Beamforming

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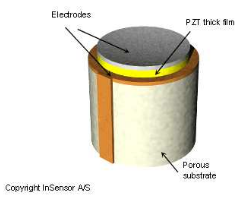
Slide 2: Beamforming

- Chapter 4: *Beamforming*
  - Delay-and-sum beamforming
  - Space-time filtering
  - Filter-and-sum beamforming
  - Frequency domain beamforming
  - Resolution
- Chapter 7: *Adaptive beamforming / Direction of Arrival (DOA) estimation*

Slide 3: Norwegian Terminology / *Norsk terminologi*

- Beamforming: *Stråleforming*
- Beampattern: *Strålingsdiagram*
- Delay-and-sum: *Forsinkelse-og-sum*

Slide 4: Focusing w/ single & directional sensor



Slide 5: Single-sensor characteristics

- Geometrical pre-focusing, *e.g.* spherical curving or lens
- Ability to distinguish between sources (lateral resolution): governed by physical size
- Simple: processing not required for focusing
- Inflexible: focusing depth fixed

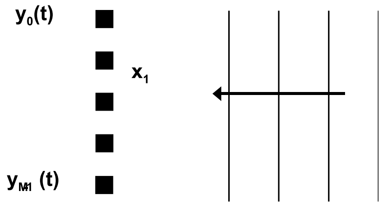
- To focus in a different direction: sensor must be physically moved

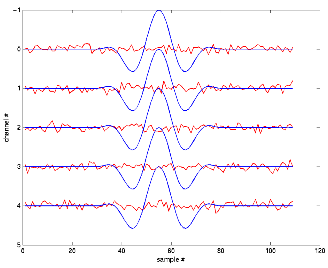
Slide 6: Array beamforming

- [Old French: *areer* = "to put in order"]

- Physical elements: apply delays & sometimes amplitude weights
- Flexible: May change focusing without altering of physical array
- Requires processing of recorded signals
- Allows for adaptive methods (chapter 7)
- On receive: Possible to aim @ more than one source "simultaneously"

Slide 7: Array gain example





$$y_m(t) = s(t) + n_m(t)$$

Slide 8: Array gain example, *continued*

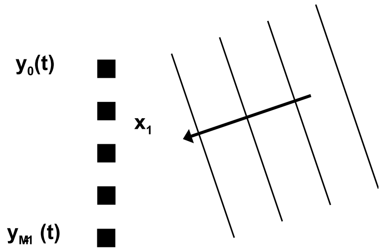
$$y_m(t) = s(t) + n_m(t)$$

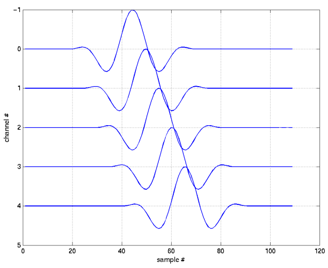
□

...

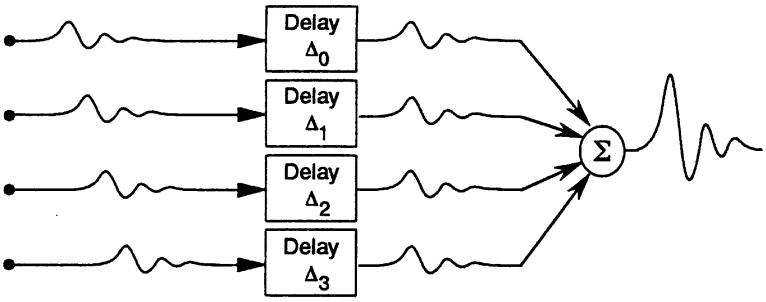
$$\mathrm{SNR}_m = \frac{\sigma_s^2}{\sigma_n^2}, \quad \mathrm{SNR} = M\frac{\sigma_s^2}{\sigma_n^2}$$

Slide 9: Non-zero angle of arrival



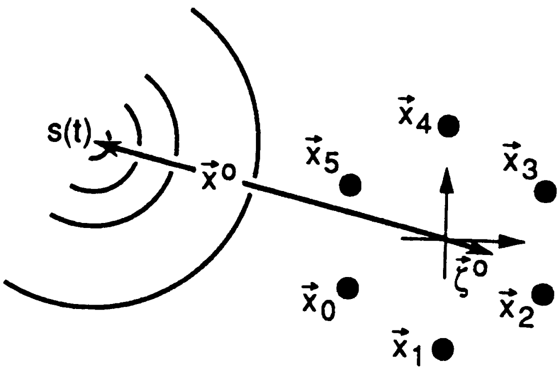


Slide 10: Delay-and-sum



stacking $\triangleq$ adjustment of $\Delta_0 \ldots \Delta_{M-1}$

Slide 11: Phase center



Slide 12: Delay-and-sum, definition

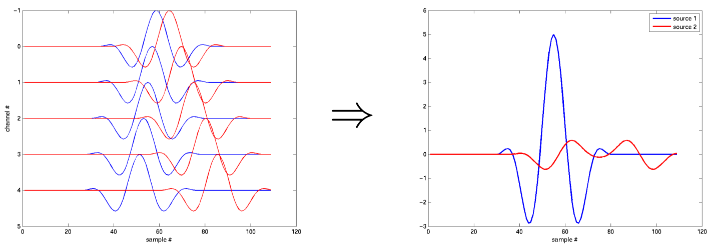
$$z(t) = \sum_{m=0}^{M-1} w_m y_m(t - \Delta_m)$$

- $z(t)$: beamformer output
- m: element #

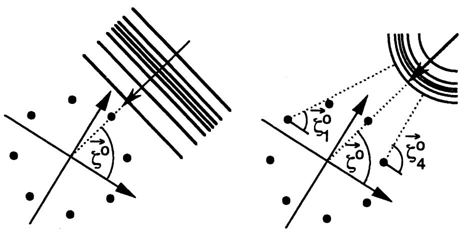
- M: # of elements
- w_m: amplitude weight $\#m$
- y_m: signal @ sensor m
- Δ_m: delay of signal @ sensor m

