Adaptive Array Processing

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Slide 2: Chapter 7: Adaptive array processing

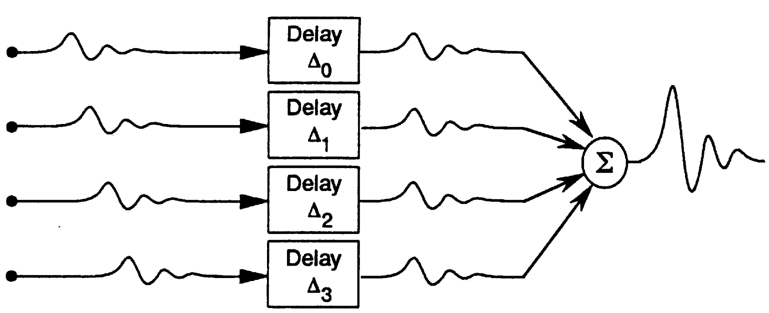
Minimum variance beamforming

- Generalized sidelobe canceler
- Signal coherence
- Spatial smoothing to de-correlate sources

Eigenanalysis methods

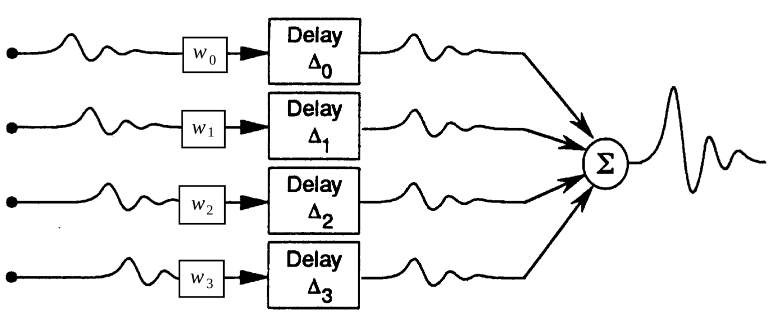
- Signal/noise subspaces
- Eigenvector method
- MUSIC

Slide 3: Delay-and-sum



stacking $\triangleq$ adjustment of $\Delta_0 \ldots \Delta_{M-1}$

Slide 4: Delay-and-sum, *continued*



- $z(t) = \sum_{m=0}^{M-1} w_m e^{-j\omega\Delta_m} y_m(t)$
- w_m: weight on signal m $\Rightarrow$ *shading* = *apodization*
- *Conventional D-A-S:* $\mathbf{w}$ *independent of recorded signal data*

Slide 5: Delay-and-sum on vector form

- Monochromatic source: $y_m(t) = e^{j(\omega t - \vec{k}^{\circ}\cdot\vec{x}_m)}$.
- Delayed signal: $y_m(t-\Delta_m) = y_m(t)e^{-j\omega\Delta_m}$

Define:

$$\mathbf{w} \triangleq \begin{bmatrix} w_0 e^{j\omega\Delta_0} \\ w_1 e^{j\omega\Delta_1} \\ \vdots \\ w_{M-1}e^{j\omega\Delta_{M-1}} \end{bmatrix} \xlongequal[\boxed{\Delta_m = -\vec{k}\cdot\vec{x}_m/\omega^{\circ}}]{\text{Delay-and-sum:}} \begin{bmatrix} w_0 e^{-j\vec{k}\cdot\vec{x}_0} \\ w_1 e^{-j\vec{k}\cdot\vec{x}_1} \\ \vdots \\ w_{M-1}e^{-j\vec{k}\cdot\vec{x}_{M-1}} \end{bmatrix}, \mathbf{Y}(t) \equiv \vec{Y} \triangleq \begin{bmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{M-1}(t) \end{bmatrix}$$

- Beamf. output: $z(t) = \sum_{m=0}^{M-1} w_m e^{-j\omega\Delta_m} y_m(t) = \mathbf{w}^{\mathrm{H}}\mathbf{Y}(t)$

□

- Power of $z(t)$:

$$\begin{aligned} \mathrm{E}\{|z(t)|^2\} &= \mathrm{E}\{(\mathbf{w}^{\mathrm{H}}\mathbf{Y})(\mathbf{w}^{\mathrm{H}}\mathbf{Y})^{\mathrm{H}}\} = \mathrm{E}\{\mathbf{w}^{\mathrm{H}}\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\mathbf{w}\} \\ &= \mathbf{w}^{\mathrm{H}}\,\mathrm{E}\{\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\}\,\mathbf{w} = \mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w} \end{aligned}$$

□

Slide 6: Delay-and-sum on vector form, *continued*

Define:

Matrix of weights:

$$\mathbf{W} \triangleq \begin{bmatrix} w_0 & 0 & \cdots & 0 \\ 0 & w_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{M-1} \end{bmatrix},$$

Steering vector:

$$\mathbf{e} \triangleq \begin{bmatrix} e^{-j\vec{k}\cdot\vec{x}_0} \\ e^{-j\vec{k}\cdot\vec{x}_1} \\ \vdots \\ e^{-j\vec{k}\cdot\vec{x}_{M-1}} \end{bmatrix}$$

