Course Script

INF 5110: Compiler construction

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Martin Steffen

Contents

Chapter 2 Scanning

What is it about?

Learning Targets of this Chapter

1. alphabets, languages
2. regular expressions
3. finite state automata / recognizers
4. connection between the two concepts
5. minimization

The material corresponds roughly to [7, Section 2.1–2.5] or a large part of [11, Chapter 2]. The material is pretty canonical, anyway.

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2.1 Introduction

The *scanner* or *lexer* is the first phase of a typical compiler, leaving out preprocessing, which is more seen as something that happens "before" the compiler does its job. What a lexer does is also called **lexical analysis**, basically chopping up the input string into smaller units (so-called **lexemes**), classifying them according to the lexical rules of the language being implemented, and handing over the results of that chopping-up-and-classification to the parser in a stream of so-called tokens.

The theory underlying lexers is that of *regular languages*. Typically, lexical aspects of a language are specified using some variant of **regular expressions**. The lexer program then has to implement that specification, in that it is able to read in the source program (it *scans* it) and the checks it for compliance with the specification. At the same time, it does the chopping-and-classification task mentioned (it **tokenizes** the input string). Checking for compliance with regular expressions is done via **finite-state machines**. Finite-state machines are equivalent to regular expressions insofar that they can describe the same class of languages. Here, "language" is meant as sequence of characters from an **alphabet**. Regular expressions are declarative in nature (hence more useful for specification), whereas finite-state automata are more *operational* in nature, hence used in implementing a scanner. We discuss how to translate regular expressions to automata. The reverse

translation is also possible (and easy), but we won't discuss that, as it's not needed for a compiler. The `lex` tool actually does just that: the users specifies the lexical aspects of the language to compile and `lex` generates from that the lexer for that language, based on the theory of regular expression and finite state automata. Actually, tools like `lex` do a bit more, which mostly has to do with with support to generate tokens and to interface properly with the parser. Parsing will be covered in subsequent chapters.

2.1.1 Scanner section overview

A scanner is the part of a compiler that takes the source code as input and translates this stream of characters into a stream of tokens (see Figure 2.1). Working with the source code, it's the first phase of a compiler (leaving aside possible pre-processing).

Figure 2.1: Input and output of a lexer

Characters are typically *language-independent*, but perhaps the encoding (or its interpretation) may vary, like ASCII, UTF-8, also Windows-vs.-Unix-vs.-Mac newlines etc. In contrast, tokens are already language-*dependent*, in particular, specific for the grammar used to describe the language. There are, however, large commonalities across many languages. Many languages support, for instance, strings and integers, and consequently, it's plausible that the grammar will make use of corresponding tokens (perhaps called `INT` and `STRING`, the names are arbitrary, like variable names, but it is a good idea to call the token representing strings `STRING` or similar...). Tokens are not just language-specific wrt. the language being implemented. They show up in the implementation, i.e., they are specific to the meta-language used to implement the compiler.

We said the input of a scanner is the source code. That's a bit unspecific. It's often a "character stream" or a "string" (of characters). Practically, the argument of a scanner is often a *file name* or an *input stream* or similar. Or the scanner in its basic form takes a character stream, but it "alternatively" also accepts a file name as argument (or even an url). In that case, of course, the string of the file name is not scanned as source code, but it's used to access the corresponding file, whose content is then read in in the form of a string or whatever. As scanner works from *left to right* to its input. That is not a theoretical necessity, but that's how also humans consume or "scan" a "source-code" text. At least those humans trained in e.g. Western languages.

Other names for a scanner are lexical scanner or **lexer** for short, or tokenizer.

The scanner reads in the input character stream, and it **segments** it and **classifies** the individual pieces. So it chops up the character stream into small pieces (called lexemes), it cleassifies lexemes, resulting in *tokens*, and returns then one after the other, in the form of a token stream (see again Figure 2.1).

In the introductory chapter, we have seen some typical lexical categories and we will see further examples later. In general, typical language aspects covered by the scanner are

- **reserved words** or **key words**
- **comments**
- **white space**
- Further standard token classes:
  - format of **identifiers**, representing variables, methods, ...
  - format of different **numerical** representations

The scanner is a piece of software to perform the sketched task. It's a straightforward task. We will look at from the outside, namely how to *specify* the different lexical aspects of a language. For that, one uses **regular expressions** (see Section 2.2).

The lexical "rules" often also involve (explicit or implicit) *priorities*. For instance, given a lexeme that match the regular expression for *identifiers* or the one for a *keyword*, it will be classified as keyword. Another priority is typically, that the scanner chooses the *longest* possible scan that yields a valid token.

The **implementation** of a lexer is based on **finite-state automata**. We will encounter different closely related variation of the concept, in particular *deterministic* and *non-deterministic* ones, and see how regular expressions can be presented by a non-deterministic finite state automata (NFAs) (see Section 2.5) and how NFAs can be turned into deterministic finite-state automata (DFA, see Section 2.6). As a *rule of thumb:* Everything about the source code which is so simple that it can be captured by regular expressions resp. finite state automata belongs into the scanner.

lexer specification = regular expressions (+ priorities), a lexer implementation is based on finite-state automata

In this lecture, we will use a parser and lexer generating tool (a variant of lex and yacc), and the representation of the specification of the tokens is specific to the chosen tool.

Let's schematically look at how scanning roughly works, for instance scanning the input string `a[index] = 4 + 2`.

Figure 2.2 illustrates the status of the scanner after reading the first character, i.e., after reading `a`. A usual invariant in such pictures (by convention) is that the arrow or reading head points to the *first* character to be *read next* and thus *after* the last character having been scanned/read last.

Concerning the *head* in the pictures: it's for illustration, the scanner does not really have a "reading head". In the scanner *program* or procedure, the reading head is represented by a specific variable. The name of the variable depends on the scanner/scanner tool. And there is an analogous invariant: the variable contains or points to the next character to be read.

The picture of a reading head may be reminiscent of the typical picture illustrating Turing machines (which is not a coincidence). But a "reading head" is is not just a theoretical construct. In the old times, program data may have been stored and read from magnetic tape. Very deep down, if one still has a magnetic disk as opposed to an SSD, the secondary

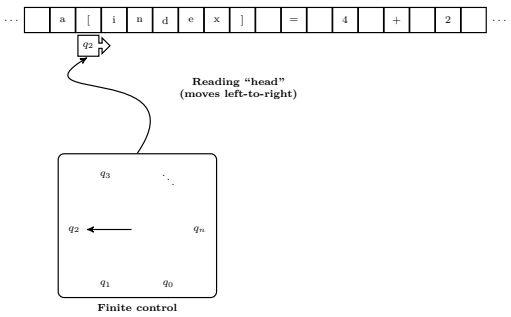


Figure 2.2: Scanner machine

storage still has "magnetic heads", only that the compiler typically does not scan or parse *directly* char by char from disk...

2.1.2 Lexical aspects & the bad(?) old times: Fortran

Standard responsibilities of the scanner is to take care of *white space* and of so-called *reserved words.* Basically, the scanner ignores the whitespaces in that it will hop over them and in particular will not turn them into "whitespace" tokens. To highlight the functionality , we have a look at Fortran and how there whitespace is dealt with, at least in older version. In a way is used to show how whitespace should *not* be treated ...

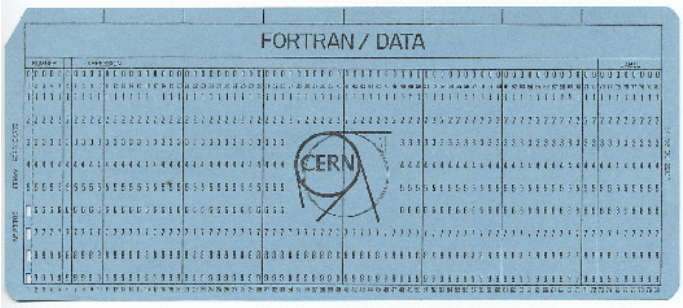


Figure 2.3: Fortran punch card

One should not forget that Fortran is a seriously ancient programming language, developed in the days of the pioneers. Main memory was seriously smaaaaaaaaaall, compiler technology was not well-developed (or not at all), and programming was for *very* few experts. There was no computer science as profession or university curriculum, so there was no textbook to consult how to do a compiler or a lexer. At that time Fortran was considered a high-level language ("wow, a language so high-level that you have to compile it ..."), indeed the first "widely" used high-level programming language.

The treatment of whitespace there was that it was completely without meaning. In the sense: it does not matter if there is a white space or not at every point in the source code. For example, the following to lines are equivalent:

`I F( X 2.  EQ. 0) TH E N` vs. `IF ( X2.  EQ.0 ) THEN`

The treatment of reserved words is another lexical aspect that one would nowadays do differently. Indeed, Fortran did *not* have *reserved* words! For instance, `IF` and `THEN` represents parts of a conditional statement, but the words are not reserved for that use. I.e., the following would be tolerated, where for instance `THEN` is used as variable.[1]

```
IF (IF.EQ.0) THEN THEN=1.0
```

Also the following are both lexically (and syntactically) correct

`DO99I=1,10` vs. `DO99I=1.10`

The comma corresponds to specifying the bounds of a loop, as in the following code

```
DO 99 I=1,10
 _
 _
99 CONTINUE
```

For the treatment of whitespace It's in a way like in the super-old days when there was no "white space" in writing (for instance ancient Latin) and it entered manuscripts (in Latin or emerging Western European languages) only slowly. It proved helpful in reading for humans, of course.

Over time, also programming languages adopted a more "helpful" treatment of lexical aspects. Remember that one core task of scanning is segmenting the input, and white space can help there. If one treats white space as "basically not there" and thus absolutely meaningless, one does human readability a big disfavor: humans are used to white space since the times of no-white-space texts are long gone. One reason why initially Fortran treated white space like that was perhaps: it may have been the easiest thing to do: if the scanner reads a white space, do nothing and proceed. Or perhaps the motivation was to allow "compact programs". It allowed the expert programmers to write programs without wasting precious memory for "white space" in the source code. Note that in the conventional interpretation of white space nowadays, white space does not *exactly* represent "nothing" in that one can put it in or out without changing the meaning. White space has no meaning by itself but *terminates* preceding non white space.

That treatment is so conventional, that most compilers use more or less the same definition of "white space" though there is typically not only one "white space" character. There is tabs, spaces, and then there is different "end-of-line" representations (carriage-return, end-of-line, newline). In a way, things like "carriage-return" `CR` and "tabulation command" is anyway a hold-over from the times of the mechanical and electrical type-writer era: at the beginning, the input and output peripheral devices connected to a computer were not just trying to behave like type-writer in software, there *were actually* sort of type-writers (and

[1] To figure out that the first `THEN` is part of a conditional and the second one a variable is not done by the scanner, that would be the task of the parser. But the scanner nowadays would classify the `THEN` as reserved word, and the parser would not allow assignments to reserved words, but to (non-reserved) identifiers only.

built in many cases by IBM at any rate...) and one needed some encoding of files to drive them or read input from them: the encoding for the "bell character" actually may result in banging on a small copper bell... Standard devices like `tty` are likewise remembrance of real hardware teletype (typewriter-style) terminals. Since those codes are part of the ASCII-code (from the 60ies), those "characters" or "symbols" are here to stay...

There are other "unconventional" ways to deal with white space. For instance, one could make the decision that indentation (via "tabbing" or otherwise) has a meaning, as opposed to be another example of whitespace. Python is an example of a language, where indentation is "meaningful" and the lexer (and parser) must be aware of that.

Proper and improper indentation is sometimes also layed down in style guides. It's more like a recommendation for programmers to follow for writing "pretty programs". Improper indentation would make no semantic difference on the compilation (as would be the case in Python), it's just frowned upon as bad taste. Perhaps the compiler would mutter some criticism or warning about the uglyness of the program. A lexer to support checking of some stylistic guidelines and proper formatting would have to distinguish between different things commonly just treated as whitespace. That's not surprising, as those tools focus on how nice the program looks (to the user). There are *formatting* tools (for instance `gofmt` for go) that transforms a program is a nicely written one that follow the stylistic guidelines.

A formatting tool is similar to a pretty printer. A pretty printing is a functionality on abstract syntax; it allows to inspect the parsed version of a program, printed out in a nice way. It's not necessary (but perhaps desirable) that the pretty-printed program formatted following stylistic guidelines (if there are any). It's not even necessary that the pretty-printed program is a program at all, the pretty-printer may add additional information, helpful for someone debugging the parser and designer of the abstract syntax tree, perhaps decorated already. Also the parser may have already do very small pieces of rearranging the code. For instance, the language may offer the user different looping constructs, like while-loops and repeat-until-loops. These are different constructs in concrete syntax and have thereby different concrete parse-trees. but the parser may output both kind of loop in the *abstract* syntax tree as one kind of loops, maybe a while-loop. That may be done for uniformity, having less cases to deal with in subsequent phases. There are limitations what can be achieved in such transformations by the lexer and the parser, so only quite simple simplifications can be done early one. In the example of the loops, one would say, repeat-until is treated as *syntactic sugar*.

Things like massaging the code slightly (or adding helpful extra information) is not done by formatting tools. At any rate: formatting tools are more for *users*, pretty printers are more for compiler writters, that want to look at internal representations (like ASTs).

The lecture will neither be concerned with the stone-age treatment of white-space as in old Fortran,[2] nor with the more elaborate ways discussed afterward. In the oblig, as simplistic "pretty printer" will be part of the task (though the pretty-ness of the output won't be a priority).

The treatment of white space is mostly a question of language *pragmatics*. Pragmatics deals with non-formal questions like "what's helpful for *humans* that program in a

[2] Also Fortran (of course) has evolved from the pioneer days ....

language", what's "user-friendly syntax". Lexers/parsers would have no problems using `while` as variable, but humans tend to. Pragmatics for instance also deals with questions like "how verbose should a language design be", how much syntactic sugar it should offer, etc. In COBOL, the designer's presumably thought it's user-friendly to write addition as `ADD y TO z GIVING x` (for `x:=y+z`), so that the programmar can avoid math notation and use "ordinary language". But not everyone prefers programming in that style.

In general, in such questions, there is mostly no commonly agreed best answer, and it depends also on what kind of user one targets and to some large part on the personal taste, education, and experience of the programmer.

Sometimes, the part of a lexer / parser which removes whitespace (and comments) is considered as separate and then called *screener*. It's not a very common terminology, though.

2.1.3 Classifying lexemes into tokens

Besides segmenting input character stream into pieces and ignoring comments and whitespace, the lexer is tasked with classifying the pieces. The terminology is not 100% uniform, but most would agree:

> **Lexemes** are the "chunks" (pieces) the scanner produces from segmenting the input source code (and typically dropping whitespace and comments). **Tokens** are the result of *classifying* those lexemes. And token (generally) is a pair consisting of the token *name* and a token token *value*.

What is a good classification depends also on later phases and it may not be clear till later, but as a rule of thumb: Things being treated equal in the syntactic analysis, i.e., by the *parser*, should be put into the same category.

On the other hand, it's not too complex either. Programming languages may be vastly different as far as their syntax, semantics, style of programming, application area(s) etc. are concerned. As far as their lexical aspects are concerned, they share a great deal of commonalities. Maybe not in details, but in principle. There will be a notion of whitespace, there will be comments, though how they are written may vary. Languages will probably have keywords and will have identifiers. Whether the language refers to them as keywords and identifiers or uses a different terminology it's not so important, but languages have variables, constants and other "things" that are referred to by name (classes, methods, functions, modules etc.).

Of course details depends on the particular language, like what exactly is allowed as identifier. Are digits, underscores, or dot's allowed as part of an identifier or not? If so, presumably an identifier is not allowed to start with those. Is it allowed to use two underscores in a row? ... The designer may make the decision to work with different classes of identifiers, like some consisting of only lower-case characters (plus digits perhaps) and with identifiers (starting with or consisting only of) capitals (plus digits perhaps). That could be done to use the small-letter identifiers for variables and functions, the capital-letter identifiers for other purposes (classes, modules).

While the details may be different, depending on the language, they all share that the classification can be done via regular expression. It's not a law of nature that, for instance, comments are done by the lexer resp. can be captured by regular expressions. Assume a language that supports comments, specifying the begin and the end of a comment. In Java, for instance one can use `/* comment */`. What is not allowed is to have *nested comments*, like

```
/* Comment text
     /* some inner comment text */
*/
```

Listing 2.1: Nested comments

There are language which allow nested comments, but the vast majority does not. If one really wanted nested comments, a consequence would be: the lexer can no longer take care of them. To deal with "nested" situations like that cannot be captured by regular expressions or finite state machines (at least if one allows arbitrary nesting instead of specifying : "comments can be nested but only up to level 32"). Dealing with nesting is very typical for the *parser*, like expressions consists of sub-expression etc, but the lexer is too weak to handle that. Nestin is part of the *syntactic* structure, not the *lexical* structure.

That virtually no language supports nesting of comments (and virtually all let the lexer deal with them) is a question of language *pragmatics*. It's just that there is not big need to have nested comments. As a consequence, of course, a good (beginner's) advice is: *don't comment out code* ...

Similar remarks apply also to identifiers and whitespace and other aspects: regular expressions are just fine to describe them, no one knows whay "nested whitespace" could be useful for.

Coming back to the classification, Table 2.1 shows one plausible one.

name/identifier	`abc123`
integer constant	`42`
real number constant	`3.14E3`
text constant, string literal	`"this is a text constant"`
arithmetic op's	`+ - * /`
boolean/logical op's	`and or not` (alternatively `/\ \/`)
relational symbols	`<= < >= > = == !=`
all other tokens: every one it its own group	`{ } ( ) [ ] , ; := .` etc.

Table 2.1: A possible classification

Note in the classification: there is an **overlap**: `"."` is here a token, but also part of real number constant. Also `"<"` is part of `"<="`. Finally, keywords like `if` and `while` are also identifiers.

