INF5410 Array signal processing. Chapter 2.5 Wavenumber-Frequency Space

Sverre Holm

Wavenumber-Frequency Space

- Four-dimensional Fourier transform:

$$S(\vec{k}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(\vec{x}, t) \exp\{-j(\omega t - \vec{k} \cdot \vec{x})\} d\vec{x}\, dt$$

- Spatial frequency variable $\vec{k}$ dual to $\vec{x}$
- Just like temporal frequency, ω, and t
- Note the different signs for the exponents, due to the concern with propagating waves

Inverse Fourier transform

$$s(\overrightarrow{x}, t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\overrightarrow{k}, \omega) \exp\{j(\omega t - \overrightarrow{k} \cdot \overrightarrow{x})\} d\overrightarrow{k} \, d\omega$$

- Any reasonable spatiotemporal signal can be decomposed into infinitely many plane waves as long as the Fourier transform converges.
- This also holds for an arbitrary, nonperiodic, nonpropagating distribution of energy in space and time. It can be represented as a superposition of periodic, propagating plane waves

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Monochromatic Plane Wave

$$s(\overrightarrow{x}, t) = c^{j(\omega_0 t - \overrightarrow{k}^0 \cdot \overrightarrow{x})}$$

- Fourier transform:

$$S(\overrightarrow{k}, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{j(\omega_0 t - \overrightarrow{k}^0 \cdot \overrightarrow{x})\} \exp\{-j(\omega t - \overrightarrow{k} \cdot \overrightarrow{x})\} d\overrightarrow{x} \, dt$$

- Separable, product of four factors:

$$\int_{-\infty}^{\infty} \exp\{-j(\omega - \omega_0)t\} dt = \delta(\omega - \omega_0)$$

- Result: A single point in $(\overrightarrow{k}, \omega)$ space

$$S(\overrightarrow{k}, \omega) = \delta(\overrightarrow{k} - \overrightarrow{k}^0) \cdot \delta(\omega - \omega_0)$$

Propagating Wave

$$s(\overrightarrow{x},t) = s(t - (\overrightarrow{k}^0/\omega) \cdot \overrightarrow{x}) = s(t - \overrightarrow{\alpha}^0 \cdot \overrightarrow{x})$$

- Only energy along a line in wavenumber-frequency space

$$S(\overrightarrow{k},\omega) = S(\omega)\delta(\overrightarrow{k} - \overrightarrow{k}^0) = S(\omega)\delta(\overrightarrow{k} - \omega\overrightarrow{\alpha}^0)$$

- This line is given by the dispersion equation for the wave in the medium

$$\overrightarrow{k}^0 = \omega\overrightarrow{\alpha}^0$$

  – dispersionless medium $|k| = \omega/c$

Wavenumber-Frequency Space

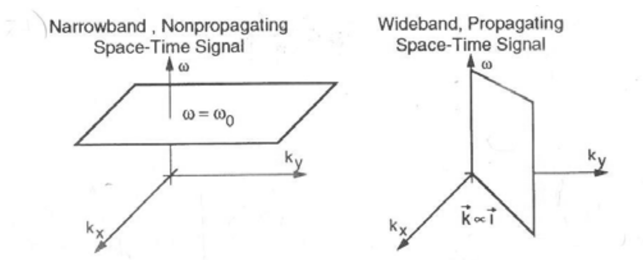

Figure 2.14 When $s(\vec{x}, t)$ contains temporal frequencies near ω_0 only (left portion), the wavenumber-frequency spectrum $S(\vec{k}, \omega)$ has significant energy only near the plane $\omega = \omega_0$ in $(\vec{k}, \omega)$ space. Here we have displayed the three-dimensional space (k_x, k_y, ω) rather than the full four-dimensional space (k_x, k_y, k_z, ω) for the purposes of illustration. If a signal consists of components propagating in a particular direction $\vec{\iota}$ (right portion), then its wavenumber-frequency spectrum is zero except for the half-plane where $\vec{k}$ is proportional to $\vec{\iota}$.

Wavenumber-Frequency Space

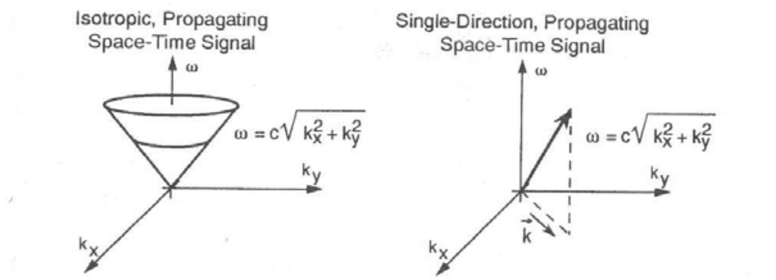

Figure 2.15 An isotropic signal, propagating in all directions with equal amplitudes and equal speeds, has a cone-shaped wavenumber-frequency spectrum. In contrast, a signal having a specific direction of propagation has spectrum defined on a line in the $(\vec{k}, \omega)$ plane.

2.5.3 Spectrum – Spherical Wave

- Only of theoretical interest
- Skip it

Filtering in Wavenumber-Frequency Space

- Straightforward extension to spatiotemporal signals:

$$Y(\overrightarrow{k},\omega) = H(\overrightarrow{k},\omega)\cdot X(\overrightarrow{k},\omega)$$

- 4-D convolution integral:

$$y(\overrightarrow{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\overrightarrow{x}-\overrightarrow{\xi},t-\tau)s(\overrightarrow{\xi},\tau)d\overrightarrow{\xi}d\tau$$

- Same considerations as with time-domain LTI-filters: Ideal filters ⇔ infinitely long impulse responses
- Ideal filters:
  - focus on one frequency $H(\cdot) = \delta(\omega-\omega_0)$
  - or focus in one direction $H(\cdot) = \delta(k-k^0)$
- Beamforming = realizable space-time filtering
  - Linear filtering in chapter 4
  - Nonlinear methods in chapter 7

2.6 Random Fields

- Will go through it if needed in chapter 7