Slide 13: Delay-and-sum example

Steering ("listening") towards source 1



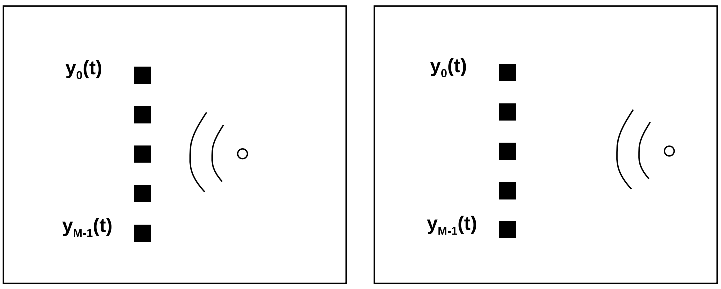
Slide 14: Near-field / far-field sources



- *Plane wave approaching* (left): [source in far-field] $\forall$ sensors: Propagation direction ($\triangleq \vec{\zeta}^\circ$): same relative to $\forall \vec{x}_m$
- *Spherical wave approaching* (right): [source in near-field] Propagation direction relative to sensor m ($\triangleq \vec{\zeta}^\circ_m$) differs.

  $\angle(\vec{\zeta}^\circ - \vec{\zeta}^\circ_m)$: discrepancy between far- & near-field @ sensor m

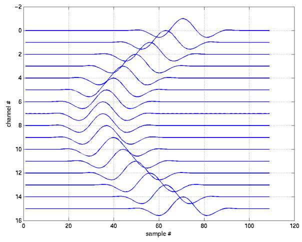
Slide 15: Near-field sources



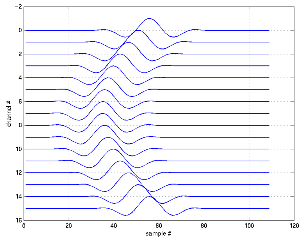
- Wavefront propagation direction $\vec{\zeta}^\circ_m$ differs with distance to source

Slide 16: Near-field sources, *continued*

Example received signals per sensor:



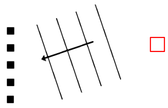
Source *close* to sensors



Source further away

Slide 17: Beamforming for plane waves

Source in the far-field



- Wavefield @ aperture:

$$f(\vec{x}, t) = s(t - \vec{\alpha}^\circ \cdot \vec{x}), \qquad \vec{\alpha}^\circ \triangleq \vec{\zeta}^\circ / c\text{: slowness}$$

- Delay @ element: $\Delta_m = -\vec{\alpha}^\circ \cdot \vec{x}_m \Rightarrow$

$$z(t) = \sum_{m=0}^{M-1} w_m s(t - \Delta_m - \vec{\alpha}^\circ \cdot \vec{x}_m) = s(t) \sum_{m=0}^{M-1} w_m$$

- Steering direction $\vec{\zeta} \Leftrightarrow \left(\vec{\alpha} = \vec{\zeta}/c\right)$: $\Delta_m = -\vec{\alpha} \cdot \vec{x}_m \Rightarrow$

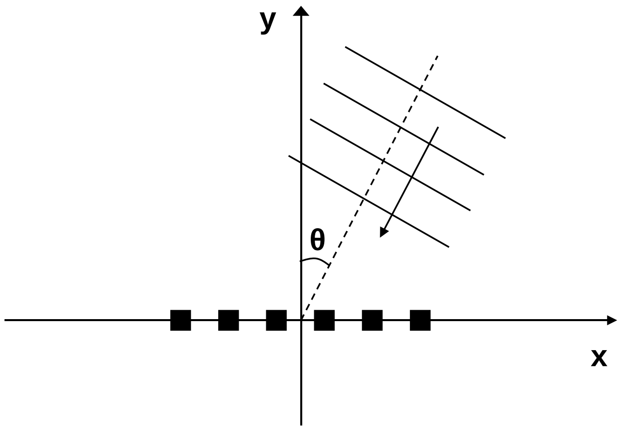
$$z(t) = \sum_{m=0}^{M-1} w_m s\left(t - (\vec{\alpha}^\circ - \vec{\alpha}) \cdot \vec{x}_m\right)$$

Slide 18: Beamforming for plane waves, *continued*

$$z(t) = \sum_{m=0}^{M-1} w_m s\left(t - (\vec{\alpha}^\circ - \vec{\alpha}) \cdot \vec{x}_m\right)$$

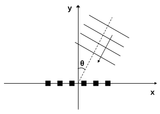
- If $\vec{\alpha} \neq \vec{\alpha}^\circ$ $\Rightarrow$ mismatch $\Rightarrow$ y_m not summed constructively $\forall$ sensors m
- Mismatch sources for incorrect $\vec{\alpha}^\circ \triangleq \vec{\zeta}^\circ / c$:
  - Assumption about $\vec{\zeta}^\circ$
  - Assumption about c
- If $\vec{\alpha}^\circ$ or c is known: Get the other by variation of c or $\vec{\alpha}$, until max energy output in $z(t)$, *e.g.*: $\max_c \int_{-\infty}^{\infty} |z|^2 \mathrm{d}t$

Slide 19: Linear array example



Slide 20: Linear array example, *continued*

- Monochromatic, plane wave: $s(t - \vec{\alpha}^\circ \cdot \vec{x}) = e^{j(\omega^\circ t - \vec{k}^\circ \cdot \vec{x})}$



□

- Steering delays: $\Delta_m = -\frac{\vec{k}}{\omega^\circ} \cdot \vec{x}_m$, & $\vec{x}_m = [x_m, 0, 0]^{\mathrm{T}}$ $\Rightarrow$

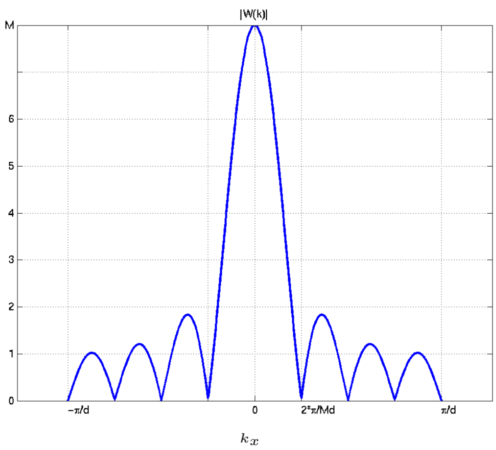
$$z(t) = e^{j\omega^\circ t} W(k_x - k_x^\circ),$$

w/ Array pattern def.: $W(\vec{k}) \triangleq \sum_{m=0}^{M-1} w_m e^{j\vec{k} \cdot \vec{x}_m}$