That refers to some aspect of the lexical analysis, that was refered to earlier, namely that the scanner has to deal with **priorities**. With the classification from Table 2.1, if one has

some sequence <=, then, without further elaboration, the < part of the string could be seen as representing the relation "less" and the subsequent symbol as `equality`. Whether or not that makes sense is not for the scanner to decide, the scanner just chops up the string into pieces, and then the subsequent parser may complain that it's syntactically not allowed to have to relation symbols side by side (or not complain, depending on the grammar).

In that particular situation, language pragmatics would suggest that <= is *not* chopped-up but treated as one chunk, i.e., one *lexeme*, namely representing the relation less-or-equal.

The same principle also apply to other entries in the classification. For instance `abc123` is intended in most languages as *one* identifier, not as `abc` followed by the number `123`, or even more weiredly by three identifiers followed by three separate digits).

So, the "priority" is: prefer **longer** lexemes over shorter ones with white-space as possible terminator, unlike as in old Fortran, as discussed earlier.

Earlier we said that a token is a pair consisting of a token name and a token value. Not all tokens actually do have a value. Especially, the reserved "words" or keywords don't have a value. For instance `if` or `while` are represented by one token each, without a value attached, same for line called "all other tokens".

For the operations and relations, it's more a matter of taste and design. And also perhaps of the language used to implement the lexer (the meta-language). In Table 2.1, for instance, the operators `+`, `-`, `*`, and `/` are listed on the same line.

One could seem them as the same token class, perhaps with a token name like `AOP` (for arithmetic operator) and one of 4 different token *values*. Alternatively, one can also see them as four different tokens (without value), perhaps with token name like *PLUS*, `MINUS`, `TIMES`, and `DIV`.

It does not make any real difference, one way or the other it's 4 different tokens in this case. Ultimately the tokens are data items that need to be implemented in the meta-language, using data structures available in that language.

If one uses an object-oriented language, one can do the following: tokens themselves defined by classes (i.e., as instance of a class representing a specific token, and the token values as attribute or instance variable in When discussing the oblig, we will give hints on how to do it and how it's concretely represented in the version of lex/yacc we suggest to use.

Often a scanner does slightly *more* than just classification, it stores names in some *table* and store a corresponding index as attribute, it store text constants or literals in some *table*, and store the corresponding index as attribute. One may even calculate numeric constants and store their value as attribute.

2.1.4 How to define lexical analysis and implement a scanner?

Even for complex languages, the lexical analysis is in principle not hard to do. So a "manual" implementation is straightforwardly possible. Likewise the *specification* of the different token classes) may be given in "prosa". However: there are straightforward formalisms and notations available as well as efficient, rock-solid tools. This makes it

easier to specify the lexical aspects unambigously. It makes it also easier to communicate the lexical definitions to others and easier to change and maintain the scanner. That's often done hand in hand with the parser; see the paragraph about parser generators below.

Prosa specification

A precise prosa specification is not so easy to achieve as one might think. For ASCII source code or input, things are basically under control. But what if dealing with unicode? Checking "legality" of user input to avoid SQL injections or similar format string attacks can involve lexical analysis/scanning. If you "specify" in English: " *Backlash is a control character and forbidden as user input* ", which characters (besides `char 92` in ASCII) in Chinese Unicode represents actually other versions of backslash? Note: unclarities about "what's a backslash" have been used for security attacks. Remember that "the" backslash-character in OSs often has a special status, like it *cannot* be part of a file-name but used as separator between file names, denoting a *path* in the file system. If one can "smuggle in" an inofficial ("chinese") backslash into a file-name, one can potentially access parts of the file directory tree in some OS which are supposed to be inaccessible. Attacks like that have been used.

Anyway, Figure 2.4 is an excerpt from some earlier year's oblig, concerning the lexical conventions for *compila 20* Anyway, it's an example of a prosa specification, and the oblig involves writing a lexer for that. Since the lexer is supposed to be based on a lex-style tool, this means, one has to capture the prosa text in the regular language format of the chosen tool (for instance JLex). The lexical part does not really change over the years, or only in minor aspects, but better wait for the official 2022 specification.

2 Lexical aspects

2.1 Identifiers and literals

- `NAME` must start with a letter, followed by a (possibly empty) sequence of numeric characters, letters, and underscore characters; the underscore is not allowed to occur at the end. Capital and small letters are considered different.
- All *keywords* of the languages are written in with lower-case letters. Keyword *cannot* be used for standard identifiers.
- `INT_LITERAL` contains one or more numeric characters.
- `FLOAT_LITERAL` contains one or more numeric characters, followed by a decimal point sign, which is followed by one or more numeric characters.
- `STRING_LITERAL` consists of a string of characters, enclosed in quotation marks (`"`). The string is not allowed to contain line shift, new-line, carriage return, or similar. The semantic *value* of a `STRING_LITERAL` is only the string itself, the quotation marks are not part of the string value itself.

2.2 Comments

Compila supports *single line* and *multi-line* comments.

1. Single-line comments start with `//` and the comment extends until the end of that line (as in, for instance, Java, C++, and most modern C-dialects).
2. Multi-line comments start with `(*` and end with `*)`.

The latter form cannot be nested. The first one is allowed to be "nested" (in the sense that a commented out line can contain another `//` or the multi-line comment delimiters, which are then ignored).

Figure 2.4: Sample prosa specification

Parser generator

Those tools are all based on the same principles, they work roughly similar and can generate parsers for the same class of languages (language in the theoretical meaning of sets of words over an alphabet): the lexers cover some (extended) form of regular expressions and the parser does some form of bottom-up parsing, known as LARL(1) parsing. This form of parser/lexer generators inspired by lex/yacc is the bread-and-butter, standard version. The overview at Wikipedia over different such tools is pretty long.

The most famous pair of lexer+parser tools is called "compiler compiler" (lex/yacc = "yet another compiler compiler") since it generates (or "compiles") an important part of the front end of a compiler, the lexer+parser. lex/yacc originate from C, there are also gnu-versions around (called flex and bison). Those kinds of tools are seldomly called compiler compilers any longer. Many other languages ship with a corresponding pair of tools. In the lecture 2020, someone from the audience mentioned Alex & Happy (for Haskell) and one oblig groups used that, ocaml has ocamllex/ocamlyacc, similar for other ML versions and other languages. Java, for some reason, does not ship with such a pair of tools; they exist though. Notably there is JLex and CUP.

2.2 Regular languages and regular expressions

Regular expressions and regular languages are a very well-known concept and, with variations, used in different applications, inside compilers or outside, for instance as input notation to search engine interfaces. They are supported by many editors and text processing facilties specifying search patterns for pattern matching, and much more. They are also part of system tools and utilities, starting from classical ones like `awk` or `sed`), but also tools like `grep` or `find` (or general "globbing" in shells). For instance, the following lists all (La)TeX files, more precisely all files whose file name starts arbitrary and ends with `.tex`.

```
find . -name "*.tex"
```

Many programming languages, notably "scripting" languages offer extensive support for working with regular expressions and extended regular expressions. Besides that, they have been studied theoretically and there are also imporant generalizations (which are outside of the scope of the lecture).

In the lecture, we start focusing on the classic, vanilla core regular expressions. For usability and convenience, one often likes to offer extra syntax, that makes the use of regular expression more convenient. The lex-style tools do that, for example. Adding more constructs to a language for convenience without really extending the expressivity of a languages in this way is sometime called "syntactic sugar" (we have mentioned that concept earlier).

Other practical extensions of regular expressions may fall outside that classification: sometimes one really likes to add *expressivity*. Those extensions are sometimes called "extended

regular expression", and those may still keep central aspects and the feel of regular expressions, but being actually more expressive, fall outside the formalisms that captures *regular languages*. We will look at some abbreviations that fall into the "syntactic sugar" category, but won't venture into genuine extensions (which are technically not longer pure regular expressions, as said).

Practically, for the oblig, one has to cope with the concrete syntax and posibilities of the chosen lexer generator, for instance, JLex.

2.2.1 The role of regular expressions inside the scanner: the bigger picture

The construction of the parser itself is conceptually split in a sequence of steps. We cover each of them later, but preview the architecture here shortly.

As mentioned and as fleshed out later, **regular expressions** are used to describe or specify a language's *lexical* aspects. Regular expressions, though, is a *declarative* or non-executable way to decribe those lexical aspects. They don't represent directly a way that can be "run" in order to scan a sequence of characters.

For that, one uses **finite state automata.** It's a formalism or machine model that includes states and transitions, and that can directly be represented in any programming language. Thus, one needs a way to translate a regular expression into such an automaton. The translation first translates to **non-deterministic** finite automan (NFA) (see Section 2.5). Being non-determistic makes that model not really useful for scanning. Thus, in a second step, the NFA is translated to a DFA, a deterministic finite-state automaton (see Section 2.6 covering **determinization**). Those deterministic machines can straightforwardly be implementated. However, typically, in a last step, one tries to reduce the size of the DFA by a process called **minimization** (see Section 2.7).

All those steps are done automatically by a "lexer generator". They also help also in other user-friendly ways of specifying the lexer: defining *priorities*, assuring that the *longest* possible lexeme is tokenized

A lexer generator may even prepare useful *error messages* if scanning (not scanner generation) fails, i.e., when running the scanner on a lexically illegal program. Of course, if the scanner generation itself fails, also there meaingful errors messages and giving reasons for the failure are welcome. A final source of error could be: the scanner generation produces a scanner, which is supposed to be a Java or C or whatever kind of program, and that one is incorrect, maybe syntactically incorrect or ill-typed. Would that imply the lexer generator tool is broken? Not necessarily. The lex and yacc tools have a mechanism to *inject* Java (or or C or whatever) code into the generated output, for instance type and class definitions and import and package specifications, and the programmer may make errors there, resulting in an incorrect scanner code.

The classification in step 2 is actually *not* directly covered by the classical results that stating reg-expr = DFA = NFA, it's something extra. The classical constructions presented here are used to *recognise* (or reject) words. As a "side effect", in an actual implementation, the "class" of the word needs to be given back as well, i.e., the corresponding *token* needs to be constructed and handed over (step by step) to the next compiler phase, the parser.

2.2.2 Alphabets and languages

Languages, regular or otherwise, are sets of words and words a sequences of characters.

Definition 2.2.1 (Alpabet Σ)**.** Finite set of elements called "letters" or "symbols" or "characters".

Definition 2.2.2 (Words and languages over Σ)**.** Given alphabet Σ, a **word** over Σ is a finite sequence of letters from Σ. A **language** over alphabet Σ is a *set* of finite *words* over Σ.

Practical examples of alphabets include ASCII, Norwegian letters (capitals and non-capitals) etc.

In this lecture: we avoid terminology "symbols" for now, as later we deal with e.g. symbol tables, where symbols means something slighly different (at least: at a different level). Sometimes, the Σ is left "implicit" (as assumed to be understood from the context).

Remark 2.2.3 (Symbols in a symbol table (see later))**.** In a certain way, symbols in a symbol table can be seen similar to symbols in the way we are handled by automata or regular expressions now. They are simply "atomic" (not further dividable) members of what one calls an alphabet. On the other hand, in practical terms inside a compiler, the symbols here in the scanner chapter live on a different level compared to symbols encountered in later sections, for instance when discussing symbol tables. Typically here, they are *characters*, i.e., the alphabet is a so-called character set, like for instance, ASCII. The lexer, as stated, segments and classifies the sequence of characters and hands over the result of that process to the parser. The results is a sequence of *tokens*, which is what the parser has to deal with later. It's on that parser-level, that the pieces (notably the identifiers) can be treated as atomic pieces of some language, and what is known as the symbol table typically operates on symbols at that level, not at the level of individual characters.

2.2.3 Some general remarks on languages and how to describe them

A language over a alphabet is a set of words of the given alphabet (see Definition 2.2.2. Σ is typically finite, words are of *finite* length, but languages are in general infinite sets of. Finite languages are boring and trivial.

Example 2.2.4. Assume a two-lettered $\Sigma = \{a, b\}$. Let's write ϵ for the empty word, i.e. the empty sequence of letters, and let ab mean " first a then b " etc.

Remark 2.2.5 (Words and strings)**.** In terms of a real implementation: often, the letters are of type character (like type `char` or `char32` ...) *words* then are "sequences" (say arrays) of characters, which *may* or *may not* be identical to elements of type `string`, depending on the language for implementing the compiler. In a more conceptual part like here we do not write words in "string notation" (like "*ab*"), since we are dealing abstractly

$\{\}$ (also written as $\emptyset$)	the empty set
$\{a, b, ab\}$	finite language with 3 words
$\{\epsilon\}$ $(\neq \emptyset)$	language containing one word
$\{\epsilon, a, aa, aaa, \ldots\}$	infinite languages, all words using only a 's
$\{\epsilon, a, ab, aba, abab, \ldots\}$	alternating a's and b's
$\{ab, bbab, aaaaa, bbabbabab, aabb, \ldots\}$	??

Table 2.2: Some languages over Σ

with *sequences* of letters, which, as said, may not actually be strings in the implementation. Also in the more conceptual parts, it's often good enough when handling alphabets with 2 letters, only, like $\Sigma = \{a, b\}$. One-letter alphabets are uninteresting, let alone 0-letter alphabets. But 2 letters are often enough to illustrate some concepts. 3 letter alphabets may not add mach as far as "theoretical" questions are concerned. That may be compared with the fact that computers ultimately operate in words over two different "bits" . □

Remark 2.2.6 (Finite and infinite words)**.** There are important applications dealing with *infinite* words, as well, or also infinite alphabets. For traditional scanners, one mostly is happy with finite Σ 's and especially sees no use in scanning infinite words. Of course, some character sets, while not actually infinite, are large or extendable (like Unicode or UTF-8). □

Languages are simply sets of words and that begs the question, how to **describe** languages concretely. The last 3 languages from Table 2.2 are infinite languages and are given using "dot-dot-dot" (...). That's not a good way to describe a language to a computer (and to humans) what is meant. And enumerating explicitly all allowed words for an infinite language does not work either.

Needed: A **finite** way of describing infinite languages (which is hopefully efficiently implementable & easily readable).

Beware: Is it apriori to be expected that *all* infinite languages can even be captured in a finite manner?

Remark 2.2.7 (A metaphor: Rational and irrational numbes)**.** The remark here draws a parallel drawn from a field you may know, numbers. Also numbers need to be represented notationally, for instance in decimal notation. Let's take the reals. They can be represented by a finite sequence of decimals, followed by a dot, followed by an *infinite* sequence of digits. Consider the following two numbers:

$$2.727272727\ldots \qquad 3.1415926\ldots \tag{2.1}$$

The first number from equation (2.1) is a *rational number*. It corresponds to the fraction

$$\frac{30}{11}\ . \tag{2.2}$$

That fraction is actually an acceptable finite representation for the "endless" notation 2.72727272... using "..." As one may remember, it may pass as a decent definition of

rational numbers that they are exactly those which can be represented finitely as fractions of two integers, like the one from equation (2.2). We may also remember that it is characteristic for the "endless" notation as the one from equation (2.1), that for rational numbers, it's *periodic.* Some may have learnt the notation

$$2.\overline{72} \tag{2.3}$$

for *finitely representing* numbers with a *periodic* digit expansion (which are exactly the rationals). The second number, of course, is supposed to be π, one of the most famous numbers which do *not* belong to the rationals, but to the "rest" of the reals which are not rational and hence called irrational. Thus it's one example of a "number" which cannot be represented by a fraction, resp. in the periodic way as in equation (2.3).

Well, fractions may not work out for π (and other irrationals), but still, one may ask, whether π can otherwise be represented finitely. That, however, depends on what actually one accepts as a "finite representation". If one accepts a finite description that describes how to *construct* ever closer approximations to π, then there is a finite representation of π. That construction basically is very old (Archimedes), it corresponds to the limits one learns in analysis, and there are computer algorithms, that spit out digits of π as long as one wants (of course they can spit them out *all* only if one had infinite time). But the *code* of the algorithm that does that is finite.

The bottom line is: it's possible to describe infinite "constructions" in a finite manner, but *what exactly* can be captured depends on what precisely is allowed in the description formalism. If only fractions of integers are allowed, one can describe the rationals but not more.

A final word on the analogy to regular languages. The set of rationals (in, decimal notation) can be seen as language over the alphabet $\{0, 1, \ldots, 9\ .\}$, i.e., the decimals and the "decimal point". It's however, a language containing *infinite* words, such as $2.727272727\ldots$. The syntax $2.\overline{72}$ is a *finite* expression but denotes the mentioned infinite word (which is a decimal representation of a rational number). Thus, coming back to the regular languages resp. regular expressions, $2.\overline{72}$ is similar to the *Kleene*-star, but *not the same.* If we write $2.(72)^*$, we mean the *language* of *finite words*

$$\{2, 2.72, 2.727272, \ldots\}\ .$$

In the same way as one may conveniently *define* rational number (when represented in the alphabet of the decimals) as those which can be written using periodic expressions (using for instance $\overline{\text{overline}}$), *regular* languages over an alphabet are simply those sets of finite words that can be written by *regular expressions* (see later). Actually, there are deeper connections between regular languages and rational numbers, but it's not the topic of compiler constructions. Suffice to say that it's not a coincidence that *regular* languages are also called *rational* languages (but not in this course). □

2.2.4 Regular expressions

Without further ado, here the definition of regular expression. Later we will represent the same definition in a different fashion, using a (context-free) grammar.

Definition 2.2.8 (Regular expressions)**.** A *regular expression* is one of the following

1. a *basic* regular expression of the form **a** (with $a \in \Sigma$), or $\boldsymbol{\epsilon}$, or $\boldsymbol{\emptyset}$
2. an expression of the form $r \mid s$, where r and s are regular expressions.
3. an expression of the form $r\, s$, where r and s are regular expressions.
4. an expression of the form r^*, where r is a regular expression.

