INF5410 Array signal processing. Chapter 2.1-2.2: Wave equation

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Chapters in Johnson & Dungeon

- Ch. 1: Introduction.
- Ch. 2: Signals in Space and Time.
  - Physics: Waves and wave equation.
    - » c, λ, f, ω, k vector,...
    - » Ideal and "real" conditions
- Ch. 3: Apertures and Arrays.
- Ch. 4: Beamforming.
  - Classical, time and frequency domain algorithms.
- Ch. 7: Adaptive Array Processing.

Norsk terminologi

- Bølgeligningen
- Planbølger, sfæriske bølger
- Propagerende bølger, bølgetall
- Sinking/sakking: $\vec{\alpha}$
- Dispersjon
- Attenuasjon eller demping
- Refraksjon
- Ikke-linearitet
- Diffraksjon; nærfelt, fjernfelt
- Gruppeantenne (= array)

Kilde: Bl.a. J. M. Hovem: ``Marin akustikk'', NTNU, 1999

Wave equation

- This is *the* equation in array signal processing.
- Lossless wave equation

$$\nabla^2 \vec{s} = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$

- $\Delta = \nabla^2$ is the Laplacian operator (del=nabla squared)
- *s* = s(x,y,z,t) is a general scalar field (electromagnetics: electric or magnetic field, acoustics: sound pressure ...)
- *c* is the speed of propagation

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Three simple principles behind the acoustic wave equation

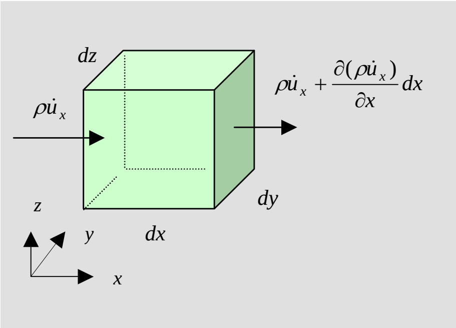

Figure: J Hovem, TTT4175
Marin akustikk, NTNU

1. Equation of continuity: conservation of mass
2. Newton's 2. law: $F = m\ a$
3. State equation: relationship between change in pressure and volume (in one dimension this is Hooke's law: $F = k\ x$ – spring)

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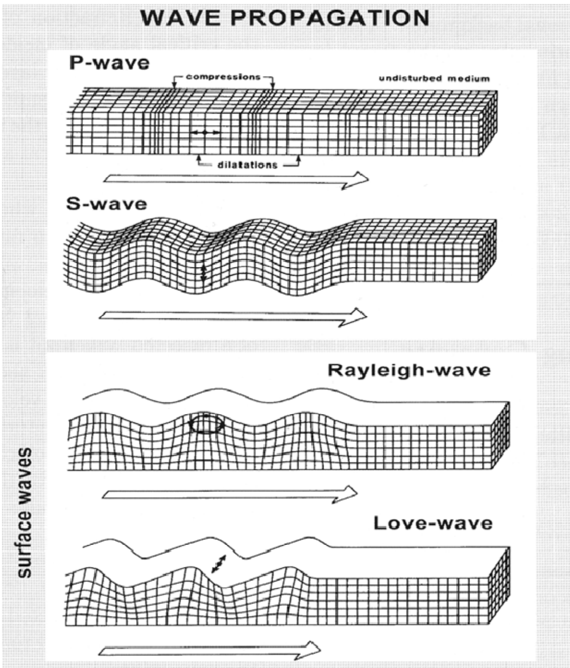

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Wave Type	Medium	Propagation Speed c	
		Formula	Value
Electromagnetic	Free space	$1/\sqrt{\epsilon\mu}$	3×10^8 m/s $\epsilon = 1/36\pi \times 10^{-9}$ C^2/N-m^2, $\mu = 4\pi \times 10^{-7}$ W/A-m
Electromagnetic	Glass	$1/\sqrt{\epsilon\mu}$	2×10^8 m/s[a]
Acoustic	Air	$\sqrt{\gamma R T_0/M}$	330.7 m/s $R = 8.3 \times 10^7$ erg/°K, $\gamma = 1.4$, $T_0 = 273$ °K, $M = 29$ g
Acoustic	Sea water	$\sqrt{\gamma B/\rho}$	1, 498 m/s[b] $\gamma = 1.01$, $B = 2.28 \times 10^9$ N/m^2, $\rho = 1.026 \times 10^3$ kg/m^3
Water waves (Shallow water)		$\sqrt{gH}$	$3.13\sqrt{H}$ m/s $g = 9.8$ m/s^2, H = water depth (m)
Acoustic (longitudinal)	Granite	$\sqrt{\frac{K+4G/3}{\rho}}$	3, 310 m/s[c]
Acoustic (transverse)	Granite	$\sqrt{G/\rho}$	5, 770 m/s[c]

Reverse as $v_l > v_t$

[a]This value is derived from the refractive index, which for glass equals about 1.5.

