

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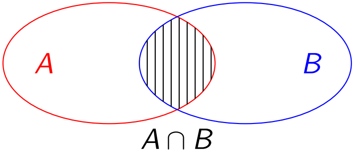

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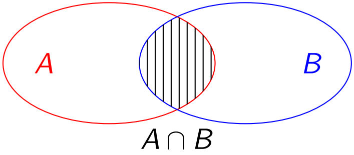

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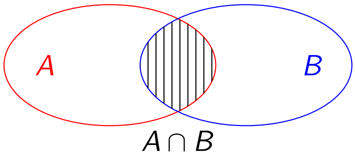

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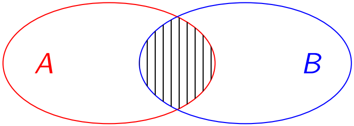

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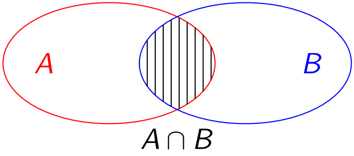

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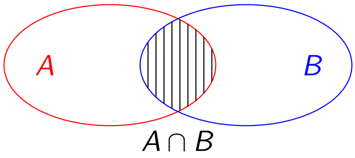

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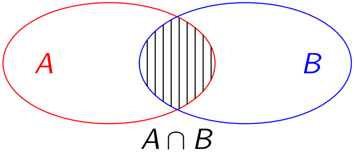

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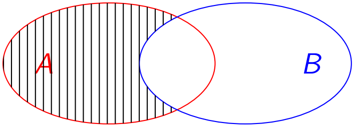

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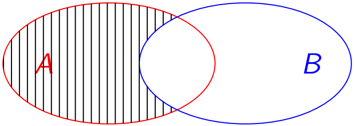

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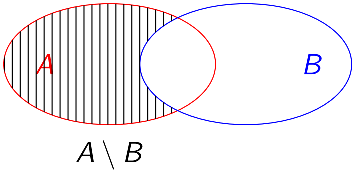

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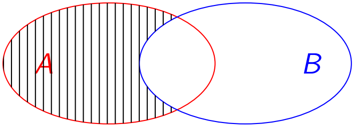

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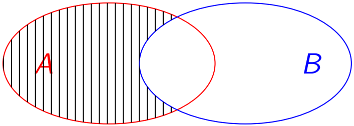

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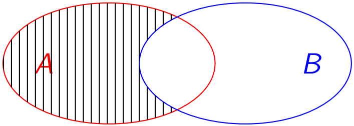

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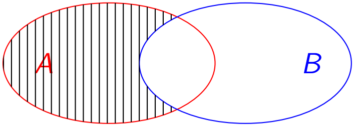

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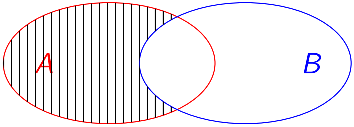

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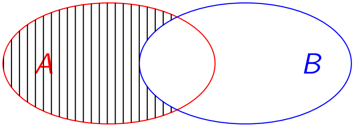

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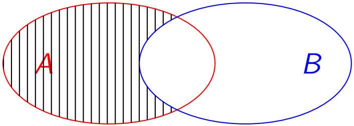

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- Example:
  - Let $L = \{\text{‘a’}, \text{‘b’}, \ldots, \text{‘z’}\}$
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Outline

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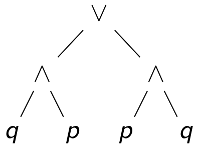

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- Let's formalize this context, a.k.a. interpretation

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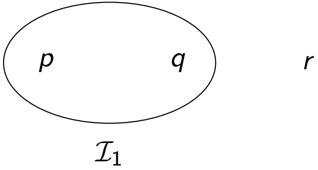

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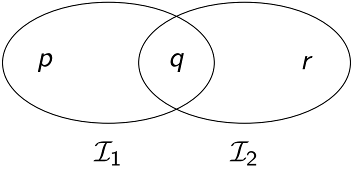

- But in $\mathcal{I}_2 = \{q, r\}$, p is false, but r is true.

Semantic Validity $\models$

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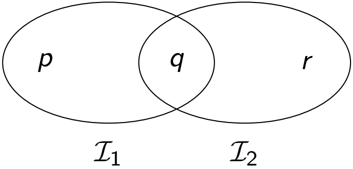

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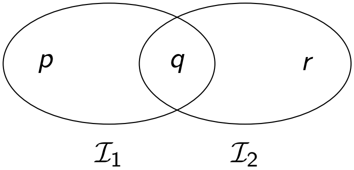

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- In other words, for all letters p:

$$\mathcal{I} \models p \qquad \text{if and only if} \qquad p \in \mathcal{I}$$

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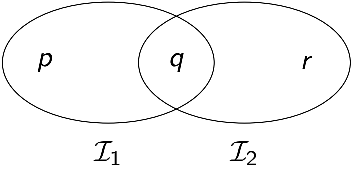

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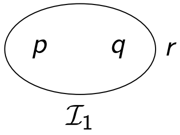

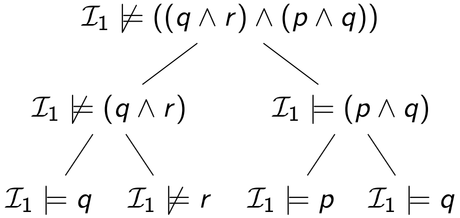

Semantics for $\neg$ and $\vee$

- The complete definition of $\models$ is as follows:
- For any interpretation $\mathcal{I}$, letter p, formulas A, B:
  - $\mathcal{I} \models p$ iff $p \in \mathcal{I}$
  - $\mathcal{I} \models \neg A$ iff $\mathcal{I} \not\models A$
  - $\mathcal{I} \models (A \wedge B)$ iff $\mathcal{I} \models A$ and $\mathcal{I} \models B$
  - $\mathcal{I} \models (A \vee B)$ iff $\mathcal{I} \models A$ or $\mathcal{I} \models B$ (or both)
- Semantics of $\neg$, $\wedge$, $\vee$ often given as *truth table*:

A	B	$\neg A$	$A \wedge B$	$A \vee B$
f	f	t	f	f
f	t	t	f	t
t	f	f	f	t
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- $(p \vee \neg p)$ is true in *all* interpretations!

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- . . . without understanding their meaning!

Checking Tautologies

- Checking whether $\models A$ is the task of SAT-solving
- (co-)NP-complete in general (i.e. in practice exponential time)
- Small instances can be checked with a truth table:

$$\models (\neg p \vee (\neg q \vee (p \wedge q))) \quad ?$$

p	q	$\neg p$	$\neg q$	$(p \wedge q)$	$(\neg q \vee (p \wedge q))$	$(\neg p \vee (\neg q \vee (p \wedge q)))$
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- Therefore: $(\neg p \vee (\neg q \vee (p \wedge q)))$ is a tautology!

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