In other textbooks, also the notation + instead of | for "alternative" or "choice" is a known convention. The | seems more popular in texts concentrating on *grammars*. Later, we will encounter *context-free* grammars (which can be understood as a generalization of regular expressions) and the |-symbol is consistent with the notation of alternatives in the definition of rules or productions in such grammars. One motivation for using + elsewhere is that one might wish to express "parallel" composition of languages, and a conventional symbol for parallel is |. We will not encounter parallel composition of languages in this course. Also, regular expressions using lot of parentheses and | seems slightly less readable for humans than using +.

Regular expressions as language

A regular language is a language that can be described by regular expressions, so regular expressions is a **notation** for regular languages. However, regular expressions is a language *itself*. Without actually adding more to the concept of regular expression as given in Definition 2.2.8, in the following we discuss issues with regular expression as notation or language. This is partly also done in preview to later chapters about grammars and parsing. So the discussion here may be appreciated better after we have introduced grammatical issues like the one we touch upon here.

What issues are we talking about? That will be (context-free) grammars, associativity and precedence, and a bit about abstract vs. concrete syntax. All those concepts will covered in more depth in the later chapter, but let's use the regular expression notation for a warm up.

Regular expressions is a notation, i.e., a language consisting of symbols over some alphabet, that can be combined to "words". Some of the words are legal regular expressions and some not. For instance

$$\mid \mid \mid \mathbf{a}$$

is a word from the alphabet of regular expression, but it's not a regular expression, it does not adhere to the **syntax** of regular expressions.

Regular expressions as notation have a syntax (and a semantics). One could write a lexer (and parser) to parse a regular language. Obviously, tools like parser generators *do* have such a lexer/parser, because their input language are regular expressions (and context free grammars), besides syntax to describe further things. One can see regular languages as a domain-specific language for tools like (f)lex (and other purposes).

Context-free grammars cover the *syntax* of a language, not the lexical aspects. So let's describe syntax of regular expression by a grammar (without actually bothering to introduce the concept of context-free grammars explicitly, that's for later).

A "grammatical" definition The essence of Definition 2.2.8 could be captured by a "rule system" as follows

$$\begin{array}{lcl} r & \to & \mathbf{a} \\ r & \to & \boldsymbol{\epsilon} \\ r & \to & \boldsymbol{\emptyset} \\ r & \to & r \mid r \\ r & \to & r\, r \\ r & \to & r^* \end{array} \tag{2.4}$$

It's an example for a context-free grammar. We will see more than enough context-free grammars in the form of equation (2.4) resp. the more compact forms that follow. They will be central to parsing and their definition and format will be explained in detail at that point. Here, we use the context-free grammar notation (known as BNF) to describe one particular notation, namely the notation known as *regular expressions*.

To save space, one typically would not list each rule or production in a single line, but could compress the presentation a bit. Later, for CF grammars, we use mostly capital letters to denote "variables" of the grammars (then called *non-terminal symbols* or *non-terminals* for short). If we like to be consistent with that convention in the parsing chapters and use capitals for non-terminals, and writing it more compactly, the grammar for regular expression looks as follows:

Definition 2.2.9 (Regular expressions as CFG)**.** Assume an alphabet Σ. The syntax of *regular expression* is given by the following grammar

$$\begin{array}{lcll} R & \to & \mathbf{a} \mid \boldsymbol{\epsilon} \mid \boldsymbol{\emptyset} & \text{basic reg. expr.} \\ & \mid & R \mid R \mid R\, R \mid R^* & \text{compound reg. expr.} \end{array} \tag{2.5}$$

This is just a more "condensed" representation of the grammar we have seen before. In particular note the two "different" versions of the | symbol: one as syntactic element for regular expressions, one as symbol used in context-free grammars on the *meta-level*, used to *describe* the syntax of regular expressions. Though these levels are clearly separated, the intended meaning of the symbol is kind of the same, it represents "or".

Symbols, meta-symbols, meta-meta-symbols ... So, regular expression is a notation or language to describe (regular languages over a given alphabet Σ (i.e. subsets of Σ^*. Regular expressions have their own alphabet, which contains $|$, $*$ among other symbols. So the is a language being described and a language used to describe the language. So there's a gap between

language $\Leftrightarrow$ meta-language

Here, the language is regular expressions as a notation to describe regular languages, and as meta-language, we use English (for instance in Figure 2.4) resp. a notation for context-free grammars, a notation here used to describe regular expressions.

With two levels of languages around, that begs the question, how to distinguish between them. If we use a prosa specification, it's not a big deal, English is English and symbols are symbols. Howecer, regular expressions and the context-free grammar notation not so much different.[3] We hinted at that already pointing out that in the grammar from Definition 2.2.9, the | is used as part of the language as well as part of the meta-language.

There is (resp will be another) level. It's not only the syntax level of the language (here regular expression) but also what regular expression *mean*, the **semantics** of regular expression. We already know what regular expressions are supposed to mean, namely regular languages, and we should have a good intuitive feeling what concrete language a concrete regular expressions expressed; the semantics will be nailed down more explicitly in Section 2.2.5 afterwards.

For the regular expressions syntax, we use **boldface** font. $\boldsymbol{a}$. For the semantics, we use non-boldface, so the meaning of the notation $\boldsymbol{a}$ is the letter a. The regular expression syntax for the empty word ϵ is $\boldsymbol{\epsilon}$, the notation for the empty set $\emptyset$ is $\boldsymbol{\emptyset}$ etc.

That may sounds like a bit hairsplitting, like saying the notation **0** stands for the number 0. Splitting hairs like this is like a *déformation professionelle* of compiler writers. The distinction between language and meta-language as well as the distinction between the language and what it represents is sharply felt.

The distinction between language and meta-language in compiler *implementations*, is very real (even if not done by typographic means as in the script here or textbooks ...): the programming language being implemented need not be the programming language used to implement that language (the latter would be the "meta-language"). For example in the oblig: the language to implement is called "Compila", and the language used in the implementation will (for most) be Java. Both languages have concepts like "types", "expressions", "statements", which are often quite similar. For instance, both languages support an integer type at the user level. But one is an integer type in Compila, the other integers at the meta-level. They may be quite similar, but, looking at the fine-print they are different, like their relationship to or compatibility with other tymes, perhaps their inner representation etc. The distinction may look blurred if one writes a compiler for a language in the language itself, something we discussed as bootstrapping a compiler, but it's still there.

Also the distinction for syntax and what it represents is very real. That's the very essence of a compiler, translate a notation into something else that executes which is the "meaning" of the syntactic program (never mind that outcome of the compilation can be talked about in a different, lower level notation, for instance representing the instruction set of some chip). That distinction already applies to a notation like **0** and what it represents (above we just wrote 0). In practice, it will be some bit pattern of some length, and the compiler has to arrange for that representation.

Remark 2.2.10 (Regular expression syntax)**.** We are rather careful with notations and meta-notations, at least at the beginning. Note:

[3] Later we make more explicit in which way actually context-free language are strictly more expressive as regular languages and in which way a regular expression can be seen as a restricted form of context-free grammars.

Later, there will be a number of examples using regular expressions. There is a slight "ambiguity" about the way regular expressions are described (in this presentation, and elsewhere). It may remain unnoticed (so it's unclear if I should point it out here). On the other hand, the lecture is, among other things, about scanning and parsing of *syntax*, therefore it may be a good idea to reflect on the *syntax* of regular expressions itself.

In the examples shown later, we will use regular expressions using parentheses, like for instance in $\boldsymbol{b(ab)^*}$. One question is: are the parentheses **(** and **)** part of the definition of regular expressions or not? That depends a bit. In the presentations like the one here typically one often would not care, one tells the readers that parentheses will be used for disambiguation, and leaves it at that (in the same way one would not bother to tell the reader that it's fine to use "space" between different expressions (like $a \mid b$ is the same expression as $a \quad \mid \quad b$). Another way of saying that is that textbooks, intended for human readers, give the definition of regular expressions as *abstract syntax* as opposed to *concrete syntax*. Those two concepts will play a prominent role later in the grammar and parsing sections and will become clearer then. Anyway, it's thereby assumed that the reader can interpret parentheses as grouping mechanism, as is common elsewhere, as well, and they are left out from the definition not to clutter it. Note also that the non-grammar-based definition of regular expression from Definition 2.2.8 does not mention parentheses.

Of course, *computers* and programs (i.e., in particular scanners or lexers for instance those of tools dealing with regular expressions), are not as good as humans to be educated in "commonly understood" conventions, So one does not explicitly inform the reader that "paretheses can be added for disambiguation if wished".) *Abstract* syntax corresponds to describing the *output* of a parser (which are *abstract syntax trees*). In that view, regular expressions (as all notation represented by abstract syntax) denote *trees*. Since trees in texts are more difficult (and space-consuming) to write, one simply use the usual *linear notation* like the $\boldsymbol{b(ab)^*}$ from above, with parentheses and "conventions" like precedences, to disambiguate the expression. Note that a tree representation represents the grouping of sub-expressions in its structure, so for grouping purposes, parentheses are not needed in abstract syntax.

Of course, if one wants to implement a lexer or to use one of the available ones, one has to deal with the particular *concrete* syntax of the particular scanner. There, of course, characters like $'('$ and $')'$ (or tokens like `LPAREN` or `RPAREN`) will typically occur.

To sum up the discussion: Using concepts which will be discussed in more depth later, one may say: whether paretheses are considered as part of the syntax of regular expressions or not depends on the fact whether the definition is wished to be understood as describing *concrete* syntax trees or *abstract* syntax trees!

See also Remark 2.2.13 later, which discusses further "ambiguities" in this context. Later (when gotten used to it) we may take a more "relaxed" attitude towards it, assuming things are clear enough by then, as do many textbooks.

□

Precedences and associativity Parentheses can be used for grouping, as discussed, but they don't need to be added. Thus leads to the second mentioned issue that will show up later when discussing grammars and parsing. That's the question of precedences and

associativity. Those concepts are used to allow to *not* use parentheses and still make clear how to interpret a piece of syntax repre means, here regular expressions. For instance clarifying what $\boldsymbol{a} \mid \boldsymbol{bc}$ means, is it $(\boldsymbol{a} \mid \boldsymbol{b})\boldsymbol{c}$ or $\boldsymbol{a} \mid (\boldsymbol{bc})$. The latter one is the common interpretation, and that is tolds the reader by saying that concatenation has a higher priority or binding power than |. Another question is, what to make out of $\boldsymbol{a} \mid \boldsymbol{b} \mid \boldsymbol{c}$, $(\boldsymbol{a} \mid \boldsymbol{b}) \mid \boldsymbol{c}$ or $\boldsymbol{a} \mid (\boldsymbol{b} \mid \boldsymbol{c})$? In this case it does no actually matter in the sense that either way it represents same regular *language* $\{a, b, c\}$. If one wants the first interpretation, one would specify "| associates to the left" or "| is left-associative".

Precedence of the operators of regular expressions is, from high to low: $*$, concatenation, |. The |-operator is left-associative, concatenation is right-associative.

By "concatenation", the third point in the enumeration is meant. It is written or represented without explicit concatenation operator, just as juxtaposition, like **ab** is the concatenation of the characters **a** and **b**, and also for concatenating whole words: $w_1\ w_2$. See also the remarks in connection with Example 2.2.13.

2.2.5 Semantics (meaning) of regular expressions

Definition 2.2.11 (Regular expression)**.** Given an alphabet Σ. The meaning of a regexp r (written $\mathcal{L}(r)$) over Σ is given by equation (2.6).

$$
\begin{array}{rcll}
\mathcal{L}(\emptyset) & = & \{\} & \text{empty language} \\
\mathcal{L}(\boldsymbol{\epsilon}) & = & \{\epsilon\} & \text{empty word} \\
\mathcal{L}(\boldsymbol{a}) & = & \{a\} & \text{single "letter" from } \Sigma \\
\mathcal{L}(rs) & = & \{w_1 w_2 \mid w_1 \in \mathcal{L}(r), w_2 \in \mathcal{L}(s)\} & \text{concatenation} \\
\mathcal{L}(r \mid s) & = & \mathcal{L}(r) \cup \mathcal{L}(s) & \text{alternative} \\
\mathcal{L}(r^*) & = & \mathcal{L}(r)^* & \text{iteration}
\end{array}
\tag{2.6}
$$

Note: left of "=": regular expressions, i.e. *syntax*, right of "=" its semantics or meaning. [4]

The definition may seem a bit over the top. One could say, the meaning of the regular expression is clear enough when described in simple prose. That may actually be the case. But seeing it as obvious just means, regular expressions and the meaning of such an expression, which is the set of words it describes, is likewise straightforward. Nonethless, we make the "effort" to define the meaning. First of all, precision does not hurt, within a compiler lecture and outside. In other situations, the question of "what does it mean", i.e., the question of semantics, become more pressing. One can ask the same question about later other formalism, like the meaning of context-free grammars. Thirdly, in this simple situation, the description of the meaning of a language hopefully makes the different levels more clear: the syntactic level (symbols) and the semantic level resp. the meta-level

[4] Sometimes confusingly "the same" notation.

(math). Of course, "math" is a discipline which has its own symbols and notations. In this particular case of regular expressions, they are pretty close. And of course the description of the semantics using math assumes that the reader is familiar with those notations, so that a definition like $\mathcal{L}(r \mid s) = \mathcal{L}(r) \cup \mathcal{L}(s)$ is helpful or more compact than an English description. But of course, it just a way of saying "the regular expression symbol $\mid$ **means** set union". Indeed, another motivation is that this form of semantic definition is a form of translation, i.e., "compilation". In this case from one notational form (regular expression) to another one (mathematical notation, whose meaning is assumed to be clear). Semantics and translations from one level of abstraction to another one are also needed for *programming languages* themselves, though we don't go there in this lecture. For instance, in the oblig, the compila language has to be translated to a lower level. We could have specified the semantics of compila more formally, though the definition would be much more complicated (and probably use different techniques) than the semantics of regular languages. We could even be more ambitious: not only define the semantics of compila, but also define the semantics of the language it is compiled to. That would be some form of "byte-code" in our lecture. After having defined both levels of semantics, one could establish that both semantics do the same. That would be the question of *compiler correctness*. There are attempts of having a provably (!) correct compiler, though that is pretty complex, And even more complex than a verified compiler would be a *verifying compiler*. That problems is on the list of the so-called *grand challenges* in computer science.

2.2.6 Examples

In the following, we assume as alphabet $\Sigma = \{a, b, c\}$, and we aready start relaxing and no longer bother sometimes to "boldface" the syntax

words with exactly one b	$(a \mid c)^* b (a \mid c)^*$
words with max. one b	$((a \mid c)^*) \mid ((a \mid c)^* b (a \mid c)^*)$ $(a \mid c)^*\ (b \mid \epsilon)\ (a \mid c)^*$
words of the form $a^n b a^n$, i.e., equal number of a's before and after 1 b	

Example 2.2.12 (Words that do not contain two b's in a row). Let's assume a three-letter alphabet $\Sigma = \{a, b, c\}$. The following regular expression

$$(b\ (a \mid c))^* \tag{2.7}$$

describes a language which contains only words that do not contain two b's in a row, but it is too restricted, in that it does not contain *all* such words over the given alphabet. One thing that's wrong with equation (2.7) is that b occurs *alternatiningly*: the words start with b, then an non-b, then b again, etc (or a prefix thereof). Furthermore, a word can never end with a b.

The following does not do strict alternation

$$((a \mid c)^* \mid (b\ (a \mid c))^*)^* \tag{2.8}$$

but the way it's formulated it allows to contain two or more b's in a row. Equation (2.9) shows a few equivalent formulations capturing the intended property.

$$\begin{array}{rl} ((a \mid c) \mid (b\ (a \mid c)))^* & \\ (a \mid c \mid ba \mid bc)^* & \\ (a \mid c \mid ba \mid bc)^*\ (b \mid \boldsymbol{\epsilon}) & \\ (notb \mid b\ notb)^*(b \mid \boldsymbol{\epsilon}) & \text{where } notb \triangleq a \mid c \end{array} \tag{2.9}$$

□

Remark 2.2.13 (Regular expressions, disambiguation, and associativity)**.** Note that in the equations in the example, we silently allowed ourselves some "sloppyness" (at least for the nitpicking mind). The slight ambiguity depends on how we *exactly* interpret definitions of regular expressions. Remember also Remark 2.2.10 on page 18, discussing the (non-)status of parentheses in regular expressions. If we think of Definition 2.2.8 on page 16 as describing abstract syntax and a concrete regular expression as representing an abstract syntax tree, then the constructor | for alternatives is a *binary* constructor. Thus, the regular expression

$$a \mid c \mid ba \mid bc \tag{2.10}$$

which occurs in the previous example is *ambiguous*. What is meant would be one of the following

$$a \mid (c \mid (ba \mid bc)) \tag{2.11}$$

$$(a \mid c) \mid (ba \mid bc) \tag{2.12}$$

$$((a \mid c) \mid ba) \mid bc\ , \tag{2.13}$$

corresponding to 3 different trees, where occurences of | are inner nodes with two children each, i.e., sub-trees representing subexpressions. In textbooks, one generally does not want to be bothered by writing all the parentheses. There are typically two ways to disambiguate the situation. One is to state (in the text) that the operator, in this case |, *associates to the left* (alternatively it *associates to the right*). That would mean that the "sloppy" expression without parentheses is meant to represent either (2.11) or (2.13), but not (2.12). If one really wants (2.12), one needs to indicate that using parentheses. Another way of finding an excuse for the sloppyness is to realize that it (in the context of regular expressions) *does not matter*, which of the three trees (2.11) – (2.13) is actually meant. This is specific for the setting here, where the symbol | is *semantically* represented by *set union* $\cup$ (cf. Definition 2.2.11 on page 20) which *is* an associative operation on sets. Note that, in principle, one may choose the first option —disambiguation via fixing an associativity— also in situations, where the operator is *not* semantically associative. As illustration, use the '−' symbol with the usal intended meaning of "subtraction" or "one number minus another". Obviously, the expression

$$5 - 3 - 1 \tag{2.14}$$

now can be interpreted in two semantically different ways, one representing the result 1, and the other 3. As said, one *could* introduce the convention (for instance) that the binary minus-operator associates to the left. In this case, (2.14) represents $(5 - 3) - 1$.