- Uniform weights, $w_m = 1$, $\forall m$ $\Rightarrow$

$$W(k_x - k_x^\circ) = \frac{\sin\left[\frac{M}{2}(k_x - k_x^\circ)d\right]}{\sin\left[\frac{1}{2}(k_x - k_x^\circ)d\right]}$$

Slide 21: Array pattern amplitude $|W(k_x)|$

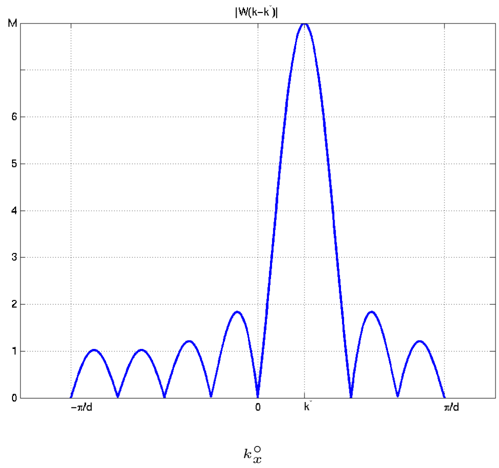


Example: $|W(k_x)|$, for $x_m = [x_m, 0, 0]^{\mathrm{T}}$

□

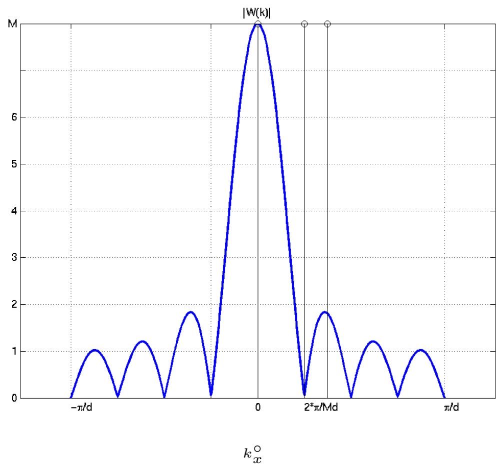
Slide 22: Beampattern amplitude $|W(\vec{k} - \vec{k}^\circ)|$

Array steered in fixed $\vec{k} = \omega^\circ \vec{\alpha}^\circ$. Plotting $|W(\vec{k} - \vec{k}^\circ)|$ for different $\vec{k}^\circ$.



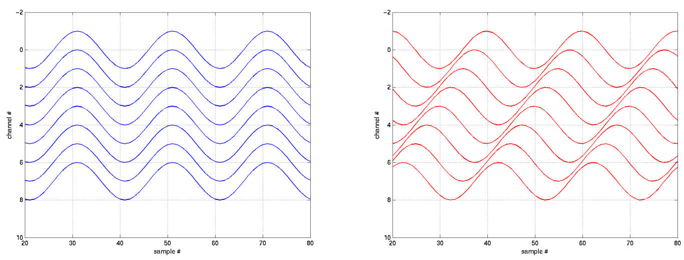
Slide 23: Example: 3 sources in different directions

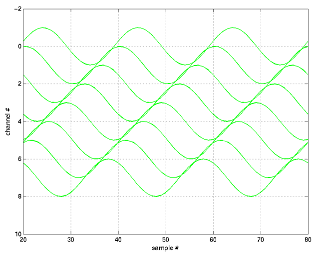
Steering: $k_x = 0$



Slide 24: Example, *continued*

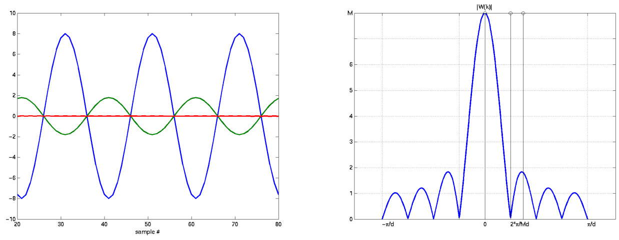
Recorded sensor signals



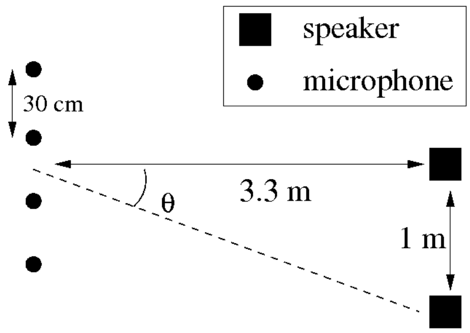


Slide 25: Example, *continued*

Delay-and-sum beamforming of each of the 3 signals. Steering: $k_x = 0$



Slide 26: Example: Microphone array



Questions

1. For what frequencies is the wavefield properly sampled? Assume $c = 340$ m/s.

2. Up to what frequency are the speakers considered to be in the array far-field?

Slide 27: Sound examples

- Single microphone
   [follow link]

- Delay-and-sum, steering to speaker 1

[follow link]

- Delay-and-sum, steering to speaker 2

  [follow link]

  http://johanfr.at.ifi.uio.no/lydlab/

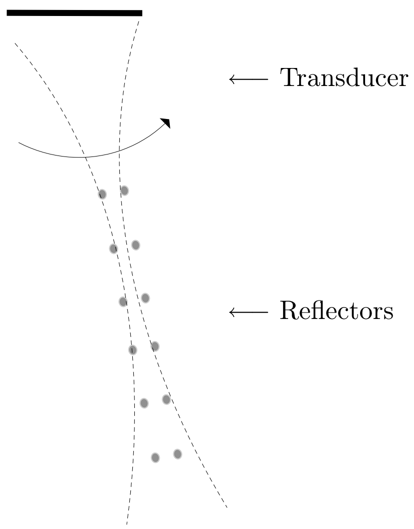
Slide 28: Beamforming for spherical waves