[b]Because of the complicated properties of most fluids, experimental, serieslike expressions for sound propagation speeds are common. For sea water, the expression is

$$c = 1,449.2 + 4.623T - 0.0546T^2 + (1.391 - 0.012T) * (S - 35) + \cdots$$

where T is temperature in °C, and S is salinity in parts per thousand.

[c]This value derived from direct measurements.

TABLE 2.1 Formulas and Values for Propagation Speed c of Various Types of Waves in Different Media.

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Wave modes – media

- Electromagnetic, E and H: transverse waves
- Mechanical bulk waves:
  - Pressure waves, longitudinal – acoustic wave in air, water, body,
  - Shear waves, transverse – only in solids
- Mechanical guided waves
  - Surface wave:
    - » Rayleigh waves (ocean waves)
    - » Stoneley waves
  - Plate modes:
    - » Lamb waves are dispersive plate waves
    - » Love waves are horizontally polarized shear waves which also exist on the surface.

Solution

Guesses:

1. Separable $s(x,y,z,t) = A \cdot s_t(t) \cdot s_x(x) \cdot s_y(y) \cdot s_z(z)$
2. Complex exponential in time: $s(t) = \exp\{j\omega t\}$
3. Complex exponential in space
   $s_x(x) = \exp\{-jk_x t\}$, (also in y and z)

Assumed solution:

$$s(\vec{x}, t) = A \exp\{j(\omega t - k_x \cdot x - k_y \cdot y - k_z \cdot z)\}$$

Solution

Insert $s(\vec{x}, t) = A \exp\{j(\omega t - k_x \cdot x - k_y \cdot y - k_z \cdot z)\}$

into $\nabla^2 \vec{s} = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 s}{\partial t^2}$

$\Rightarrow k_x^2\, s(\cdot) + k_y^2 s(\cdot) + k_z^2\, s(\cdot) = \omega^2 s(\cdot)/c^2$

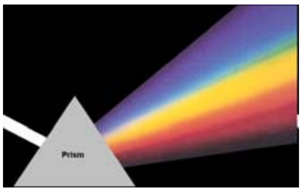
or $k_x^2 + k_y^2 + k_z^2 = |k|^2 = \omega^2/c^2$ or $|k| = \omega/c$
which is the condition for this guess to be a solution

Temporal behavior: Monochromatic

For a sensor placed in one point in space:

s(t) = exp{jωt} = Acosωt + j·Asinωt

- Single frequency
- Monochromatic = one color



Spatial behavior: Plane wave

$$s(\overrightarrow{x}, t) = A \exp\{j(\omega t - k_x \cdot x - k_y \cdot y - k_z \cdot z)\}$$

At a given time instant, the solution is the same for all points on

$$\overrightarrow{k} \cdot \overrightarrow{x} = k_x \cdot x + k_y \cdot y + k_z \cdot z = C$$

= constant phase = equation for a plane.

The vector $\overrightarrow{k}$ is perpendicular to the planes of constant phase

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Propagating wave

- If this is a propagating wave, the plane of constant phase moves by $\delta\vec{x}$ in time δt :

$$s(\vec{x}+\delta\vec{x}, t+\delta t) = s(\vec{x}, t)$$

=> $\omega(t+\delta t) - \vec{k}\cdot(\vec{x}+\delta\vec{x}) = \omega t - \vec{k}\cdot\vec{x}$

or $\omega\delta t - \vec{k}\cdot\delta\vec{x} = 0$

May take directions of k and δx vectors to be the same (minimizes length of δx): $\omega\delta t - |\vec{k}|\cdot|\delta\vec{x}| = 0$

Speed of propagation: $\frac{|\delta\vec{x}|}{\delta t} = \frac{\omega}{|\vec{k}|}$ and with |k| = ω /c

= speed of wave: $\frac{|\delta\vec{x}|}{\delta t} = c$

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Wavelength – spatial frequency