Whether or not in such a situation one wants symbols to be associative or not is a judgement call (a matter of language pragmatics). On the one hand, disambiguating may make expressions more readable by allowing to omit parentheses or other syntactic markers which may make the expression or program look cumbersome. On the other hand, the "light-weight" and "easy-on-the-eye" syntax may trick the unsuspecting programmer into misconceptions about what the program means, if unaware of the rules of associativity and priorities. Disambiguation via associativity rules and priorities is therefore a double-edged sword and should be used carefully. A situation where most would agree associativity is useful and completely unproblematic is the one illustrated for | in regular expression: it does not matter anyhow semantically. Decisions concerning when to use ambiguous syntax plus rules how to disambiguate them (or forbid them, or warn the user) occur in many situations in the scannning and parsing phases of a compiler.

Now, the discussion concerning the "ambiguity" of the expression $(a \mid c \mid ba \mid bc)$ from equation (2.10) concentrated on the |-construct. A similar discussion could obviously be made concerning concatenation (which actually here is not represented by a readable concatenation operator, but just by juxtaposition (= writing expressions side by side)). In the concrete example from (2.10), no ambiguity wrt. concatenation actually occurs, since expressions like ba are not ambiguous, but for longer sequences of concatenation like abc, the question of whether it means $a(bc)$ or $a(bc)$ arises (and again, it's not critical, since concatenation is semantically associative).

Note also that one might think that the expression suffering from an ambiguity concerning combinations of operators, for instance, combinations of | and concatenation. For instance, one may wonder if $\boldsymbol{ba \mid bc}$ could be interpreted as $(ba) \mid (bc)$ and $b(a \mid (bc))$ and $b(a \mid b)c$. However, in Definition 2.2.11 on page we stated *precedences* or priorities, stating that concatenation has a *higher* precedence over |, meaning that the correct interpretation is $(ba) \mid (bc)$. In a text-book the interpretation is "suggested" to the reader by the typesetting $ba \mid bc$ (and the notation it would be slightly less "helpful" if one would write $ba|bc\ldots$ and what about the programmer's version `a␣b|a␣c`?). The situation with precedence is one where difference precedences lead to semantically different interpretations. Even if there's a danger therefore that programmers/readers mis-interpret the real meaning (being unaware of precedences or mixing them up in their head), using precedences in the case of regular expressions certainly is helpful, The alternative of being forced to write, for instance

$$((a(b(cd))) \mid (b(a(ad)))) \quad \text{for} \quad abcd \mid baad$$

is unappealing even to hard-core Lisp-programmers (but who knows ...).

A final note: all this discussion about the status of parentheses or left or right assocativity in the interpretation of (for instance mathematical) notation is mostly is over-the-top for most mathematics or other fields where some kind of formal notations or languages are used. There, notation is introduced, perhaps accompanied by sentences like "parentheses or similar will be used when helpful" or "we will allow ourselves to omit parentheses if no confusion may arise", which means, the educated reader is expected to figure it out. Typically, thus, one glosses over too detailed syntactic conventions to proceed to the more interesting and challenging aspects of the subject matter. In such fields one is furthermore sometimes so used to notational traditions ("multiplication binds stronger than addition"), perhaps established since decades or even centuries, that one does not even think about

them consciously. For scanner and parser designers, the situation is different; they are requested to come up with the notational (lexical and syntactical) conventions of perhaps a *new* language, specify them precisely and implement them efficiently. Not only that: at the same time, one aims at a good balance between expliciteness ("Let's just force the programmer to write all the parentheses and grouping explicitly, then he will get less misconceptions of what the program means (and the lexer/parser will be easy to write for me...)") and economy in syntax, leaving many conventions, priorities, etc. implicit without confusing the target programmer. □

2.2.7 Additional "user-friendly" notations

$$\begin{aligned} r^+ &= rr^* \\ r? &= r \mid \epsilon \end{aligned}$$

Special notations for *sets* of letters:

$$\begin{aligned} [0-9] &\quad \text{range (for ordered alphabets)} \\ \sim a &\quad \text{not } a \text{ (everything except } a) \\ . &\quad \text{all of } \Sigma \end{aligned}$$

naming regular expressions ("regular definitions")

$$\begin{aligned} \mathit{digit} &= [0-9] \\ \mathit{nat} &= \mathit{digit}^+ \\ \mathit{signedNat} &= (+|-)\mathit{nat} \\ \mathit{number} &= \mathit{signedNat}(\text{"."}\mathit{nat})?(\texttt{E}\ \mathit{signedNat})? \end{aligned}$$

The additional syntactic constructs may come in handy when using regular expressions, but they don't extend the expressiveness of the formalism. That's pretty obvious by the way the extensions are defined. Note that we don't explain the meaning or semantics of the new constructs in the same way as for the core constructs (defining $\mathcal{L}$ and giving their mathematical interpretation). Instead, we expand the new constructs and express them in terms of the old syntax. They are treated as *syntactic sugar*, as one says.

Tools, utilities, and libraries working with regular expression (like `lex`) typically support sugared versions, though the exact choice of notation for the construct may vary.

As mentioned, there are also so called *extended* regular expressions, where the extensions make the formalism more expressive than the core formalism. Consequently, those extensions are not syntactic sugar then.

One could look at the collection of constructors for the syntax of regular language, including the sugar, and wonder whether there aren't some missing. For example, we have in the language a form of "or" (disjunction), written |, one could ask, why not an "and" (conjuction, intersection), for instance. That's indeed interesting, insofar it is an example which is not syntactic sugar on the one hand, but on the other hand does not extend the expressiveness for real. If one had regular expressions containing an "and", then one can alway find a different regular expression with the same meaning, without the "and". However, the transformation would not be of the same nature than for the syntactic sugar

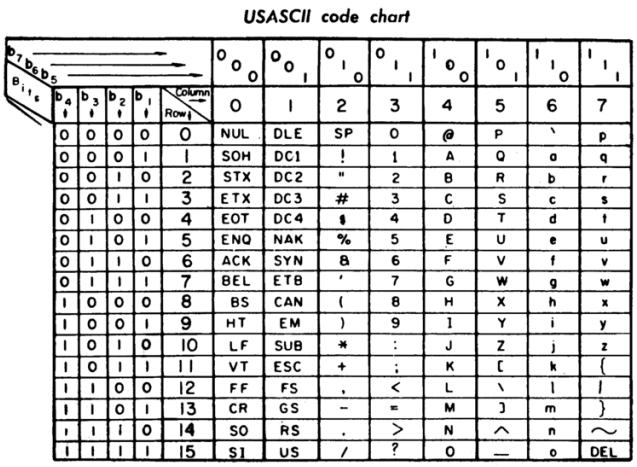
we added: the conjunction cannot just be expanded away; consequently one would not call that addition syntactic sugar. There exists other constructs that are non-sugar but do not add expressiveness (negation or complementation for example).

Mostly, such constructs like intersection or complementation are *not* part of the regular expression syntax, though theoretically, one would not leave the class of regular languages (= languages that can be expressed by regular expressions). Why are those then left out? It's probably a matter of pragmantics. One does not really need them for many things one want to do with regular expressions, like describing lexical aspects of a language for a lexer (for other applications that may be different). One wants to classify strings, and one is content by saying "It's whitespace (which is is this or this or this), *or* it's a number, *or* it's an identifier, *or* a bracket ...". Given also the fact that adding conjuction or negation or other non-sugar ingredients would make the some following constructions more complex, there is no real motivation to support conjunction. By the "following constructions" I mean basically the translation of a regular expressions into a (non-deterministic) finite-state automaton. This translation, called *Thompson's construction*, will be covered later in this chapter. A tool like `lex` does this construction (followed by other steps). The construction is fairly simple, but adding conjuction and complementation would drive up the size of the resulting automata. For intersection, for instance, one would needed a form of *product construction*, which is also conceptually more complex than the straightforward, compositional algorithm underlying Thompon's construction. Actually, it would not be so bad, since if one avoid using conjuction or negation, the size of the result would not blow up, so the reason, why regular expressions don't typically support those more complex operators is that pragmatically, no one misses them for the task at hand, at least not for lexers.

Ordered alphabet

We have defined an alphabet as a (finite) *set* of symbols. In practice, alphabets or character sets are not just sets, which are unordered, but are seen as ordered. Each symbol of the alphabet has a "number" associated to it (a binary pattern) which corresponds to its place in order in the sequence of symbols. One of the simplest and earliest established ordered alphabets in the context of electronic computers is the well-known ascii alphabet. See Figure 2.5.

Having the alphabet ordered is one thing, having a "good" order or arrangement is a different one. The reference card shows some welcome properties, for instance, that all lower-case letters are contiguous and in the "expected" order, same for the capital case letters. Since the designers of ascii arranged it in that way, one can support specifying all lower-case letters as `[a-z]` and capital case letters as `[A-Z]`. What does *not* work is having all letters as `[a-Z]`, since, in ascii, the letters are not arranged like that. The capital letters come before the lower case letters, but also `[A-z]` would not work as intended, as there is a "gap" of other symbols between the lower-case and the upper-case letters. Isn't that stupid? Actually not, the arrangement as made clear in the figure, is such that the operation of turning a lower-case letter to a upper-case latter a matter of flipping bits. Another rational decision is to place the decimal numbers "align" partly with their binary representions. It's not that 0, 1, etc. are exactly the corresponding bit patterns, but at least parts of the word correspond to the binary pattern. Anyway,



Bits b_4	b_3	b_2	b_1	Row ↓ / Column → 0	0	1	2	3	4	5	6	7
0	0	0	0	0	NUL	DLE	SP	0	@	P	`	p
0	0	0	1	1	SOH	DC1	!	1	A	Q	a	q
0	0	1	0	2	STX	DC2	"	2	B	R	b	r
0	0	1	1	3	ETX	DC3	#	3	C	S	c	s
0	1	0	0	4	EOT	DC4	$	4	D	T	d	t
0	1	0	1	5	ENQ	NAK	%	5	E	U	e	u
0	1	1	0	6	ACK	SYN	&	6	F	V	f	v
0	1	1	1	7	BEL	ETB	'	7	G	W	g	w
1	0	0	0	8	BS	CAN	(	8	H	X	h	x
1	0	0	1	9	HT	EM	)	9	I	Y	i	y
1	0	1	0	10	LF	SUB	*	:	J	Z	j	z
1	0	1	1	11	VT	ESC	+	;	K	[	k	{
1	1	0	0	12	FF	FS	,	<	L	\	l	|
1	1	0	1	13	CR	GS	-	=	M	]	m	}
1	1	1	0	14	SO	RS	.	>	N	^	n	~
1	1	1	1	15	SI	US	/	?	O	_	o	DEL

Figure 2.5: ASCII reference card

details like that don't matter too much for us, but one has to be aware of the concept of ordered alphabets as such, in order to specify, for example, all letters as `[a-zA-Z]` (or `[A-Za-z]`). Many encodings are nowadays extensions or variations of ascii, and also for those, specifications like `[a-z]` work. For instance, UTF-8. As a side remark: Ken Thompson (the one from Thompon's construction) was involved in working out UTF-8, an encoding that includes ASCII insfar that it's identical with ASCII in its first part. Of course, there are very many variations of UTF (and Unicode symbol set, of which UTF is a encoding scheme),

There are however, also *alternatives* to ascii, not just extension. One is *Extended Binary Coded Decimal Interchange Code (EBCDIC)* (actually also for EBCDIC, there are many variation). EBCDIC is perhaps mostly of historic interests as it is supported mainly by IBM mainframes and larger such computers. I mention EBCDIC here, because the encoding has the unfortunate property, that, for instance, capital letters are *not* contiguous (what where the thinking...). For such an encoding, of course, things like `[A-Z]`, make no sense. The encoding would have other negative consequences, for instance, sorting a list of words is more tricky (or at least less efficient). However, EBCDIC still lives on, there exist Unicode encodings based on that (as opposed to based in extensions of ascii like UTF-8), which concequently are called UTF-EBCDIC. Once, some things are standardized, they never die out completely (and actually EBCDIC just inherits properties of punchcards, which existed before the modern electronic computer, to the new area. In that context, it's not a coincidence that IBM which was a big name in "punchcard processing equipment" and stuck to aspects of the encoding when it became a big name in electronic computers and mainframes.

2.3 Finite state automata (DFAs and NFAs)

In this section and the following we introduce the very central notion of finite state automata and cover their close relation to regular expression. Finite state automata are

well-studied and play an important role also beyond their use for lexing. There are many different variations of finite state automata, also under different names. Such automata in their classic form are pretty simple objects, basically some graphs with labelled edges, and some nodes are singled out as start or initial nodes and some as final or accepting nodes. What makes such "graphs" automata or machines is their operational interpretation, they are seen as mechanisms that "run" or do steps. The nodes of the "graph" are seen as *states* the machine can be in. The edges are *transition*. It's assumed that "executions" of the machine starts in one of the initial states, and when in one final state, the execution ends, more precisely, *can end*. The mental picture of some entity, being in some discrete state, starting somewhere, doing steps or transitions one after the other is of course super-general and very unspecific. Basically all mechanized computing can be thought of operationally that way, going from one state to a next one and so on.

As the name indicates, specific here is that the number of states are *finite*. That's a strong restriction. Finite state automata are an important model of computation. It is also a model for hardware circuits, more specifically discrete, "boolean" circuits not analog hardware. It's clear that a binary circuit can be only in a finite number of states, and finite state machine are a good model for describing such hardware. The automata in that case are a bit more elaborate than the ones we use here, in particular, one would use automata that don't have a unstructured alphabet, but one would conceptually distinguish between input and output (though possible on the same alphabet). There are different ways one can do that. Typically, the edges carry the output, whereas on can connect the output to the states, or alternatively to the edges, as well. Those two styles of finite-state input/output automata are called Moore-machines (= output on the state) resp. Mealy-machines (= output on the transitions). The two different models would also require different styles of hardware realization, but those things are not important for us.

For lexing, we are handling automata with an unstructured alphabet, without distinguishing input from output. Such single-alphabet automaton can be "mentally seen" that the edges *generate* the letters (i.e., the letters are the output). With this view, a given automaton *generates* a language, i.e., the set of all sequences of letters that lead from an initial to an accepting state. Alternatively, one can see the letters on the edges as input; in this view, such machines are seen as *recognizers* or *acceptors*. The final states of an automaton are also called *accepting* states. Anyway, that view of acceptors is also the appropriate one for lexing or scanning. The letters of the alphabet are the characters from the input and the machine moves along and accepts a word (a lexeme of the language being scanned), and the accepting state corresponds to the token classes (for instance, an indentifier, or a number etc.).

Coming back to the issue of finite state I/O automata we brushed. Actually, the lexer in the context of a compiler *can* be seen as involving input *and* output. The characters are the input, and the token (token class and token value) are outputs and parsing a file means making iterated use of that arrangement, handing over a token stream to the parser.

We (as basically all compiler books) focus on the classic theory of finite state automata, ignoring as far as the theory is concerned, the token-output part. This is also the part, which connects with the regular expression from before. Regular expressions specify the lexical aspects of the language, and finite state automata are the execution mechanism to *accept* the corresponding lexemes. Of course, concretely, tools like `lex` need to arrange

also for the token-output part, but if one has the input-part under control, there is not much to understand there.

One important aspect is the question determinism vs non-determinism. Determinism in computational situations mean: there is (at most) one next state (or one reaction, on possible result ...). Non-determinism means, there is potentially more than one, the future is not determined. For finite state automata, it's more precisely as follows: Given a state and given an (input) symbol, say a, there is (at most) one successor reachable via an a-transition. One can also say: there is at most one a-successor. In other words, the current state *and* the input *determines* the next state (if any).

That's highly-desirable in a lexer: the lexer scans one letter after the other, and its not supposed to make guesses how to proceed. Doing so would lead do the danger of backtracking: in case the guess turns out to rejecting the input later down the line the lexer has to try to explore alternatives to find out if any of this could lead to accepting the input nonetheless. That's a horrible way to scan the input.

The good news is: one can avoid that. Intuitively the way to do it is to replace an non-determistic automaton by a different, but equivalent one, that conceptually explores all alternatives "at the same time". The determinization algorithm is known as *powerset construction* and is pretty straightforward and pretty natural.

Side remark 2.3.1 (Determinisation of automata-like formalisms)**.** Determinization of FSAs will be covered in Section 2.6. Here, already, as some side remarks, As we will see, the construction is, as said, straightforward and natural However, strangely perhaps, it works not universally. For instance, there are other automata-based formalisms that look quite similar. One such is finite-state automata that doen't work in finite words (as we do) but infinite words. Or finite-state automata that work in trees (either working top-down or bottom-up). We will not encounter those.

What we will encounter, though, is a particular form of "infinite state automaton" known as *push-down automaton*. Those, by having an infinite amount of memory, they are more expressive than finite state automata. They are central for *parsing* (not lexing) of context-free languages. The amount of memory for push-down automata is infinite, but not "random access", i.e. one can only access the top of a stack (by pushing and popping content), and this restriction fits with context-free languages (in the same way that the finite-state restriction fits with regular languages).

Anyway, for all those automata-like constructions, there are deterministic and non-deterministic variants in that the respective input determines their reaction or not. However, non-deterministic versions for those are strictly more expressive than deterministic ones (with exception of bottom-up tree automata where determinism vs. non-determinism does not matter, and the powerset construction would not work for those. Perhaps also interesting: for Turing machines, which can be seen as machines with finite control and infinite amount of random access memory (not just a stack), again determinism is not a restriction.