- Single-source in array near-field:
  Maximum beamformer output ↔ a *spatial location*
  [far-field case: ↔ a *propagation direction*]
- $\Delta_m = f(\vec{x}_{\text{source}})$
- Adjustments of Δ_m
  ⇒ may focus array to $\vec{x}$ in the near-field

□

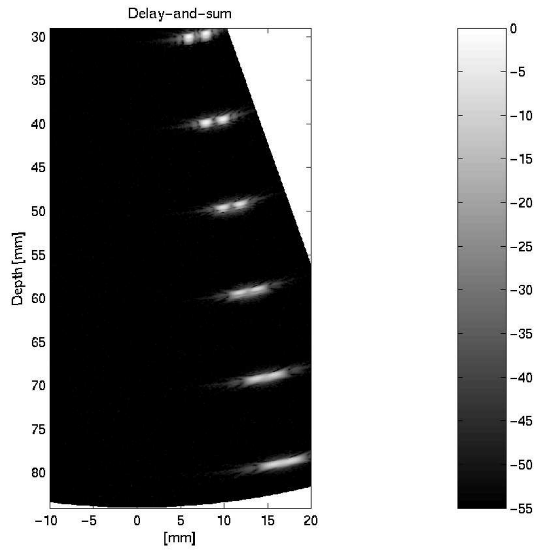
Slide 29: Example: Ultrasound imaging

Experimental setup



Slide 30: Example: Ultrasound imaging, *continued*

Resulting image



Slide 31: Section 4.2–4.6

- Space-time filtering
  - Array pattern
  - Wavenumber-frequency response of delay-and-sum beamformer
  - Beampattern & steered response
- Filter-and-sum beamforming
- Frequency domain beamforming
- Resolution

Slide 33: 4.2 Space-time filtering

Array pattern

- Response to a monochromatic wave after delay-and-sum beamforming
- $\leftrightarrow$ the wavenumber-frequency response of a spatio-temporal filter
- Determines the directivity pattern of an array

Slide 34: Far-field array pattern

Response to monochromatic *plane* wave after delay-and-sum beamforming

$$W(\vec{k}) = \sum_{m=0}^{M-1} w_m e^{j\vec{k}\cdot\vec{x}_m}$$

Slide 35: Near-field array pattern

Definitions:

$\vec{x}^\circ$		source position
$\vec{x}$		focus position
$\vec{x}_m$		sensor m position
r°	$\triangleq \lvert\vec{x}^\circ\rvert$	dist. phase center → source
r	$\triangleq \lvert\vec{x}\rvert$	dist. phase center → focus
r_m°	$\triangleq \lvert\vec{x}_m - \vec{x}^\circ\rvert$	dist. sensor m → source
r_m	$\triangleq \lvert\vec{x}_m - \vec{x}\rvert$	dist. sensor m → focus

□

$$\mathcal{W}(k, \vec{x}, \vec{x}^\circ) = \sum_{m=0}^{M-1} w_m \frac{r^\circ}{r_m^\circ} e^{jk[(r^\circ - r) - (r_m^\circ - r_m)]}$$

Slide 36: Wavenumber-frequency response $H(\vec{k}, \omega)$

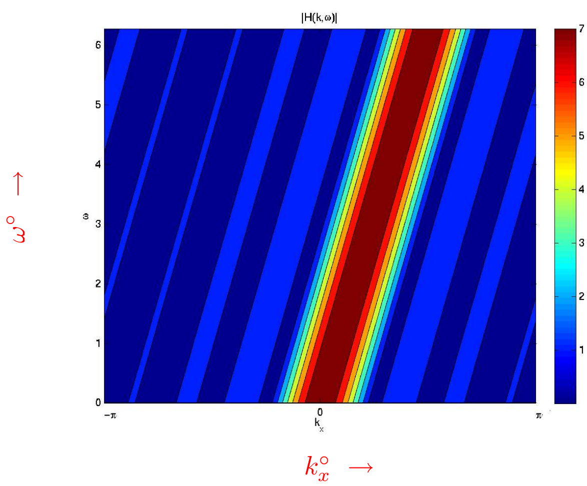
for delay-and-sum beamformer

□

$$H(\vec{k}^\circ, \omega^\circ) = \sum_{m=0}^{M-1} w_m e^{j(\omega^\circ \vec{\alpha} - \vec{k}^\circ)\cdot \vec{x}_m} \overset{\text{D-A-S}}{=\!=\!=} W(\omega^\circ \vec{\alpha} - \vec{k}^\circ)$$

Slide 37: Example: Wavenumber-frequency response

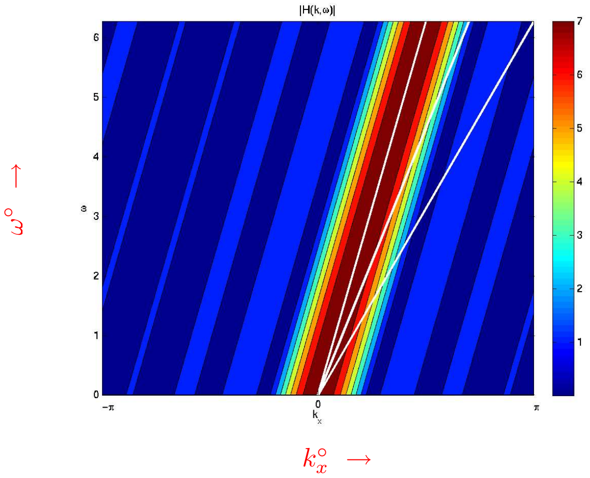
$$\left|H(\vec{k}^\circ, \omega^\circ)\right| = \left|W(\omega^\circ \vec{\alpha} - \vec{k}^\circ)\right| = \left|W(\omega^\circ \alpha_x - k_x^\circ)\right| = \left|\frac{\sin\left[\frac{M}{2}(\omega^\circ \alpha_x - k_x^\circ)d\right]}{\sin\left[\frac{1}{2}(\omega^\circ \alpha_x - k_x^\circ)d\right]}\right|$$



Slide 38: Example: Wavenumber-frequency response (2)