- Propagation in space in one period, $T=2\pi/\omega$:
- Wavelength
  $\lambda = \delta x = c\cdot\delta t = c\cdot T = 2\pi\cdot c/\omega = 2\pi/|k|$
- Interpretation of wave number vector $\vec{k}$:
  - The number of cycles in radians per meter
  - = Spatial frequency
  - Angular frequency ω is no of cycles in radians per second
- Unit vector for direction of propagation (zeta): $\vec{\zeta}^0 = \frac{\vec{k}}{|\vec{k}|}$

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Slowness vector

- Alternative notation

$$s(\vec{x},t) = A\exp\{j(\omega t - \vec{k}\cdot\vec{x})\} = A\exp\{j\omega(t - \vec{\alpha}\cdot\vec{x})\} = As(t - \vec{\alpha}\cdot\vec{x})$$

- Expressed as a function of a single variable
- $|\alpha| = |k|/\omega = 2\pi/(\omega\lambda) = 1/c$
- This is the slowness vector (Norsk: sinking)
  - Points in the direction of propagation
  - Has units of reciprocal velocity (s/m)
  - Parallels optical index of refraction: $n=c_0/c$, except there is no free-space c_0 in acoustics.

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Wave equation and arbitrary solutions

- The wave equation is linear
- Solution may be a sum of complex exponentials
- Almost any signal may be expressed as a sum of complex exponentials using Fourier theory
- Therefore any signal, no matter its shape, may be a solution to the wave equation – and the shape will be preserved as it propagates
- Propagating waves are therefore ideal carriers of information
- Modified by the boundary conditions – to determine which components that are excited
- Propagation is determined by the deviations of the medium from ideal

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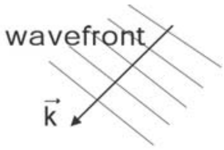

Plane waves

- Propagating plane wave: $s(t - \vec{\alpha} \cdot \vec{x})$
- Propagating sinusoidal plane wave: $\sin(\omega t - \vec{k} \cdot \vec{x})$
- Slowness vector: $\vec{\alpha} = \vec{k}/\omega, \ |\vec{\alpha}| = 1/c$
- Dispersion relation: $\omega = c \cdot k$
- Wavenumber vector: $\vec{k} = \omega \vec{\alpha}, \ |\vec{k}| = 2\pi/\lambda$
- Frequency and wavelength: $c = \lambda \cdot \omega / 2\pi = \lambda \cdot f$

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Wave equation, spherical coordinates

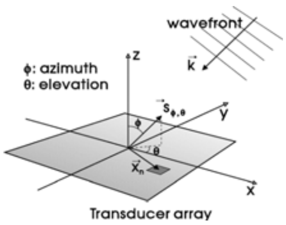
Assumption: Solution exhibits spherical symmetry:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial s}{\partial r}) = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$

Monochromatic solution, spherical wave, propagating away from origin:

$$s(r,t) = \frac{A}{r}\exp\{j(\omega t - kr)\}$$

Another soluton propagating towards the origin is found by replacing '-' with '+'. Also valid, boundary conditions determine which ones that exist



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Spherical solution

- Distance between zero-crossings, $\cos(\omega t - kr)/r$, is given by $kr=\pi \Leftrightarrow r=\pi/k = \pi/(2\pi/\lambda)=\lambda/2$
- Distance between peaks – problem 2.4

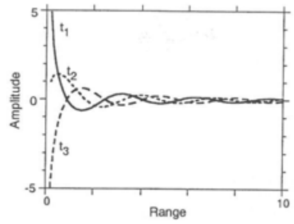

Figure 2.4 A plot of $s(r,t) = [\cos(\omega t - kr)]/r$ as a function of r for three time instants $t_1 < t_2 < t_3$.