All that is meant just as a cautening not to assume that the powerset construction can be transported "obviously" to other settings... □

Now to the basic definition of *finite state automata*.. Variations of FSA's exist in many flavors and under different names, other well known names nclude finite-state machines, finite labelled transition systems. Generally "state-and-transition" representations of programs or behaviors (finite state or else) are wide-spread as well, for instance as state diagrams, Kripke-structures, I/O automata, Moore & Mealy machines .... As mentioned earlier, the logical behavior of certain classes of electronic circuitry with internal memory ("flip-flops") is described by finite-state automata and indeed, historically, the design of electronic circuitry (not yet chip-based, though) was one of the early very important applications of finite-state machines.

Definition 2.3.2 (FSA)**.** A FSA $\mathcal{A}$ over an alphabet Σ is a tuple $(\Sigma, Q, I, F, \delta)$
- Q: finite set of states
- $I \subseteq Q$, $F \subseteq Q$: initial and final states.
- $\delta \subseteq Q \times \Sigma \times Q$ transition relation

The final states are also called **accepting** states. The transition relation can *equivalently* be seen as function

$$\delta : Q \times \Sigma \to 2^Q$$

For each state and for each letter, give back the set of sucessor states (which may be empty). We often use a more suggestive notation, for instance

$$q_1 \xrightarrow{a} q_2 \quad \text{for} \quad (q_1, a, q_2) \in \delta$$

We also use freely —self-evident, we hope— things like

$$q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \ .$$

The definition given is fairly standard and whether one see δ as relation of function is, of course, equivalent. One often uses graphical representations to illustrate such automata; we will encounter numerous examples.

Figure 2.6 shows a graphical representation of an FSA, using convention we will use throughout. The *initial* states, in this case only one, are marked by an incoming arrow. The accepting states, in this case only one, as well, are marked by with a double ring. The automaton is deterministic, though not complete. We could interpret the automaton as complete DFA, with one extra non-accepting state.

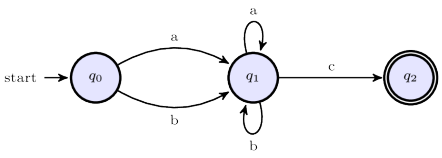


Figure 2.6: Graphical representation of an FSA

The intended **meaning** of an FSA over an alphabet Σ is the set of all the finite words, the automaton **accepts**.

Definition 2.3.3 (Accepted words and language of an automaton)**.** A word $c_1 c_2 \dots c_n$ with $c_i \in \Sigma$ is *accepted* by automaton $\mathcal{A}$ over Σ, if there exists states $q_0, q_2, \dots, q_n$ from Q such that

$$q_0 \xrightarrow{c_1} q_1 \xrightarrow{c_2} q_2 \xrightarrow{c_3} \quad \dots \quad q_{n-1} \xrightarrow{c_n} q_n \ ,$$

and were $q_0 \in I$ and $q_n \in F$. The *language* of an FSA $\mathcal{A}$, written $\mathcal{L}(\mathcal{A})$, is the set of all words that $\mathcal{A}$ accepts.

Remark 2.3.4 (Finite states)**.** The distinguishing feature of FSA (as opposed to more powerful automata models such as push-down automata, or Turing-machines), is that they have " *finitely many states* ". That sounds clear enough at first sight. But one has too be a bit more careful. First of all, the set of states of the automaton, here called Q, is finite and fixed for a given automaton, all right. But actually, the same is true for pushdown automata and Turing machines! The trick is: if we look at the illustration of the finite-state automaton earlier, where the automaton had a *head*. The picture corresponds to an *accepting* use of an automaton, namely one that is fed by letters on the tape, moving internally from one state to another, as controlled by the different letters (and the automaton's internal "logic", i.e., transitions). Compared to the full power of Turing machines, there are *two* restrictions, things that a finite state automaton cannot do

- it moves on one direction only (left-to-right)
- it is *read-only.*

All non-finite state machines have some *additional* memory they can use (besides $q_0, \dots, q_n \in Q$). Push-down automata for example have additionally a stack, a Turing machine is allowed to *write* freely (= moving not only to the right, but back to the left as well) on the tape, thus using it as external memory. □

2.3.1 FSA as scanning machine? (Determinism vs. non-determinism)

General FSA have slightly unpleasant properties when considering them as decribing an actual program (i.e., a scanner procedure/lexer), given the "theoretical definition" of acceptance:

Mental picture of a scanning automaton: Starting in an *initial* state, the automaton eats one character after the other, and, when reading a letter, it *moves* to **a** successor state, if any, of the current state, depending on the character at hand. Once hitting an accepting state, the automaton *accepts* the processed word.

There are 2 problematic aspects in that.

- **non-determinism**: what if there is more than one possible successor state?
- **undefinedness**: what happens if there's no next state for a given input?

The 2nd one is *easily* repaired, the 1st one requires more thought. In [7], the concept of **recogniser** corresponds to a DFA.

Let's discuss the first point, **non-determinism**, a bit. We touched upon the issue in the introduction of the chapter already: non-determinism is "problematic". One could try **backtracking**, but, you definitel don't want that in a scanner. And even if you think it's worth a shot: how do you scan a program directly from magnetic tape, as done in the bad old days? Magnetic tapes can be rewound, of course, but winding them back and forth all the time destroys hardware quickly. How should one scan network traffic, packets etc. on the fly? The network definitely cannot be rewound. Of course, buffering the traffic would be an option and doing then backtracking using the buffered traffic, but maybe the packet-scanning-and-filtering should be done in hardware/firmware, to keep up with today's enormous traffic bandwith. Hardware-only solutions have no dynamic memory, and therefore actually *are* ultimately finite-state machine with no extra memory. As hinted at in the introducton: there is a way to turn a non-deterministic finite-state automaton into a deterministic version.

We start by first defining the concept of determinism, resp. what constitutes a deterministic automaton

Definition 2.3.5 (DFA)**.** A *deterministic, finite automaton* $\mathcal{A}$ (DFA for short) over an alphabet Σ is a tuple $(\Sigma, Q, I, F, \delta)$

- Q: finite set of states
- $I = \{i\} \subseteq Q$, $F \subseteq Q$: initial and final states.
- $\delta : Q \times \Sigma \to Q$ transition function

The transition function is a special case of the transition relation: it is deterministic, but also left-total ("complete"). For a relation, being *left-total* means, for each pair q, a from $Q \times \Sigma$, $\delta(q, a)$ is defined. When talking about functions (not relations), it simply means, the function is *total*, not partial.

Some people call an automaton where δ is not a left-total but a deterministic relation (or, equivalently, the function δ is not total, but partial) still a deterministic automaton. In that terminology, the DFA as defined here would be deterministic *and* total.

Remark 2.3.6 (Transition function and totality)**.** Depending on which text one consults, the definition of DFA slightly disagrees. It's not a fundamental disagreement, it's more a question of terminology. It concern if being a deterministic automaton includes "totality" of the transition relation/transition function or not. Or in other words: for each state and each letter a, is there *exactly one* a-successor or *at most one*.

One could make the argument, determinism means the latter: at each state, and for each input, the reaction is fixed: one either moves to one particular successor state, or else is "stuck." That corresponds to a definition where δ is a *partial function*, unlike the definition given, where δ is a *total function*. So, our definition of DFA means, the automaton is deterministic and total. Some would say a deterministic finite state automaton need not be total (being a separate aspect the automaton enjoys or not).

Actually, it's a terminology question and does not matter much, basically it says: A DFA is a determistic *and* total finite state automaton (but we won't bother to call it DTFA

or something). The reason why it does not matter much is that really there is no much difference anyway. A automaton with a partial transition function can always be completed into a total one by adding an extra non-accepting state, covering the situations when the partial automaton would otherwise be "stuck". That's so obvious, that one need bother talk about it much. Also later, when showing graphical representation of automata: when talking about DFA (and when we want to really stress that they are total), we still might leave out to show the extra state in the figure, it's just assumed that one understands that it's there.

As far as implementations of automata is concerned (for instance for lexing purposes): the "partial transition function" is also not too realistic. If the lexer eats one symbol which, at that point, is illegal, and for which there is no successor state, the lexer (and the overall compiler) whould not simply stop or deadlock or crash. It will eat the symbol and inform the surrounding program (the parser, the compiler) that this situation occured. It's indicates a form of error (a lexical error in the input), since we are dealing with an *deterministic* automaton, so there cannot be an alternative reading of the input that would have avoided that the lexer is stuck (or moved to a non-accepting state, or raised an exception etc). So, turning an automaton into a "total" or "complete" one is a non-issue, but removing non-determinism from an automaton is an issue. We will discuss *determinization* later. □

2.3.2 Some examples of finite state automata, mostly deterministic

Example 2.3.7 (Identifiers). Consider the following regular expression specifying indentifiers.

$$identifier = letter\ (letter \mid digit)^* \ . \tag{2.15}$$

Figure **??** shows finite-state automata that accept the corresponding regular language. The one from **??** is deterministic, but not complete or total. The second one from Figure **??** is additionally complete, i.e., the has a *total* transition function. In general, any deterministic finite-state automaton can be made complete (and still deterministic) by adding one extra non-accepting state and the appropriate additional transitions, as shown. The extra state is here called *error*, but the names of the states don't matter, they are only there to help the human reader. □

(a) deterministic

(b) deterministic + complete

Figure 2.7: Identifiers

The example shows an automaton for identifiers, as they could appear in this or similar form in the lexer specifications for typical programming languages. They are specified using regular expressions, so it's also an illustration how regular expression can be translated

into an automaton. The exact construction (which will be presented in three stages) will be covered later in this chapter, but the example is so simple, that one can easily come up with a deterministic automaton corresponding to the regular expression.

Example 2.3.8 (Natural numbers). Consider the following regular expression for natural numbers.

$$\begin{aligned} \mathit{digit} &= [0-9] \\ \mathit{nat} &= \mathit{digit}^+ \end{aligned} \tag{2.16}$$

One might say, it's not really the natural numbers, it's about a decimal *notation* of natural numbers (as opposed to other notations, for example Roman numeral notation). Note also that initial zeroes are allowed here. It would be easy to disallow that. Another remark: we make use of some user-friendly aspect supported in many applied versions of regular expression, some form of syntactic sugar. That's the possiblity to use *definitions* or *abbreviations*. We give a name to the regular expression $[0-9]$ and that abbreviation *digit* is used for defining *nat*. That certainly makes regular expressions more readable, and we will continue that form of building larger concepts from simpler ones in the following.

Also, using $[0-9]$ makes sense only if we assume an *ordered* alphabet.

Note the treatment of *digit* in the automaton. Officially, transitions are to be labelled by letters from the alphabet, but here we labelled some by *digit*, which abbreviates $[0-9]$. It's easy to see that it can be seen simple as a shorthard for writing 10 individually labelled transisition. Actually we did the same already for the identifiers in the automata from Figure **??**. In Figure **??** we used *any* and *other* to represent any letter from the alphabet resp. all letters not yet covered by other transitions. Actually, one could even allow edges to be labelled by regular expressions, without leaving the regular languages, but we won't make use of that. □

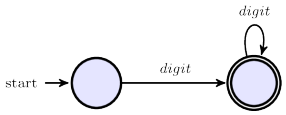


Figure 2.8: (Deterministic) FSA for natural numbers

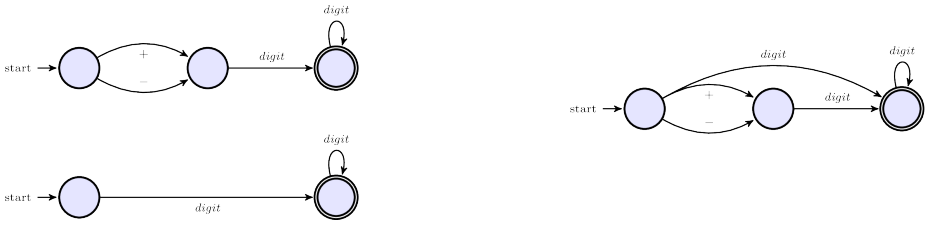
Example 2.3.9 (Signed natural numbers). Extending the definition of natural numers from Example 2.16, we can define signed natural numbers, natural numbers preceded optionally by a sign. with the following regular expression:

$$\mathit{signednat} = (+ \mid -)\mathit{nat} \mid \mathit{nat} \tag{2.17}$$

Figure 2.9a shows a non-deterministic FSA and Figure 2.9b a deterministic one for that regular expression.

The first automaton is non-deterministic by the fact that there are two initial states. Basically, one informally does two "constructions", the "alternative" in the regular expression is simply writing two automata side by side, i.e., one automaton which consists of the union of the two automata. In this example, it therefore has two initial states.

Again the automata are not complete. □



(a) non-deterministic

(b) deterministic

Figure 2.9: Signed natural numbers

Example 2.3.10 (Fractional numbers).

$$frac \quad = \quad signednat(\text{“.”}nat)? \tag{2.18}$$

Figure 2.10 shows the corresponding deterministic automaton. Note the “optional” clause

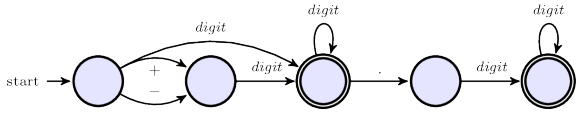


Figure 2.10: Automaton for fractional numbers

using **?** in the regular expression and the corresponding fact that the automaton has multiple accepting states. □

Remark 2.3.11 (Non-determinism in automata vs. scanners)**.** As mentioned, the automaton from Figure 2.10 has two accepting states. Note that this does not count as *non-determinism*. If one considers the automaton as some abstract form describing a scanner, one could make the argument that there is non-determinism involved. It's true that, if one gives the automaton a sequence of letters, that *determines* its end-state (and thus whether the word is accepted or not). However, lexer's task will *also* have to segment the input and decide when a word if done (and then tokenized) and when not. In the given automaton, after having reached the first accepting state, one can make the argument that, if there is a dot following, the automaton has to make the decision whether to accept the word or to continue. That sounds like a **non-deterministic choice** (actually, seen like that it *would be* a non-determistic choice).

It just means, a deterministic automaton is not in itself a scanner. A scanner deals *repeateadly* with accepting words (and segmenting), a finite state automaton is dealing only with the question, whether a given word (seen as already segmented, so to say) is acceptable or not. The current section deals just with acceptance or rejection of *one* word, and also the standard, classical definition of determisim for automata is only concerned with that question. □

Example 2.3.12 (Floats).

$$\begin{aligned} \mathit{digit} &= [0-9] \\ \mathit{nat} &= \mathit{digit}^{+} \\ \mathit{signednat} &= (+\mid -)\mathit{nat} \mid \mathit{nat} \\ \mathit{frac} &= \mathit{signednat}(\text{“.”}\mathit{nat})? \\ \mathit{float} &= \mathit{frac}(\texttt{E}\ \mathit{signednat})? \end{aligned} \tag{2.19}$$

Figure 2.11 shows the corresponding deterministic automaton. □

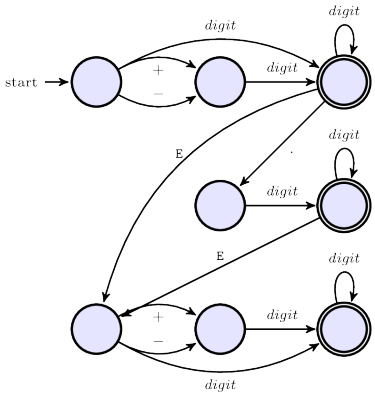


Figure 2.11: Automaton for fractional numbers

Remark 2.3.13 (Recursion)**.** Louden [11] points out that regular expressions do *not contain recursion*. This is for instance stated at the beginning of Chapter 3 in [11] where the absence of recursion in regular expressions is pointed out as the main distinguishing feature of regular expressions compared to context-free grammars (or even more expressive grammars of formalisms, see later).

When considering regular expressions as being “without recursion”, not everyone would agree. Looking at the defining equations in (2.19), the series of equations “culminates” in the one for floats, the last one listed. Furthermore, each equation makes use on its *right-hand* side only of definitions defined strictly *before* that equation (“strict” means, that a category defined in the left-hand side equation may also not depend directly on itself by mentioning the category being defined on the defining right-hand side). In that sense, the definition clearly is without recursion

For context-free grammars, that restriction will not longer apply. This absence of at least explicit recursion when defining a regular expression for instance for floats allows that one can consider the definitions as given as simply useful “abbreviations” to assist the reader’s or designer’s understanding of the definition. They therefore play the role of *macro definitions* as for instance supported by C-style preprocessors: the “real” regular expression can easily obtained by literally replacing the “macro names” as they appear in some right-hand sides by their definition, until all of them have disappeared and one has reached the “real” regular expression, using only the syntax supported by the original, concise definition of regular expressions.[5] Textual replacement, by the way, is also the way,

[5] Additional syntactic material that is added for the convenience of the programmer *without* adding expressivity of the language and which can *easily* be “expanded way” is also known as syntactic sugar.

pre-processors deal with macro definitions. Clearly this easy way of replacing mentioning of left-hand sides by their corresponding right-hand sides works *only in absence of recursive definitions.*

That supports the case that regular expressions don't contain recursion. There is, however, a different angle to the issue. Recursion, very generally, is a way to describe *infinite* structures, behavior etc. Regular languages are, in general, *infinite*, i.e., infinite sets of words, and the way to capture those infinite sets in a *finite* way is via the Kleene star. In the automata, infinitely many different words are represented by "loops". Thus, the Kleene star *does* allow to express (implicitly) a form of recursion, even if it's a more restricted form than that allowed by context-free grammars. The point may become more clear if we replace the definition for natural numbers from equation (2.16), using + for "one or more iterations" by the following recursive one:

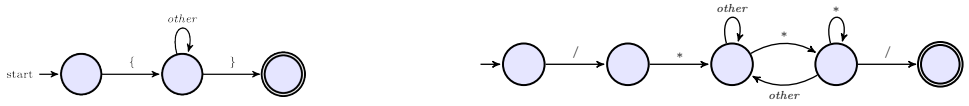
$$nat = digit\ nat \mid digit\ . \tag{2.20}$$

Compared to the more general definitions for context-free grammars later, the recursive mentioning of *nat* on the right-hand side of the definition for regular language is *restricted.* The restriction will become clearer once we have covered context-free grammars in the context of parsing. Suffices for now, that the restrictions are called **right-linear** grammars (alternatively light-linear grammars, both are equally expressive), where *linear* refers to the fact that at most one of the "meta-variables" in a grammar (such as *nat* from above) allowed to occur on the right-hand side of a *rule*, and right-linear would mean, it's allowed to occur only *at the end* of a right-hand side.[6] Another, basically equivalent, point of view or terminology is that the definitions may use **tail-recursion** (but not general recursion).Tail-recursion corresponds to right-linear definitions (as the "tail" is considered to be at the right-and end.)