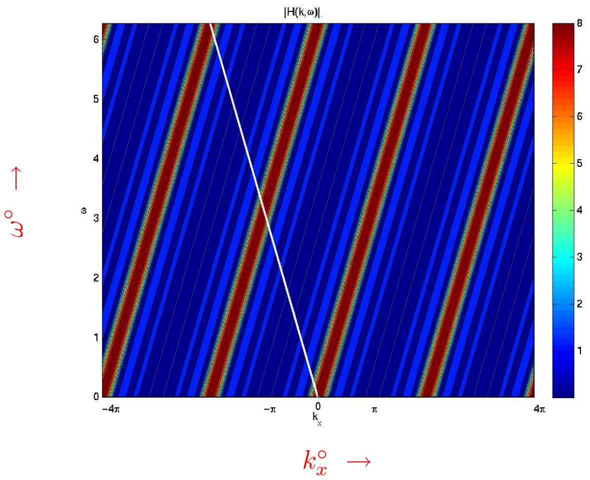
$$\left|H(\vec{k}^\circ, \omega^\circ)\right| = \left|W(\omega^\circ \vec{\alpha} - \vec{k}^\circ)\right| = \left|W(\omega^\circ \alpha_x - k_x^\circ)\right| = \left|\frac{\sin\left[\frac{M}{2}(\omega^\circ \alpha_x - k_x^\circ)d\right]}{\sin\left[\frac{1}{2}(\omega^\circ \alpha_x - k_x^\circ)d\right]}\right|$$

$$k_x^\circ = \omega^\circ \frac{\zeta_x^\circ}{c} \Leftrightarrow \zeta_x^\circ = \frac{c k_x^\circ}{\omega^\circ}$$



Slide 39: Example: Wavenumber-frequency response (3)

$$\left|H(\vec{k}^\circ,\omega^\circ)\right| = \left|W(\omega^\circ\vec{\alpha}-\vec{k}^\circ)\right| = |W(\omega^\circ\alpha_x - k_x^\circ)| = \left|\frac{\sin\left[\frac{M}{2}(\omega^\circ\alpha_x - k_x^\circ)d\right]}{\sin\left[\frac{1}{2}(\omega^\circ\alpha_x - k_x^\circ)d\right]}\right|$$



Slide 40: Beampattern & steered response

Sum-up

- *(Far-field) array pattern:*

$$W(\vec{k}) = \sum_{m=0}^{M-1} w_m e^{j\vec{k}\cdot\vec{x}_m}$$

- *Wavenumber-frequency response:*

$$H(\vec{k}^\circ,\omega^\circ) = \sum_{m=0}^{M-1} w_m e^{j(\omega^\circ\vec{\alpha}-\vec{k}^\circ)\cdot\vec{x}_m} \overset{\text{D-A-S}}{=\!=} W(\omega^\circ\vec{\alpha}-\vec{k}^\circ)$$

- *Beampattern:*

$$W(\omega^\circ\vec{\alpha}-\vec{k}^\circ), \quad \text{for fixed } \vec{\alpha}$$

- *Steered response:*

$$W(\omega^\circ\vec{\alpha}-\vec{k}^\circ), \quad \text{for fixed } \omega^\circ \text{ \& } \vec{k}^\circ$$

Slide 41: Beampattern & steered response, *continued*

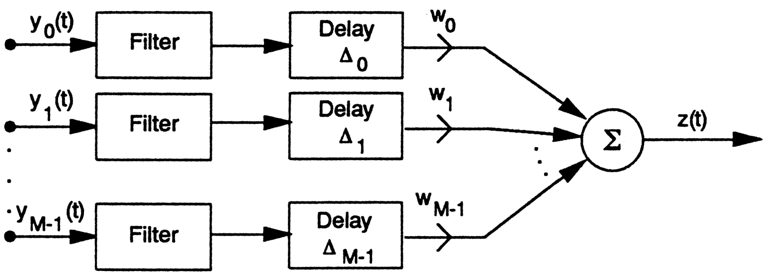
- *Beampattern:*

  When focusing in a particular direction: beampattern $\triangleq$ array output as f of incoming wavefield parameters

- *Steered response:*

  With wavefield parameters fixed: steered response $\triangleq$ array output as f of steering direction

Slide 43: 4.3 Filter-and-sum beamforming



- Linear filter of each $y_m(t)$ *before* delay-and-sum beamforming
- Often: same filter $\forall m$

Slide 44: Temporal filtering

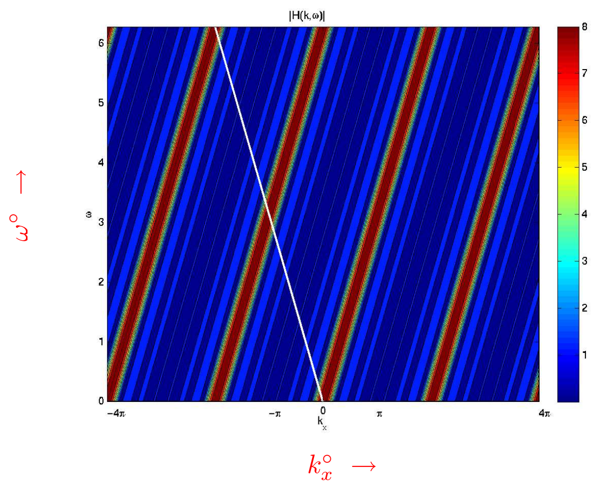
- Before beamforming: filter the frequency components of received signal
- May adjust filter to stop frequencies appearing in array grating lobes

□

- $\therefore$ For $\forall$ temporal filters equal: resulting total spatiotemporal filter is a multiplication: [filter response] & [array response]

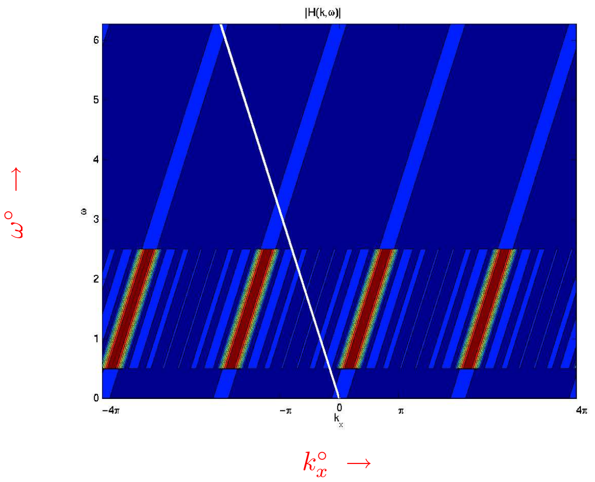
Slide 45: Example: Wavenumber-frequency response