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Doppler shift

- $f_0+f_D \approx f_0(1+v/c)$ where v is the component along the wave propagation
- Christian A. Doppler (1803-1853), Austria
- Determination of the velocity of blood flow (medical ultrasound)
- Air plane velocity by radar
- 1 page derivation (better than Johnson & Dungeon): A. Donges, "A simple derivation of the acoustic Doppler shift formulas," Eur. J. Phys. 19 467, 1998

http://www.iop.org/EJ/abstract/0143-0807/19/5/010

Doppler 1, v<<c, moving observer

- Source: y_s=Asin(ωt)
- Propagation time to observer: τ=x/c
- Observer at rest: y_o=Asin(ϕ_o),
  - Only phase $\phi_o=\omega(t-\tau)= \omega(t-x/c)$ is shifted
- Observer moves with v_o away from source
  - Distance increases: $x=x_o+v_ot$
- Observed phase: $\phi_o=\omega(t-x/c) = \omega(t-\{x_o+v_ot\}/c) = \omega(1-v_o/c)t - \omega x_o/c$
- Observed angular frequency is $\omega' = \omega(1-v_o/c)$

Doppler 2, v<<c, moving source

- Source moves towards observer at v_s
  - Distance decreases: $x=x_o-v_st$
  - Apparent velocity > c, not valid for EM-waves (Einstein)
- During the propagation time, τ, the source travels a distance $\Delta x=v_s\tau$
- Propagation time is now not τ=x/c, but $\tau=\{x+v_s\tau\}/c$
  - Lasts longer since source is approaching
  - solve for τ: $\tau = x/\{c-v_s\}$
- Observed phase: $\phi_o=\omega(t-\tau) = \omega(t-x/\{c-v_s\}) = \omega(t-\{x_o-v_st\}/\{c-v_s\}) = \omega/\{1-v_s/c\}t - \omega x_o/\{c-v_s\}$
- Observed angular frequency is $\omega' = \omega/\{1-v_o/c\}$

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Doppler effect

- Nonrelativistic: Combine the two former derivations:

$$\omega' = \omega \left(\frac{1-\vec{\alpha}\cdot\vec{v}_{sensor}}{1-\vec{\alpha}\cdot\vec{v}_{source}} \right)$$

- Approximation: 1/(1-y) ≈ 1+y for y <<1
- Combine: (1-x)/(1-y) ≈ (1-x)(1+y) = 1-(x-y)-xy ≈ 1-(x-y)
- Small velocities: v<<c:

$$\omega \approx \omega \left(1 - \vec{\alpha} \cdot (\vec{v}_{sensor} - \vec{v}_{source})\right)$$

- Equation to remember: $f_0+f_D \approx f_0(1+v/c)$ where frequency increases when source and observer move towards each other

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Doppler example

- Echo Doppler imaging: $f_0+f_D \approx f_0(1+2v/c)$
  1. Observer = scatterer moves
  2. Source = scatterer moves

  (Problem 2.5)
- Ultrasound, f=3 MHz, blood flow 1 m/s
  – Ultrasound beam is parallel to blood flow
- $f_D = f_0 \cdot 2v/c$ =3e6 · 2· 1/1560 = 3846 Hz (Audible)
- Often beam is almost perpendicular to blood flow ⇒ must multiply by cosθ
  – Ex: θ=75 deg (15 deg from perpendicular) ⇒ $f_D = f_0 \cdot 2v \cdot \cos\theta / c$ = 995 Hz

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Array Processing Implications (1)

Whenever the wave equation applies, the following is valid:

- Propagating signals are functions of a single variable, s(·), with space and time linked by the relation $t - \vec{\alpha} \cdot \vec{x}$
  - A bandlimited signal can be represented by temporal samples at one location or
  - by spatial samling at one instant
- The speed of propagation depends on physical parameters of the medium.
  - If the speed is known, direction can be found
  - If the direction is known, speed can be found

Array Processing Implications (2)

- Signals propagate in a specific direction $\vec{\zeta}^0$ represented equivalently by either $\vec{\alpha}$ or $\vec{k}$
  - Can find direction from waveform from properly sampled locations
- Spherical waves describe the radiation pattern of most sources (at least near their locations):
  - Far-field: resemble plane waves
- The Superposition Principle applies, allowing several propagating waves to occur simultaneously without interaction.
  - Spatiotemporal filters may separate multiple sources

Deviations from simple media

1. Dispersion: $c = c(\omega)$
   - Group and phase velocity, dispersion equation: $\omega = f(k) \neq c \cdot k$
   - Evanescent (= non-propagating) waves: purely imaginary k
2. Loss: $c = c_{\Re} + jc_{\Im}$
   - Wavenumber is no longer real, imaginary part gives attenuation.
   - Waveform changes with distance
3. Non-linearity: $c = c(s(t))$
   - Generation of harmonics, shock waves
4. Refraction, non-homogenoeus medium: $c=c(x,y,z)$
   - Snell's law