To summarize: the dividing line between regular languages vs. context-free languages may well be described as allowing tail-recursion vs. general recursion (not as without or with recursion as in [11]). For those who followed the lecture *IN2040* (functional programming) may remember the distinction between **recursive** and **iterative** procedures. When programming, one sometimes distinguishes between *iteration* (when using a loop construct) on the one hand and *recursion* on the other, where iteration corresponds to a restricted form of recursion, namely *tail-recursion:* tail-recursion is a form of recursion, where *no stack* is needed (and which therefore here can be handled by finite-state automata), in contrast to context-free grammars, which cannot be handled by FSA's, one needs equip them with a *stack*, after which they are called *push-down* automata, most oftenly. □

Example 2.3.14 (DFAs for comments). Without bothering this time to give a regular expression specification, Figure 2.12 contains deterministic automata for comments, one in the style of Pascal, one in the style supported by C, C^{++}, Java ... □

[6] As a fine point, to avoid confusion later: The definition from equation (2.20) would count as *two* rules in a grammar, not one, corresponding to the two alternatives. The restrictions for linearity etc. apply *per rule/branch* individually.



(a) Pascal

(b) C, C^{++}, Java

Figure 2.12: Comments

2.4 Implementation of DFAa

DFAa underly the implentation of lexers. The notion as such is simple enough, but a concrete lexer has to cover slightly more things than the theoretical coverage so far. One is that the lexer needs to be coupled up with the parser, feeding it with one token after the other. I.e., in the implementation, the automaton is not just a *recognizer*, it has not just sequences of characters as input which it has to decide on whether to accept or not. It also needs to produce sequences of tokens as output. Related to that, the lexer is not just the implementation of one single DFA, but it's a **loop** that repeatedly "invokes" DFAa. Another aspect of the regular expressions resp. the DFA is the need for **priorities**. We have mentioned the issue when discussing regular expression, for instance, when confronted with the string <=, then that is conventionally scanned as less-or-equal, and not as a < followed by a =.

2.4.1 Longest match

That aspect of **longest scan** is not really covered by the notion of DFA (nor by non-deterministic automata). That has to do with the way, automata *accept* words (we touched upon that already in Remark 2.3.11). They start in their initial state, eat through the input, but when it come to acceptance in a lexer, it misleading to think like that: if you hit an accepting state, *accept the word* (and return the corresponding token). Sure, if a automaton hits an accepting state, this word seens so far is accepted, resp. belongs to the language the automaton describes. But there may be *another* run of the automaton (even if it is a deterministic one and is fed the same word as prefix), that reaches the accepting state, *and then continues*, perhaps accepting later down the road *a longer word* which extends the one the automaton could accept right now. Of course there may not be a guarantee that there exists a longer word. Anyway, a lexer explores one word only and makes decisions "on the spot" (being deterministic), preferring longer scans over shorter. In case of hitting an accepting state, it checks if one can proceed, still accepting the (extended) word. If *not* it accepts the word as is. This priority to proceed as long as possible and favoring longer words over shorter prefixes is *not* directly covered by the theoretical treatment of FSAa so far, but it's an aspect an implementation would have to do.

This section brushes also on **data structures** one can use to implement DFAs, resp. scanners. We don't go too deep, bascially we sketch how one could use **tables** to represent automata (which can be realized for instance by two-dimensional arrays or in other ways). Tools like `lex` allows the compiler writer to ignore details there, as that's what those

tools do: generate appropopriate data structures representing the DFA, taking care also of the other aspects mentioned, and interfacing with the parser component of a compiler. Later in this section, we introduce a notation labelling edges of the DFA with `[` and `]`, for instance, writing `[other]`. The meaning will be, that is a way to describe the "longest match" discipline. For instance, assume an automaton designed to accept a word defined as a sequence of letters; that could be described as $[a-zA-Z]$*, making use of ranges in ordered alphabets. The edge-label *other* will be used to abbreviate all "other" symbols. More technically, in a state with labels on outgoing edges labeled by some symbols, an outgoing edge labelled *other* represents all symbols *not* covered by the outgoing edges. It should be self-evident, especially for a determistic automaton, that can be only one outgoing edge labeled *other*. So far, that has nothing yet to do with the point discussed here, namely prioritizing longer matches. To do that, one uses edges, annotated with [and], as said. Note: it may be unfortunate, but the notation is meant to do something else than defining a letter as regular expression $[a-zA-Z]$*. What is meant then? Well, a transition labeled $[a]$ means: if a is next, move to the next state **without actually consuming or "eating" the** a **as input**. Concretely, we use often transitions labelled $[other]$, and moving to an accepting state. That is the way to represent the longest match or longest possible scan. We continue eating symbols, like lower and upper-case letters, but without accepting the string as yet. When we hit a symbol *other* than a letter, we proceed and accept the string, but the very last symbol is *not* part of the word we just processed. That can be seen as a form of "look-ahead". The letters or expressions in brackets are checked to make a decision without actually consuming them. That peek into the immediate future makes it a form of look-ahead. Looking ahead into the future of the lecture, the notion of **look-ahead** with play also a prominent role when we will talk about parsers: the amount of look-ahead one is willing to give to a *parser* influences its expressiveness: obviously, more look-ahead, more powerful.

Example 2.4.1 (Indentifiers). Remember the DFA for identifiers from Figure **??** in Example 2.3.7. We had two versions, an incomplete one and *complete* one with an extra "error" state.

The one from Figure 2.13 is *deterministic*, but it's *not* total or complete. The transition function is only *partial*. The "missing" transitions are not shown, as we often did earlier as well to make the pictures more compact. It is then implicitly assumed, that encountering a character not covered by a transition leads to some extra "error" state which simply is not shown.

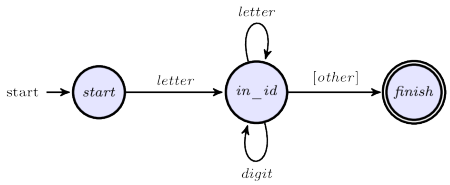


Figure 2.13: Deterministic automaton

As explained before, the [ ] around the transition *other* at the end means that the scanner does **not move forward** on the input there (but the automaton proceeds to the accepting state). That is something that is *not* 100% in the mathematical theory of FSA, but is how the *implementation* in the scanner will behave. Note also that the accepting state has

changed: we have an extra state what we move to by the special kind of transition [*other*]. As the name implies, "other" means all symbols different from the ones already covered by the other outgoing edges. This is used to realized the *longest prefix*: The shown DFA not just accepts "some" identifier it spots on the input, i.e., an arbitrary sequence of letters and digits (starting with a letter). More precisely, it takes as many letters and digits as possible until it encounters a character *not* fitting the specification but not earlier. Only at that point does the automaton accepts but without advancing the input, in that this character will have to be scanned and classified as the "next chunk" and this "the next automaton". □

2.4.2 Implementations

The following shows rather "sketchy" pseudo-code about how part of a lexer can be programmed or represented. It's one loop and it represents how to accept one lexeme. As mentioned at the beginning of this section, the task of a lexer in the context of a compiler is to *repeatedly* accept one lexeme after the other (or reject an input and stop) and hand over a corresponding stream of tokens. This need of repeated acceptance does not mean that there is another loop around the while-loop shown in the pseudo-code. At least the mentioned outermost loop is not part of the lexer. The lexer and the parser work hand in hand, and often that's arranged in that lexer works "on demand" from the parser: the parser invokes the lexer "give me a new token", the lexer has of course remembered the position in the input from the last invokation, and, starting from there tries to determine the next lexeme and, if successful, gives back the corresponding token to the parser. The parser determines if, at that point in the parsing process, the token fits to the syntactic description of the language, and if so, adds a next piece (at least implicitly) in building the parse tree, and then asks the lexer for the next token, etc. The pseudo-codes of the lexer therefore contain only one loop, the one for accepting *one* word.

```
state := 1  { start }
while state = 1 or 2
do
  case state of
  1:  case input character of
      letter: advance the input;
              state := 2
      else state := .... { error or other };
      end case;
  2: case input character of
     letter,digit: advance the input;
                   state := 2; { actually unessessary }
     else          state := 3;
     end case;
  end case;
end while;
if state = 3 then accept else error;
```

Listing 2.2: Explicit state representation

The state is here represented as some integer variable. The reaction of the autmaton is a nested case switch, first one which state one is in, and secondly on which input. One could of course also do the "case nesting" the other way around, or making one flat case switch, with all combinations of state and input on the same level. We also see that the "error" state in the *complete* DFA is also represented in some form: there is a else-case if the previous case(s) don't match.

In the following slides, we show how the decision-information can be "centralized" in one table. In the table, empty slots represent missing reactions, i.e., the move to an error state.

Table 2.3 represents the DFA in tabular form and Listing 2.3 sketches some pseudo-code that makes use of that table

state \ input char	letter	digit	other	accepting
1	2			no
2	2	2	[3]	no
3				yes

Table 2.3: Table representation of the DFA

```
state := 1  { start }
ch := next input character;
while not Accept[state] and not error(state)
do

while state = 1 or 2
do
  newstate := T[state,ch];
  {if Advance[state,ch]
   then ch:=next input character};
  state := newstate
end while;
if Accept [state] then accept;
```

Listing 2.3: A table-based implementation

2.5 From regular expressions to NFAs

We start by re-visiting finite state machines (see Definition 2.3.2) and generalize them slightly

Here we kind of repeat the definition, with δ slightly differently, but ultimately equivalently represented. What we add, however, are so-called ϵ-transitions, which allows the machine to move to a new state without eating a letter. That is a form on "spontaneous" move, not being triggered by the input, which renders the automaton non-deterministic. It will turn out, that by adding this kind of transitions does not matter, as far the expressiveness of the NFAs is concerned. Why do we then bother adding them? Well, ϵ-transitions come in handy in some situations, in particular in the construction we will present afterwards: how to turn a regular expression into an NFA. It's slightly more convenient when one allows

such transitions. It's easy to understand also why. As a preview to that construction: it will be a *compositional* construction. To construct the automton for a compound regular expression, for instance for the sequential composition $r_1 r_2$, one assumes one has the automata for the component regular expressions r_1 and r_2, and then one glues them together with ϵ-transitions, i.e., connects the accepting states of r_1 with the initial states of r_3 with empty transitions. That's pretty easy, so the use if those transitions facilitates a straightforward, *compositional* construction.

Definition 2.5.1 (NFA (with ϵ transitions))**.** A *non-deterministic* finite-state automaton (NFA for short) $\mathcal{A}$ over an alphabet Σ is a tuple $(\Sigma, Q, I, F, \delta)$, where

- Q: finite set of states
- $I \subseteq Q$, $F \subseteq Q$: initial and final states.
- $\delta : Q \times \Sigma \to 2^Q$ transition function

In case, one uses the alphabet $\Sigma + \{\epsilon\}$, one speaks about an NFA with ϵ-transitions.

It is assumed that *emptyword* is not a symbol of Σ. That's why we write $\Sigma + \{\epsilon\}$ and not $\Sigma \cup \{\epsilon\}$, with $+$ representing *disjoint* union. In a way, ϵ is not meant as letter at all it represents the absence of a latter. The version of NFA presented here includes ϵ-transitions. Depending on which book one consults, the notion of NFA may or may not include such transitions. We call them here explicitly NFA with empty transitions. It does not matter anyhow, as far as the expressiveness is concerned.

Whowever, the ϵ is *treated different* from the "normal" letters from the alphabet Σ.

δ can *equivalently* be interpreted as *relation* (as we did in the formulation from Definition 2.3.2) $\delta \subseteq Q \times \Sigma \times Q$ (transition relation labelled by elements from Σ).

Remark 2.5.2 (Terminology (finite state automata))**.** There are slight variations in the definition of (deterministic resp. non-deterministic) finite-state automata. For instance, some definitions for non-deterministic automata might not use ϵ-transitions, i.e., defined over Σ, not over $\Sigma + \{\epsilon\}$. Another word for FSAs are finite-state machines. Chapter 2 in [11] builds in ϵ-transitions into the definition of NFA, whereas in Definition 2.5.1, we mention that the NFA is not just non-deterministic, but "also" allows those specific transitions. Of course, ϵ-transitions lead to non-determinism, as well, in that they correspond to "spontaneous" transitions, not triggered and determined by input. Thus, in the presence of ϵ-transition, and starting at a given state, a fixed input may *not determine* in which state the automaton ends up in.

Deterministic or non-deterministic FSA (and many, many variations and extensions thereof) are *widely used,* not only for scanning. When discussing scanning, ϵ-transitions come in handy, when translating regular expressions to FSA, that's why for instance [11] directly builds them in.

The **language** of an automaton with ϵ-transitions is analogous to the language of automata without (see (Definition 2.3.3 on page 30). Run the manchine on a given word, see if this way one *can* reach a final state (also called accepting state). If so, the word belongs to the language, otherwise not. As far as ϵ-transitions are concerned: ϵs simply *do not count* (representing "no character").

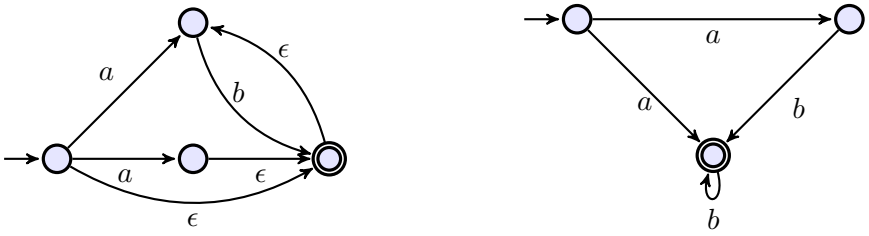
Definition 2.5.3 (Acceptance with ϵ-transitions)**.** A word w over alphabet Σ is *accepted* by an NFA with ϵ-transitions, if there exists a word w' which is accepted by the NFA with alphabet $\Sigma + \{\epsilon\}$ according to Definition 2.3.3 and where w is w' with all occurrences of ϵ removed.

The behavior of such a machine can also be seen as follows: $\mathcal{A}$ reads one character after the other (following its transition relation). If in a state with an outgoing ϵ-transition, $\mathcal{A}$ can move to a corresponding successor state *without* reading an input symbol.

2.5.1 NFA vs. DFA

We have mentioned, that, NFA's are bad as machinery for implementing a lexer, being *non-deterministic.* They have, on the other hands, also some positive aspects. The most important one being for us is: it's easier to translate regular expressions into non-deterministic machines (and if one allows ϵ-transitions, it will make it even more straightforward).

- *NFA*: often easier (and smaller) to write down, esp. starting from a regular expression
- non-determinism: not *immediately* transferable to an *algo*



The example is used as illustration of an NFA and a corresponding DFA. In this small example, it's straightforward to come up with a deterministic version of the automaton. In a later section, we discuss a systematic way of turning an NFA to a DFA, i.e., an algorithm.

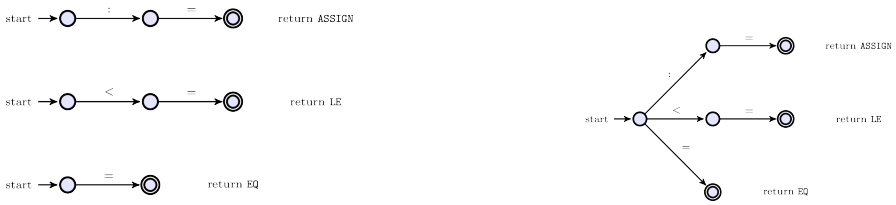
Before showing the construction itself, we show a few examples, highlighting some regular expression and corresponding NFAs.

Example 2.5.4. The task is to recognize :=, <=, and = as three different tokens. That would correspond to a regular expression involing "or":

$$:= \mid <= \mid = .$$

That's easy enough to represent as NFA, see Figure 2.14a. A deterministic version is not much harder to obtain, see Figure 2.14b.

□



(a) non-deterministic

(b) deterministic

Figure 2.14: Illustration for Thompson's construction

The example was straightforward to turn into a deterministic one. It's not always so trivial. The next example shows that, and makes clear why the example just shown is so simple to deal with.

The words in previous Example 2.5.4 share some common parts, there is some overlap: all three strings contain the character =. However, they don't share a common *prefix*, i.e., a common initial common segment). That's shown in the following example.