Delay-and-sum



Slide 46: Example: Wavenumber-frequency response

Filter-and-sum



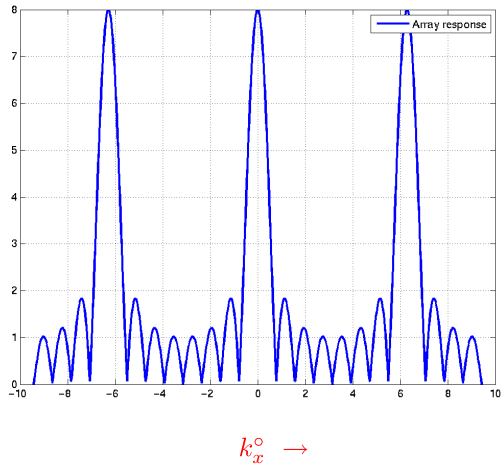
Slide 47: Spatial filtering

- Elements with spatial extension ⇒ element directivity:

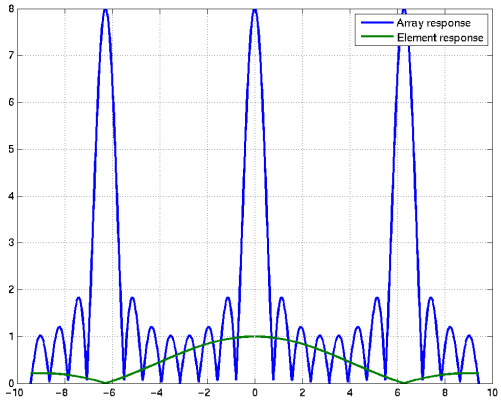
⇒ additional wavenumber-domain filtering

- Resulting spatial filter: multiplication: (array response) · (element response)
- May reduce grating lobe impact, see following slides

Slide 48: Example: Array pattern

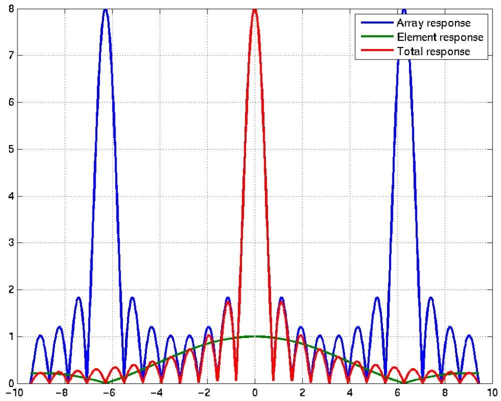


Slide 49: Example, *continued*, element response for $d = \lambda$



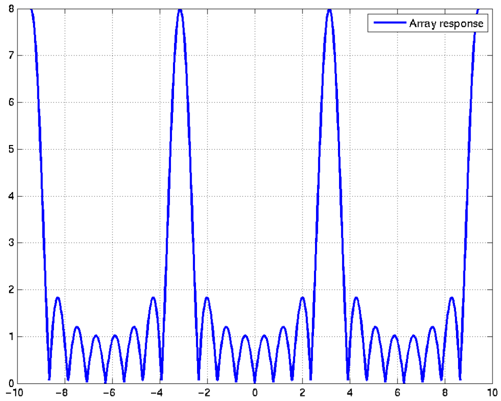
$k_x^\circ \;\rightarrow$

Slide 50: Example, *continued*, element response for $d = \lambda$



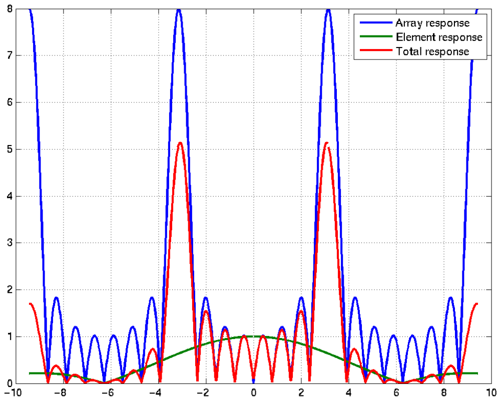
$k_x^\circ \;\rightarrow$

Slide 51: Steered array response



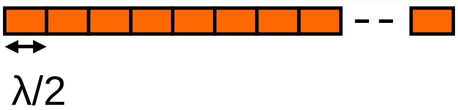
$k_x^\circ \;\rightarrow$

Slide 52: Total steered response



$k_x^\circ \rightarrow$

Slide 53: Example: Ultrasound transducer



Example: 96 elements

Slide 54: Example: Seismic streamer

- Array: groups of subarrays with hydrophones
- Signals summed within each subarray, with equal weight

Slide 55: Spatio-temporal filtering

- Filter both the temporal and spatial dimension (t and $\vec{x}$)

□

- May write filter-and-sum-response of an array of subarrays as:

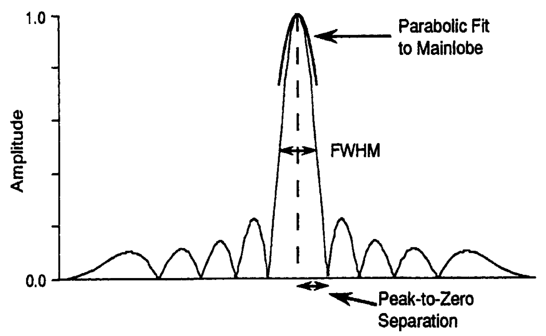
$$\left(\begin{array}{c} \text{array pattern w/ point-sensors} \\ \text{@ subarray phase-centers} \end{array} \right) \cdot (\text{subarray pattern})$$

Slide 57: 4.6 Resolution

1. Wavenumber resolution
2. Angular resolution

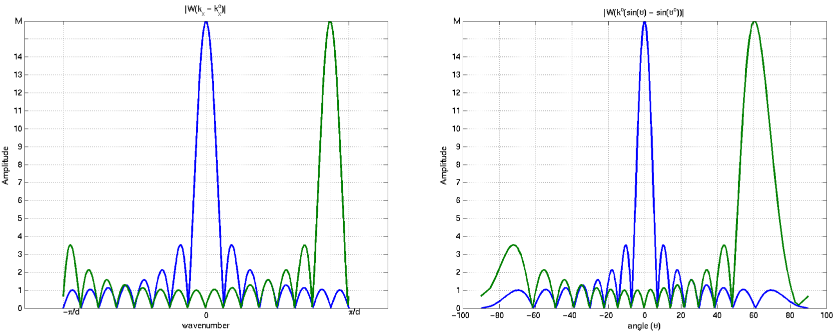
Slide 58: Wavenumber resolution

- Typical array pattern:



- Wavenumber resolution ≜ 1/ [width of mainlobe]
- Full-width, half-maximum (FWHM) ≜ beamwidth @ half of max amplitude

Slide 59: Wavenumber / angular resolution



□

Slide 60: Frequency-domain beamforming

- *Sensor signal m:*

$$y_m(t) \overset{\mathcal{F}}{\longleftrightarrow} Y_m(\omega), \qquad y_m(t-\Delta_m) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega\Delta_m} Y_m(\omega)$$

- *Beamformer output:*

$$z(t) = \sum_{m=0}^{M-1} w_m y_m(t-\Delta_m) \overset{\mathcal{F}}{\longleftrightarrow} Z(\omega) = \sum_{m=0}^{M-1} w_m Y_m(\omega) e^{-j\omega\Delta_m}$$

- *Short-time Fourier Transform:*

$$\mathcal{F}_s\{y(t)\}(t,\omega) = Y_m(t,\omega) \triangleq \int_{\infty}^{\infty} \tilde{w}(\tau) y_m(\tau+t) e^{-j\omega\tau} \mathrm{d}\tau e^{-j\omega t}$$

$$\mathcal{F}_s\{z(t)\} \triangleq \hat{Z}(t,\omega) = \sum_{m=0}^{M-1} w_m \overbrace{\underbrace{\int_{-\infty}^{\infty} \tilde{w}(\tau) y_m(\tau+t) e^{j\omega\tau} \mathrm{d}\tau}_{\text{“cheat-}Y_m(t,\omega)\text{” with “standard F.T.”} \triangleq \tilde{Y}_m(t,\omega)} \; e^{-j\omega t}}^{Y_m(t,\omega)} e^{-j\omega\Delta_m}$$

Book definition:

$$Z(t,\omega) \triangleq \sum_{m=0}^{M-1} w_m \underbrace{Y_m(t,\omega) e^{j\omega t}}_{\text{“cheat-”}Y_m(t,\omega)=\tilde{Y}_m(t,\omega)} e^{-j\omega\Delta_m} = \hat{Z}(t\omega) e^{j\omega t}$$

Inverse short-time Fourier Transform: $z(t)\tilde{w}(t-\hat{t}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\hat{Z}(\hat{t},\omega) e^{j\omega t}}_{\triangleq Z(\tau,\omega)} \mathrm{d}\omega.$

Slide 61: Review of basic statistics

- *Mean (expected value):*

$$m_x = \mathrm{E}\{x\} \triangleq \begin{cases} \int_\alpha \alpha f_x(\alpha)\,\mathrm{d}\alpha, & \text{for continuous variables} \\ \sum_\alpha \alpha p_x(\alpha), & \text{for discrete variables} \end{cases}$$

$$\hat{m}_x = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Slide 62: Basic statistics, *continued*

- *Variance:*

$$\sigma_x^2 \triangleq \mathrm{E}\{(x-m_x)(x-m_x)^*\} \xrightarrow[m_x=0]{\text{for zero-mean } x} \mathrm{E}\{xx^*\} = \mathrm{E}\left\{|x|^2\right\}$$

- *Covariance:*

$$\begin{aligned}\operatorname{cov}(x,y) &\triangleq \mathrm{E}\{(x-m_x)(y-m_y)^*\} \\ &= \mathrm{E}\left\{xy^* - xm_y^* - ym_x + m_x m_y^*\right\} \\ &= \mathrm{E}\{xy^*\} - m_x m_y^* - m_x m_y + m_x m_y^* \\ &= \mathrm{E}\{xy^*\} - m_x m_y \\ \Big(&= \mathrm{E}\{xy^*\} - \mathrm{E}\{x\}\,\mathrm{E}\{y\} \Big)\end{aligned}$$

Slide 63: Basic statistics, *continued*

- *Correlation:*

$$\operatorname{cor}(x,y) \triangleq \mathrm{E}\{xy^*\} \underset{m_x=m_y=0}{\overset{\text{for zero-mean variables}}{=\!=\!=\!=\!=\!=}} \operatorname{cov}(x,y)$$

- *Correlation Coefficient / "Pearson's Correlation":* (should rather be named covariance coefficient?)

$$\rho(x,y) \triangleq \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y} \overset{\text{for zero-mean } x,y}{=\!=\!=\!=\!=} \frac{\operatorname{cor}(x,y)}{\sigma_x \sigma_y} = \frac{\mathrm{E}\{xy^*\}}{\sigma_x \sigma_y},$$

$$\text{with } \sigma_x = \sqrt{\sigma_x^2}$$

Always: $|\rho(x,y)| \leq 1$

Slide 64: Basic statistics, *continued*

- *If x, y are correlated* $\Rightarrow$
  $0 < \rho(x,y) < 1$
- *If x, y are coherent* $\Rightarrow$
  $|\,\rho(x,y)\,| = 1$
- *If x, y are un-correlated* $\triangleq \operatorname{cov}(x,y) = 0 \quad \Rightarrow$
  - $$\begin{aligned}\operatorname{cov}(x,y) &= \mathrm{E}\{xy^*\} - \mathrm{E}\{x\}\,\mathrm{E}\{y\} = 0 \\ &\Leftrightarrow \\ \mathrm{E}\{xy^*\} &= \mathrm{E}\{x\}\,\mathrm{E}\{y\} = m_x m_y\end{aligned}$$
  - Correlation coefficient: $\rho(x,y) \triangleq \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y} = 0$