Example 2.5.5. Again three lexemes are to be recognised, represented by the following regular expressions:

$$<= \mid <> \mid < .$$

They can be straightforwardly by the NFA of Figure 2.15a, similar as we did in the previous example. One can then, as in the previous example, use a single initial state instead (see Figure 2.15b), but that does not turn the automaton into a deterministic one. Figure 2.16

(a) non-deterministic (1)

(b) non-deterministic (2)

Figure 2.15: Illustration for Thompson's construction

then shows a deterministic automaton

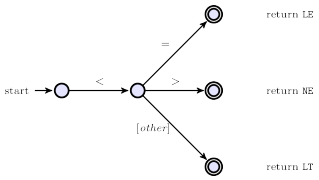


Figure 2.16: Illustration for Thompson's construction: deterministic

□

The examples were not directly Thompson's construction, i.e., how to go from regular expressions to FSAa, notably NFAa. They just perhaps made plausible that it might be a good idea not do attempt to do a deterministic automaton directly, but construct a non-deterministic one first, and determinize that in a second step. The examples basically illustrated the non-deterministic construction for alternatives, i.e., regular expressions constructed by |, is really strightforward.

But it is just an illustration. What is needed is a *systematic* translation, an algorithm, best an efficient one. Conceptually easiest is to translate to non-deterministic automata, actually translated to automata with ϵ-transitions; the latter aspect was not illustrated by the examples from before, it was easy enough without that in the considered examples. Determinization is postponed to a second step, likewise minimization.

Example 2.5.6 (Illustration for ϵ-transitions). Revisiting Example 2.5.4, one can use ϵ-transitions for the automaton. The automaton has only one initial state but is non-

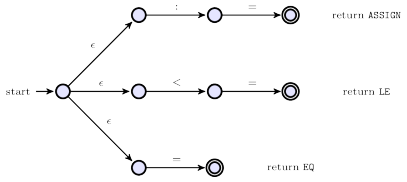


Figure 2.17: The use of ϵ-transitions

deterministic, since it uses ϵ on its transitions.

2.5.2 Thompson's construction

Regular expression are given inductively, there are *basic* regular expressions and *compound* expressions (see Definition 2.2.8 or grammar from Definition 2.2.9).

> Goal: Give NFA for the **basic** regular expression. Construct an NFA of **compound** regular expression is given by taking the NFAa of the immediate subexpressions and connect them appropriately.

The construction is due to Ken Thompson [13]. It will make ample use of ϵ-transitions and the construction slightly[7] simpler, if one constructs with one start and one accepting state.

Remark 2.5.7 (Compositionality)**. Compositionality** and compositional concepts (definitions, constructions, analyses, translations, ...) are *immensely* important and pervasive in compiler techniques (and of course beyond). One example already encountered was the definition of the language of a regular expression (see Definition 2.2.11 on page 20). The design goal of a compositional translation here is the underlying reason why to base the construction on *non-deterministic* machines.

[7] It does not matter much, though.

Compositionality is also of practical importance ("component-based software"). In connection with compilers, *separate compilation* and (static / dynamic) linking (i.e. "composing") of separately compiled "units" of code is a crucial feature of modern programming languages/compilers. Separately compilable units may vary, sometimes they are called modules or similarly. Part of the success of C was its support for separate compilation (and tools like `make` that helps organizing the (re-)compilation process). For fairness sake, C was by far not the first major language supporting separate compilation, for instance FORTRAN II allowed that, as well, back in 1958.

Btw., Ken Thompson, who first described the regular-expressions-to-NFA construction, is one of the key figures behind the UNIX operating system and thus also the C language (both went hand in hand). Not suprisingly, considering the material of this section, he is also the author of the `grep`-tool ("globally search a regular expression and print"). He got the Turing-award (and many other honors) for his contributions. □

The **base cases**, for basic, i.e., non-composed regular expressions $\boldsymbol{\epsilon}$ and $\boldsymbol{a}$ (for all $a \in \Sigma$) are shown in Figure 2.18.[8]

(a) Empty word ϵ (b) Single character

Figure 2.18: Thompson's construction: base cases

The **inductive cases**, for **compound** expressions, are shown in Figures 2.19a, 2.19b, and 2.19c. In the pictures from Figure 2.19, the rectangular box(es) represent the automata for the immediate subexpression(s). By convention, the state on the left is the unique initial one, the state on the right is the unique final one. By building the larger automaton, the "status" of the initial states and final states may change of course. For instance, in the case of | in Figure 2.19b, a *new* initial state and a *new* accepting state are introduced for the compund automaton, but the initial and final states from the two component automata loose their special status thereby, of course.

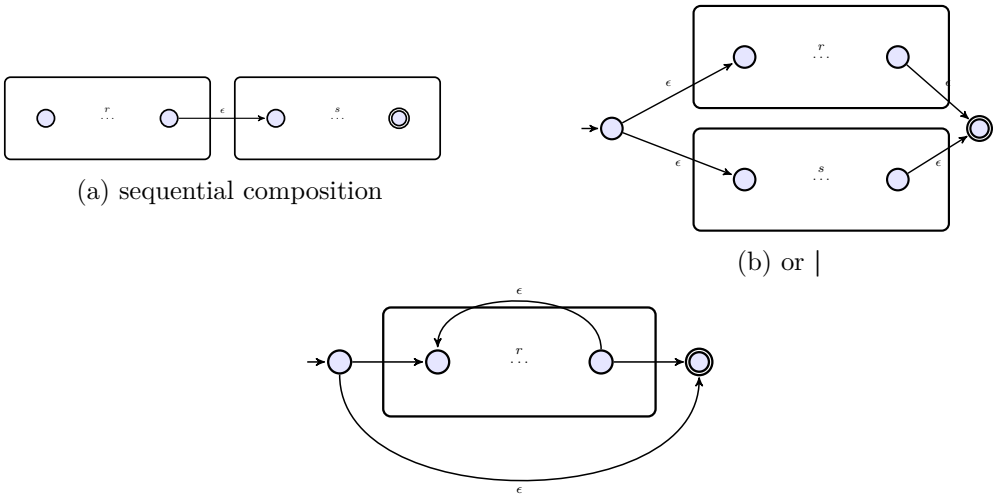
Example 2.5.8 ($\boldsymbol{ab \mid a}$). Here is a small example illustrating the construction. In the exercises, there will be more.

□

2.6 Determinization

The section covers the second step in the translation from regular expressions to DFAs: we have to get rid of non-determinism and make the machine deterministic. It's not too hard, after seeing the idea. Deterministic or not, the automaton eats symbols and moves to another state. In the case of a non-deterministic automaton, it may non-deterministically move to one of many possible sucessor states. That's problematic, because if the wrong choice is made, the automaton would need to **backtrack** and try alternative routes.

[8] The base care for $\emptyset$ is not practically useful, and the "construction" is left out here.

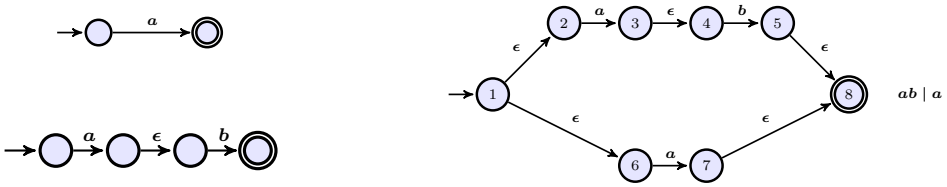


(a) sequential composition

(b) or |

(c) Iteration with Kleene star

Figure 2.19: Thompson's construction: inductive cases



(a) ***a*** and ***ab***

(b) ***a*** | ***ab***

Figure 2.20: Thompson's construction

That can be avoided, if, when presented with alternative successor states, the machine can explore all of them at the same time. So, instead of moving non-deterministically to one successor state, the automaton **moves to *all* possible successor states,** which is a set. Of course, that would be a different automaton, but it is a deterministic one: there is *exactly one* set of successor states. This alternative automaton, working on **sets of states** of the original NFA can be systematically constructed. This construction is the determinization procedure.

2.6.1 Determinization: the subset construction

The task is, given a non-deterministic automaton $\mathcal{A}$, construct an equivalent DFA $\overline{\mathcal{A}}$. The deterministic one, instead of *backtracking*, explores all successors at the same time. Each state q' in $\overline{\mathcal{A}}$ represents a *subset* of states from $\mathcal{A}$, and given word w feeding that to $\overline{\mathcal{A}}$ leads to *the* state representing **all** states of $\mathcal{A}$ *reachable* via w. The procedure is known as **powerset construction** (by Rabin and Scott [12]).

The construction itself is straightforward enough. Analogous constructions works for some other kinds of automata, as well, but for still others, the approach does *not* work: For

some forms of automata, non-deterministic versions are strictly more expressive than the deterministic one, for instance for some automata working with languages on infinite words, not finite words as here.

Definition 2.6.1 (ϵ-closure, a-successors)**.** Given a state q, the *ϵ-closure* of q, written $close_\epsilon(q)$, is the set of states reachable via zero, one, or more ϵ-transitions. We write q_a for the set of states, reachable from q with one a-transition. Both definitions are used analogously for sets of states.

We often call states like q, and sets of states then Q. So the notations for the ϵ-closure of a set Q of states is $close_\epsilon(Q)$ and Q_a represent the a-successors of Q.

It may also be worth remarking: later, when it comes to parsing, we will encounter the phenomenon again: some steps done treating symbols from a context-free grammar will be done "eating symbols" (for parsing, those symbols will be called "terminals" or "terminal symbols" and correspond to tokens). Consequently, in the context of parsing and "parsing automata" (which are supposed to be deterministic, as well), we will likewise encounter the notion of an ϵ-closure which is analogous to the concept here.

Input: NFA $\mathcal{A}$ over a given Σ
Output: DFA $\overline{\mathcal{A}}$

1. the *initial* state: $close_\epsilon(I)$, where I are the initial states of $\mathcal{A}$
2. for a state Q in $\overline{\mathcal{A}}$: the *a-successor* of Q is given by $close_\epsilon(Q_a)$, i.e.,

$$Q \xrightarrow{a} close_\epsilon(Q_a) \tag{2.21}$$

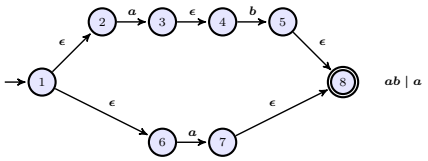
3. repeat step 2 for all states in $\overline{\mathcal{A}}$ and all $a \in \Sigma$, until no more states are being added
4. the *accepting* states in $\overline{\mathcal{A}}$: those containing *at least* one accepting state of $\mathcal{A}$

Next we show a few examples. More are covered by the exercises. In the figures, we show the resulting deterministic automata. However, we don't show the *complete* or total version, i.e., the extra state sometimes needed to obtain a total successor function is not shown. The state can be seen as being "marked" with the empty set {}

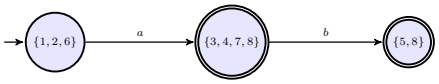
Example 2.6.2 ($ab \mid a$). Figure 2.21a shows the automaton corresponding to 2.21a as produced by Thompson's construction. It has one initial state, one final state, and makes ample use of ϵ-transitions. Figure 2.21b shows the result of the determinization (without showing the "error" state). That states use sets of states from the NFA as "name". □

Example 2.6.3 (Identifiers). Remember the regular expression for for identifiers from equation (2.15). Figure 2.22a shows the generated NFA and Figure 2.22b the DFA after determinization.

One can compare the DFA here with the automata from Figure 2.7a and 2.7b from before. Those are also determininistic; the one from Figure 2.7b is complete in that it shows an error state. The previously shown automata were hand-made, the one from Figure

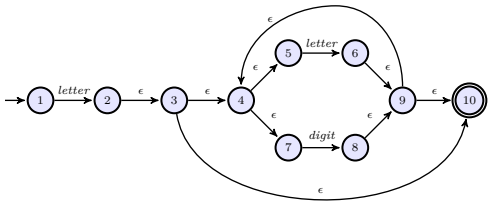


(a) NFA

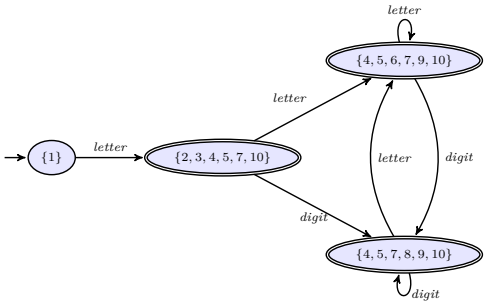


(b) DFA

Figure 2.21: $ab \mid a$



(a) NFA



(b) DFA

Figure 2.22: Identifiers

2.22b is the result of the determinization algorithm and has more states. This means, the determinization resulted not in an deterministic automatong as small as possible. □

2.7 Minimization

This then is the last stage of the construction, minimizing a DFA. It should be clear why that is useful: less states means more compact representation (but perhaps not necessarily speed-up in lexing). Minimal means, with the least number of states. It's clear that there exists an automaton with the least number of states. But what is perhaps more surprising there exists *exactly one* automaton with the minimal number of states. A priori, there might be two different automata with a minimal amount of states, but that is not the case. Of course, being the same automaton means *up-to isorphism*. Isomorphic means "structually identical" basically it means, the "names" of the states don't matter, but otherwise the automata are the same.

We learn the algorithm that systematically calculates the minimal DFA from a given DFA. Previously, we downplayed the question, whether a DFA is complete or not, because if not complete, one can easily think of it as complete, assuming an extra error state. In the minimization here, it's important to indeed have a *complete* deterministic automaton, all states participate in the construction, including an extra error state that may be necessary to complete the DFA.

The concstruction presented here is only one of different ways to achieve the goal. It's

known as *Hopcrofts's partitioning refinement algorithm*. A modern version of that is described in Hopcroft [9].

Properties of minimization:

Minimality: the resulting DFA has *minimal* number of states.

Canonicity: all DFA for the same language result in the *same* minimal DFA.

These properties have a positive side-effect, in that it solves also *equivalence* problems: Given two DFA, do they accept the same language? That can be checked by turning them into the minimal form and check if they are the same (isomorpic). One one likewise check if 2 regular expressions described the same language, turning them into their minimal deterministic automata representations.

Now to *Hopcroft's partitioning refinement* construction. As said, the starting point is a **complete DFA**, i.e., *error*-state possibly explicitly needed.

The algorithm will be explained to some extent in the following (mostly by way of examples).

I would expect, when tasked oneself with the problem of minimizing a given DFA, most would try to approach the problem the following way. One would look at an the automaton and look for situations when one could save some state. That's a natural way of thinking, also when one try concrete examples on pen and paper.

For instance, one could look at the DFA for identifiers from Figure 2.22b. It has three accepting states, but it does not take long to realize that only one is enough. One can "**merge**" the two states on the right because the "do the same". By "doing the same" it's meant that both accept the same language, when one starts in them for accepting words. That language can be described by the regular expression (*letter* | *digit*)*. Collapsing the two states into one makes (and perhaps afterwards the third one) the automaton smaller without changing the accepted language. That could be a core step for an algorithm: hunt for pairs or sets of equivalent states, collapse them, then then hunt for further opportunities and continue until only non-equivalent states remain, and then stop.

That's a valid idea. One would check some aspects before being sure it works. Termination is obvious. Another issue would be: is it important in which order to do the collapsing. It is a priori clear whether the algo would be independent from the strategy which pairs to collapse first. It could be that by choosing "wrong", one gets a smaller automaton, but somehow get stuck before reaching the really minimal one. That would be an unpleasant property of the approach (and would lead to backtracking). One also would have to solve the problem of checking when are two states equivalent (and that might be computationally complex).

But, as said, it's a valid idea (and I am rather sure, the approach would be indepedent from the order of collapsing). The algorithm we describe below works the other way around in that it's not based on merging equivalent states, but by starting out with a "collapsed automaton", where all states are collapsed, and the splits them repeatedly (based on a criterion described later). The algo is not very obvious. In the merge-based, naive one, one

starts with a DFA and in each steps, it gets smaller, but the algo maintains as invariant that all the intermediately constructed DFAs accept the same language as the original. Thereby it's clear that the result is likewise equivalent. And since we stop, when there are no more non-equivalent states, it's also plausible, that the result is minimal.

The **partition refinement** construction works the other way around! instead of collapsing equivalent states:

Initialization: start by collapsing as much as possible (details apply).

Iteration: iteratively, detect *non-equivalent* states, and then *split* a "collapsed" state

Termination: stop when no violations of "equivalence" are detected

Why is it called partitioning refinement? A partitioning of some set, here a set of states, means dividing the set up in different subsets, called partitions, such that every element (here every state) belongs to exactly one partition. So all original elements are covered without overlap of the partitions. It's called partition refinement, because, starting from a very coarse partitioning, the iteration splits partitions, refining them until the criterion for splitting no longer applies.

We can consider the partition containing sets of states as state in some collapsed automaton. Initially, basically fully collapsed, with 2 states only, it will be generally **not equivalent** to the DFA, it accepts a different and larger language. It will also be **smaller** than the minimal one. And actually it won't be **determinist**. The algorithm proceeds by splitting collapsed states as long as the splitting criterion is fulfilled. So all the intermediate automata are non-equivalent to the targeted DFA and all of them are smaller than the minimal one and non-deterministic. Once the splitting-criterion is no longer satisifed, one stops, and one has reach the *first* automaton in this process which, surprise, surprise, *is* equivalent to the targeted one, and at the same time is the minimal one, and is deterministic.

That is the high-level idea of the algorithm. The explanation pretends that the partitions are states of some automaton. That is a helpful picture to understand how it works. The code of the algorithm may not explicity construct the intermediate automaton, it simply refines the partitions. The splitting-criterion can be thought of checking if the current partitioning, when interpreted as states of an automaton is still **non-determinstic**. If so, repair that instance of *non-determinsim* by splitting. If deterministic, stop.

If the refinement ends with the fine-grained partitioning, the one where each state form it's own partion, then we are back at the original deterministic automaton, and the procedure showed that the DFA we started with was already minimal.

Let's do the **partition refinement a bit more concrete**.