Slide 65: Basic statistics, *continued*

Examples

- $x = -y$, *zero-mean* $y \quad \Rightarrow$

$$\begin{cases} \mathrm{E}\{xy^*\} = \mathrm{E}\{-yy^*\} = -\mathrm{E}\{yy^*\} = -\sigma_y^2, \\ \sigma_x^2 = \mathrm{E}\{xx^*\} = \mathrm{E}\{-y - y^*\} = \sigma_y^2 \end{cases}$$

$$\Rightarrow \rho(x,y) \triangleq \frac{\mathrm{E}\{xy^*\}}{\sigma_x \sigma_y} = -1, \quad |\rho| = 1 \quad \therefore\ x, y \text{ coherent}$$

- $x = ye^{j\Delta}$, *zero-mean* $y \quad \Rightarrow$

$$\begin{cases} \mathrm{E}\{xy^*\} = \mathrm{E}\{ye^{j\Delta}y^*\} = e^{j\Delta}\,\mathrm{E}\{yy^*\} = e^{j\Delta}\sigma_y^2 \\ \sigma_x^2 = \mathrm{E}\{ye^{j\Delta}y^*e^{-j\Delta}\} = \mathrm{E}\{yy^*\} = \sigma_y^2 \end{cases}$$

$$\Rightarrow \rho(x,y) \triangleq \frac{\mathrm{E}\{xy^*\}}{\sigma_y^2} = \frac{e^{j\Delta}\,\mathrm{E}\{yy^*\}}{\sigma_x\sigma_y} = e^{j\Delta}$$

$$\Rightarrow |\,\rho(x,y)\,| = 1$$

Slide 66: Basic statistics, *continued*

- *Stochastic process:*

$$X = \{x(1), x(2), \cdots\}$$

- *Autocorrelation function:*

$$r_X(k,l) = \mathrm{E}\{x(k)x(l)^*\}$$

Slide 67: Basic statistics, *continued*

- *Stationary process:* r_X depends only on the difference $(k-l)$:

$$r_X(m) = \mathrm{E}\{x(k)x(k-m)^*\}, \quad \text{where } m = k-l$$

- *Variance:* (for zero-mean x)

$$r_X(0) = \mathrm{E}\{x(k)x(k-0)^*\} = \mathrm{var}(x)$$

Slide 68: Basic statistics, *continued*

- *Cross-correlation:*

$$r_{XY}(k,l) = \mathrm{E}\{x(k)y(l)^*\}$$

- Example: white noise, $n(k)$ (uncorrelated if $l \neq k$)
  Autocorrelation function:

$$r_n(m) = \mathrm{E}\{n(k)n(k-m)^*\} = \sigma_n^2\delta(m)$$

Slide 70: 4.5 Array gain

□

Higher SNR using array of sensors
Gain:

$$\boxed{G = \frac{\mathrm{SNR}_{\mathrm{array}}}{\mathrm{SNR}_{\mathrm{sensor}}}}$$

Slide 71: Spatial correlation matrix, remarks

$$\vec{Y}(t) = \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{M-1}(t) \end{bmatrix} \Rightarrow$$

$$\vec{Y}\vec{Y}^{\mathrm{H}} = \begin{bmatrix} y_0(t)y_0(t)^* & y_0(t)y_1(t)^* & \cdots & y_0(t)y_{M-1}(t)^* \\ y_1(t)y_0(t)^* & y_1(t)y_1(t)^* & \cdots & y_1(t)y_{M-1}(t)^* \\ \vdots & \vdots & \ddots & \vdots \\ y_{M-1}(t)y_0(t)^* & y_{M-1}(t)y_1(t)^* & \cdots & y_{M-1}(t)y_{M-1}(t)^* \end{bmatrix}$$

Slide 72: Spatial correlation matrix, remarks, *continued*

Correlation (covariance) matrix:

$$R_Y = \mathrm{E}\left\{\vec{Y}\vec{Y}^{\mathrm{H}}\right\}$$

$$= \begin{bmatrix} r_y(0,0) = \mathrm{E}\{y_0(t)y_0^*(t)\} & r_y(0,1) & \cdots & r_y(0,M-1) \\ r_y(1,0) = \mathrm{E}\{y_1(t)y_0^*(t)\} & r_y(1,1) & \cdots & r_y(1,M-1) \\ \vdots & \vdots & \ddots & \vdots \\ r_y(M-1,0) = \mathrm{E}\{y_{M-1}(t)y_0^*(t)\} & r_y(M-1,1) & \cdots & r_y(M-1,M-1) \end{bmatrix}$$

$$\overset{\text{stationary}}{\underset{\text{process}}{=\!=}} \begin{bmatrix} r_y(0) & r_y^*(1) & \cdots & r_y^*(M-1) \\ r_y(1) & r_y(0) & \cdots & r_y^*(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ r_y(M-1) & r_y(M-2) & \cdots & r_y(0) \end{bmatrix}$$

□

Slide 74: 4.9 Averaging in space and time

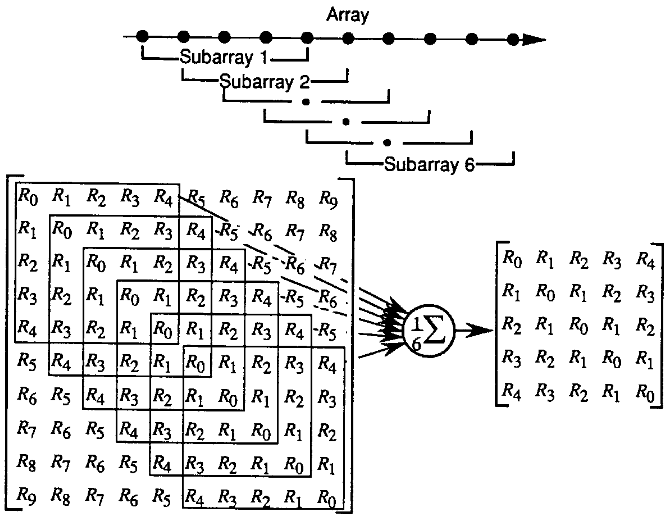
Spatial covariance matrix estimation methods:

1. Temporal averaging

□

2. Spatial / subarray averaging [next slide]

Slide 75: Subarray averaging (spatial smoothing)



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