- **Initial** partitioning: 2 partitions: set containing all *accepting* states F, set containing all *non-accepting* states $Q \backslash F$
- **Loop** do the following: pick a current equivalence class Q_i and a symbol a
  - if for all $q \in Q_i$, $\delta(q, a)$ is member of the *same* class $Q_j \Rightarrow$ consider Q_i as **done** (for now)
  - else:
    - **split** Q_i into $Q_i^1, \ldots Q_i^k$ s.t. the above situation is repaired for each Q_i^l (but don't split more than necessary).
    - be aware: a split may have a "cascading effect": other classes being fine before the split of Q_i need to be reconsidered $\Rightarrow$ *worklist* algo
- **stop** if the situation stabilizes, i.e., no more split happens (= worklist empty, at latest if back to the original DFA)

The initialization, as mentioned before, starts with an (almost completely) collapsed automaton. It's not totally collapsed to a one-state representation, but consists of 2 states, no matter how big the original automaton is.

The algo speaks about *partitions* and operates by refining them. A partition is a technical term about sets, is splitting up a set into different (non-empty) subsets in such a way, that each element of the original set is in exactly *one* of the subsets, and the the union of all subsets is the original set. Alternatively (and equivalently), a partition on a set can be seens as equivalence relation on the set (an equivalence relation being a binary relation which is reflexive, transitive, and symmetric). We won't dig into mathematical depth here, so let's just illustrate it in a very examples. Assume a five-element set

$$A = \{1, 2, 3, 4, 5\} \ .$$

We can partition it into two subsets

$$\{\{1, 2, 3\}, \{4, 5\}\}$$

Let's call the two subsets A_1 and A_2.

Equivalently, one can see that partition as considering 1, 2, and 3 and "equivalent" and likewise 4 and 5. In other words, the partition corresponds to an equivalence relation. If one likes to spell the equivalence relation $\sim \subset A \times A$ in full detail as set of pairs, it would be

$$\sim = \{(1,1), (1,2), (2,1), (2,2), (1,3), (3,1), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$$

which corresponds to $A_1^2 + A_2^2$. Both views are interchangable. Seen as equivalence relation, one can also view the algorithm as refining a equivalence relation instead of a partitioning. Remember when discussing the naive "merging approach", we merged "non-equivalent" states. So, also there, we were working with an equivalence relation, what was meant

there was semantical language equivalence. To states are equivalent, if they accept the same language, when starting acceptance runs from there.

Of course here, during the run of the automaton, the equivalence relation that corresponds to the current partition is *not* yet semantic language equivalence, it's a more coarse-grained equivalence relation, considering states currently as equivalent (grouped together in the same subset of the partition), when in fact, semantically, they are not equivalent. When the algo stops, though, the equivalence relation coincides with the intended langugue equivalence.

Here, we are working with partitons of the set of states of the given DFA, and we start with a partition, consisting of to subsets: the set of states is split into two parts: the accepting states and the non-accepting states. The algorithm works in one direction only: namely by taking a subset, i.e., one element in the current partition of Q, and splits that, if needed. The partition gets more finegrained with each iteration, until no more splitting can be done.

If one looks at some partition during the run of the algo, one can conceptually *interpret* the partition as an automaton: Each subset of the partition forms some "meta-state" consisting of sets of states, and there are transitions between those meta-states in the obvious. In this way, the algo not just steps through a sequence of partitions it refines, but at least conceptually, to a sequence of automata. This is a way of "thinking" about the run of the algo, the algo itself does not explicitly construct sequences of automata, it works on a sequence of partitions that gets more and more finegrained during the run.

However, thinking in terms of intermediate automata helps to interpret the splitting-condition: when (and how) should the algo split a meta-state, and when can it stop. As mentioned earlier, starting from the initial 2-state automaton, the intermediate automata are generally smaller than the minimal one, and they are accept a language different from the one of the target automaton (a larger language, actually). There is a third aspect, not mentioned so far: at and intermediate stage, the automaton with the meta-states is generally *non-deterministic*. It's clear that if one takes a DFA and collapses some states into one meta-state, the result will no longer be deterministic. That is also the splitting-condition. The algo looks at meta-states (i.e., a subset in the current partition) and if that meta-state violates the requirement that it should be *determistic*, then it splits it. Actually, the algo checks whether or not a meta-state is determistic per *symbol*, i.e., the algo checks where some meta-state Q and a symbol a behave deterministically or not.

If the meta-state behaves non-deterministically, we have to repair that, and that's by splitting the that meta-state, so the the resulting split behaves deterministically (with respect to that symbol a). Howerver, we split only as much as we need to repair the non-determistic violation, but not more. For instance, one does not simply "atomize" the meta-states into its individual original states. Those would surely behave deterministic as the starting point had been DFA, but this way, we won't get the minimal automaton in general, as we would do more splits than actually necessary.

So far so good. Of course, one need to treat more than a, it may be necessary, after splitting a meta-state wrt. a, that one need to split the result further wrt. b. That's clear, and let's not talk about that, let's focus on one symbol. More interesting is the question: after having split one meta-state in the way described, making the fragments

deterministic, am I done with the fragments, or will I have to split them further? The answer is: doing it one time may not be enough. The reason is as follows. Splitting a meta-state in the way described may have a *rippling effect* on other meta-states. For instance, if one has a stituation like

$$Q \xrightarrow{a} Q'$$

and the meta-state Q' happens to be split in, say, two refined meta-states Q'_1 and Q'_2, then the predecessor state Q suddenly has 2 outgoing a-transitions even if we assume that sometimes earlier, Q was the result of some splitting step, making it determistic at that earlier point. That mean, splitting a state may affect that other states have to be split again, that is the mentioned rippling effect.

A good way to organize the splitting task is to put all the current meta-states that have not been checked wether they need a split or not into a *work list*. It may not technically be a list, but could be a queue or stack, or in general a collection data type, but the algo would still be called worklist algorithm. Anyway, with this data structure, one can remove one piece of work, a current meta state out from the work-list, splits it, if necessary, removes the piece of work, but (re-)adds predecessor states, as they need to be rechecked and re-treated.

Side remark 2.7.1 (Partition refinement vs. merging equivalent states)**.** We started earlier by claiming that a naive approach would probably try to *merge* equivalent states starting from the given DFA (with would be a "partition coarsening"), as that seems more obvious. Now, why is the partition refinement algo intuitvely a better idea (without going into algorithmic complexity considerations)?

In a way, the two approaches (refinement vs. coarseing) look pretty similar. One merges states resp. split states, until no more merging resp. splitting is neccessary, and then stops. It's also not easy to say, which is the shorter route, i.e., which approach needs on average the least amount of iterations (perhaps in the special case where the automaton comes via Thompson's construction and determinzation).

There *is* a significant difference, though, that that's the condition to decide when to stop (resp. if still merging resp. splitting is necessary). In Hopcrof's refinement approach, the check is *local*. The condition concerns *the next single edges originating in a (meta)-state*. If they violate the determinism-requirement: then split, otherwise not.

The condition on the merging approach is *not-local*. They require to check wether to states accept the same language. That cannot be checked by the looking one step ahead, checking the outgoing edges. That involves checking all reachable states, and is a much more complicated condition. Perhaps some memoization (remembering and caching (partial) earlier checks) can help a bit, but Hopcroft's partitioning refinment seems not only more clever, it looks also superior. □

Next is a "illustration" of the elementary split step. Namely the situation when it's not *trivial*, in the sense that something happens.

Actually, the algo is pretty simple. Perhaps it works also in a slightly surprizing manner. If one would try to solve the problem oneself, one perhaps would think along the following

lines. One is given a deterministic (and complete) automaton, and one want to find the minimal equivalent one. The starting point in general is not minimal yet, so the task is to make it *smaller* but still accepting the same language. A natural idea might then be: find out, somehow, which two states in the automaton are "equivalent" (insofar that they accept the same language, if one takes them as initial state). Of one detects two or more such equivalent states, one may identify them, i.e., use one state instead of two or more states, making the automaton a bit smaller but still remain equivalent. And then repeat the "identify-equivalent-states" step until all states are non-equivalent.

The algo presented here *works the other way around*, insofar that it does not glue states together but splitting states. So it is a but more mysterious: it starts with a very small automaton (only 2 states, no matter how big the input of the algo is). The 2-state initial automaton is in general smaller than the minimal equivalent one and it is not equivalent. The splitting step in the algorithm make the automaton in the construction *larger* in each iteration, until splitting is no longer done. During that process, the intermediate automata (which are non-deterministic) are all *not equivalent* to the target automaton. However, the first one encountered where no split is possible any longer, that one is not just deterministic, it's also *equivalent* to the input automaton and it's the *minimal one*.

Now, how does the splitting work? As said, it's pretty straightforward: One takes one current partitioning. One is thus dealing with sets of sets of states, let's call them "meta-states" and use Q as letter (as on the slides). The algo chooses one meta-state Q and a letter a in the alphabet and ask the question:

> Does this meta-state behaves Q deterministic wrt. a?

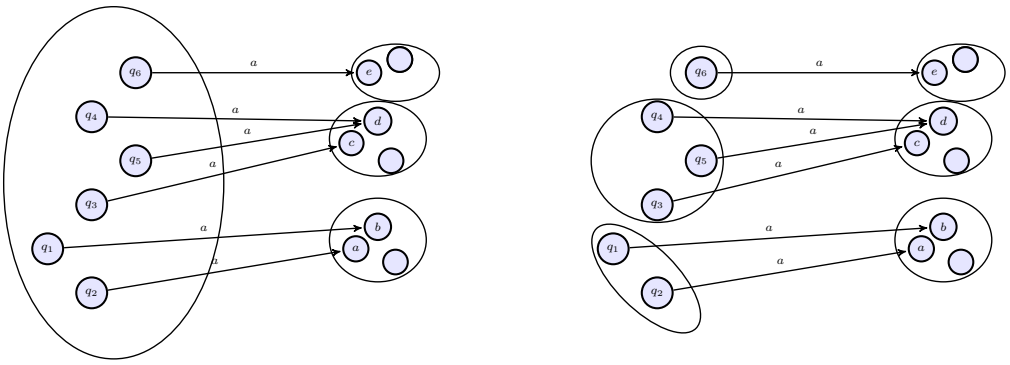
If yes, the step is done. If not, *make the meta-state deterministic* wrt. a, by splitting it. However, don't split it more than necessary, just enough to make the result deterministic wrt. a. And then repeat the step.

That's basically all. However, one can organize the process of splitting in a more clever way, more clever at least, than *randomly picking* in each iteration a meta-state Q together with a letter a. Obviously, if one has treated the pair (Q, a) in one iteration, it's stupid to pick the same pair in the next iteration again. One should remember (Q, a) as "done" and pick another one. The ones that one have *not yet been explored*, that's the **worklist** mentioned on the slides. However, it's not 100% trivial. Trivial would be: One makes a list of all pairs (Q, a) in advance, and then one ticks off one element of that list after the other, until finished. That would be so simple that one would not bother to call it a *worklist algorithm*.

The complication comes from the *cascading effect* of splitting. If in one step of the iteration, the algo splits some "meta-state", then some previously treated pairs may have to be reconsidered: they may have previously be deterministic (and thus removed from the worklist), but the splitting now may or may not have changed that. Therefore, they need to be re-added to the worklist.

The basic step is illustrated in Figure 2.23. In Figure 2.23a, the step targets the partition consisting of $\{q_1, \ldots, q_6\}$. In this illustation, all involved states have outgoing edges labelled a. Of course, each of the individual states q_i has exactly one successor state (the automaton being deterministic and complete). However, as shown in the picture,

considering the partions as state, the partition $\{q_1, \ldots, q_6\}$ has *three* a-successors, the the partitions shown on the right. Therefore, the step splits the partition to make that part of the construction deteriministing but not splitting it more than necessessary (for instance by "atomizing" the partition into individual states). The result of that split is shown in Figure 2.23b.



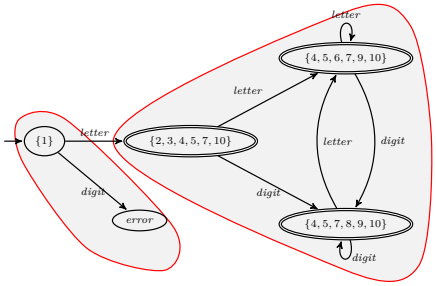
(a) Before the split

(b) After the split

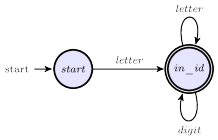
Figure 2.23: Splitting the partition $\{q_1, \ldots, q_6\}$ on a

Let's have a look at a few more examples. The following examples are shown in overlays in the slides. They unfolding of the overlays is not done for the script version here.

Example 2.7.2 (Again: DFA for identifiers). We have seen the DFA for identifiers earlier. It's repeated in Figure 2.24b, with the initial partitioning indicated by the red bubbles. The only partition that needs to be splitted is the one of the left with two states, and the split is on the label *letter*. The minimal automata (which we seen before) is shown in Figure **??** (where we have omitted the error state,



(a) Completed automaton



(b) Minimized (error state omitted)

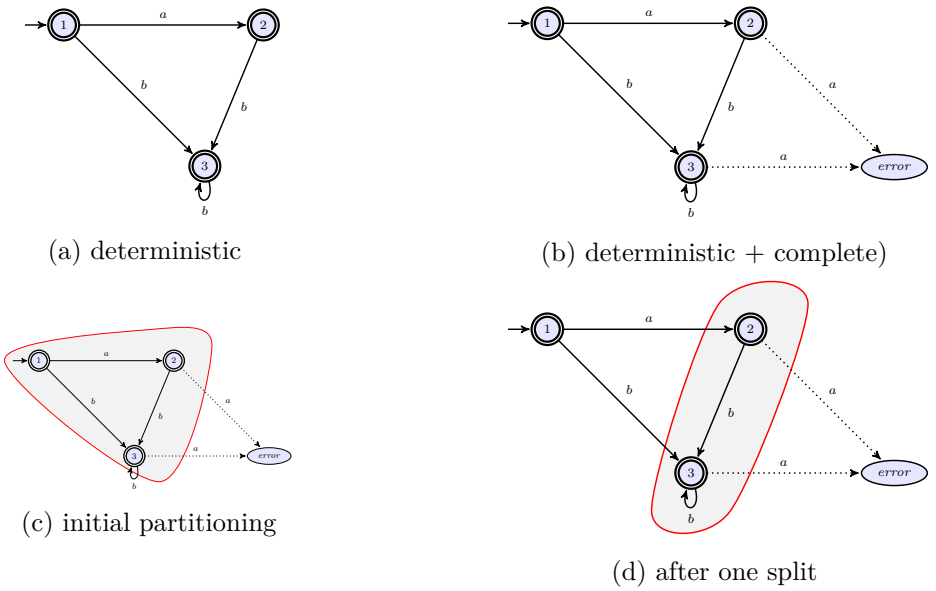
Figure 2.24: DFAs for identifiers

□

Example 2.7.3 (Partition refinement). Consider the following regular expression.

$$(\boldsymbol{a} \mid \boldsymbol{\epsilon})\boldsymbol{b}^* \tag{2.22}$$

The minimization via partitioning refinement is shown in Figure 2.25. Trivial partitions that only contain a single state are not marked by a red bubble.



(a) deterministic

(b) deterministic + complete)

(c) initial partitioning

(d) after one split

Figure 2.25: DFAs for the regular expression from equation (2.22)

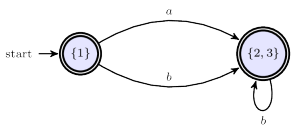


Figure 2.26: Minimal DFA, error state omitted

□

2.8 Scanner implementations and scanner generation tools

This last section contains only rather superficial remarks concerning how to implement as scanner or lexer. A few more details can be found in [7, Section 2.5]. The oblig will include the implementation of a lexer/scanner, so one will get a hands-on experience how to use tools like that

Scanners are simple and well-understood part of compiler, so hand-coding is certainly possible. Mostly, however, one is better off with a generated scanner (and parser). Standard tools are tools **lex** / **flex** (also in combination with *parser* generators, like **yacc**/**bison**. The original ones are for C, but this pair of tools exits for many implementing languages. The scanner part is based on the results covered in this chapter.

The scanner generator and the parser generator work hand in hand, to they generate a lexer and a scanner, where the output of lexer/scanner is the input for parser. As discussed, programmer specifies regular expressions for each *token*-class and corresponding actions (and whitespace, comments etc.).

The specification language typically offers some conveniences, like extended regular expression. The parser part allows to specify priorities, associativities etc.) to ease the task.

The lexer specification is then automatically translated to NFA (e.g. Thompson), turned to an DFA, and then minimized (with a little care to keep the token classes separate). The DFA us implemented usually with the help of some table representation.

Tokens and tje actions of a parser will be covered later. For example, identifiers and digits as described by the regular expressions, would end up in two different token classes, where the actual string of characters (also known as *lexeme*) being the value of the token attribute.

Here is a small example

```
1
2  DIGIT      [0-9]
3  ID         [a-z][a-z0-9]*
4
5  %%
6
7  {DIGIT}+     {
8               printf( "An integer: %s (%d)\n", yytext,
9                       atoi( yytext ) );
10              }
11
12 {DIGIT}+"."{DIGIT}*         {
13              printf( "A float: %s (%g)\n", yytext,
14                      atof( yytext ) );
15              }
16
17 if|then|begin|end|procedure|function         {
18              printf( "A keyword: %s\n", yytext );
19              }
```

Listing 2.4: Sample flex file (excerpt)

The example is taken from `http://dinosaur.compilertools.net/flex/flex_5.html#SEC5`

The example shown not the how to specify lexer that works together with a parser. The outcome of a detecting a lexeme here is not to produce a token that is handed over to the parser. Here, the lexer will simply print some diagnostic message in the “action parts”, the parts in the curly braces.

The exact syntax for the scanner generator depends on the particular tool, the example is flex-syntax. The JLex tool for compilers written in Java is pretty similar, as most such tools, as they all have comparable functionality.

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