□

- $\Rightarrow \mathbf{w} = \mathbf{W}\mathbf{e}$
- $z(t) = \mathbf{w}^{\mathrm{H}}\mathbf{Y} = [\underbrace{\mathbf{W}\mathbf{e}}_{\mathbf{w}}]^{\mathrm{H}}\mathbf{Y} = \mathbf{e}^{\mathrm{H}}\mathbf{W}^{\mathrm{H}}\mathbf{Y} \xlongequal{\mathbf{W}^{\mathrm{H}}=\mathbf{W}} \underbrace{\mathbf{e}^{\mathrm{H}}\mathbf{W}}_{(\mathbf{w}^{\mathrm{H}})}\mathbf{Y}(t)$
- Power of $z(t)$:

$$P(\mathbf{e}) \triangleq \mathrm{E}\{|z(t)|^2\}(\mathbf{e}) = \underbrace{\mathbf{w}^{\mathrm{H}}}_{\mathbf{e}^{\mathrm{H}}\mathbf{W}}\mathbf{R}\underbrace{\mathbf{w}}_{\mathbf{W}\mathbf{e}} \xlongequal{\text{if } w_m=1,\forall m} \mathbf{e}^{\mathrm{H}}\mathbf{R}\mathbf{e}$$

Slide 7: Delay-and-sum on vector form, *continued*

About $\mathbf{e}$:

- *Steering vector*

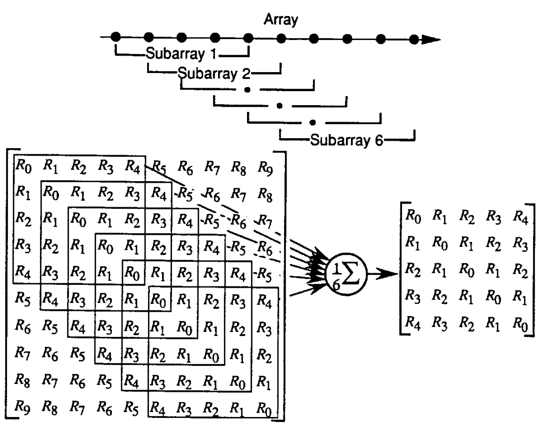
$$\mathbf{e} \triangleq \begin{bmatrix} e^{-j\vec{k}\cdot\vec{x}_0} \\ e^{-j\vec{k}\cdot\vec{x}_1} \\ \vdots \\ e^{-j\vec{k}\cdot\vec{x}_{M-1}} \end{bmatrix}$$

- Contains delays to focus in specific direction
- Represents unit amplitude signal, propagating in $\vec{k}$ direction

In conventional D-A-S: $\mathbf{w}$ *independent of received signal data*

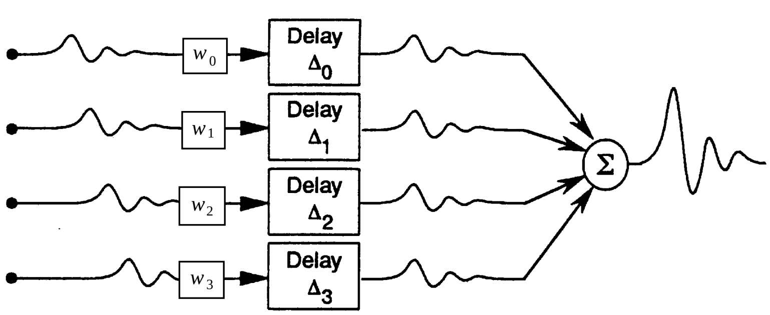
Slide 8: Estimation of spatial covariance matrix

- Averaging in time
- Averaging in space = spatial smoothing = subarray averaging



Slide 9: Minimum variance beamforming

Assume narrow-band signals



- Adaptive method. [Latin: *adaptare* "to fit to"]
- Allow w_m to also be complex and/or negative
- Sensor weights defined *not only as function of problem geometry, but also as function of received signals*

Slide 10: Minimum variance beamforming, *continued*

- "Minimum variance" = "Capon" = "Maximum likelihood" (beamforming)
- Constrained optimization problem
- New steering direction $\mathbf{e} \Rightarrow$ new calculation of element weights w_m

□

Minimum-variance key equations

$$\boxed{1} \quad \mathbf{w} \triangleq \begin{bmatrix} w_0 e^{-j\vec{k}\cdot\vec{x}_0} \\ \vdots \\ w_{M-1} e^{-j\vec{k}\cdot\vec{x}_{M-1}} \end{bmatrix} \Rightarrow \boxed{\begin{array}{c} \min_{\mathbf{w}} \left(\mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w}\right), \\ \text{constraint:} \\ \mathbf{w}^{\mathrm{H}}\mathbf{1} = 1 \end{array}} \Leftrightarrow \boxed{\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{1}}} \Rightarrow \boxed{P(\mathbf{e}) = \frac{1}{\mathbf{1}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{1}}}$$

$$\boxed{2} \quad \mathbf{w} \triangleq \begin{bmatrix} w_0 \\ \vdots \\ w_{M-1} \end{bmatrix} \Rightarrow \boxed{\begin{array}{c} \min_{\mathbf{w}} \left(\mathbf{w}^{\mathrm{H}}\mathbf{R}\mathbf{w}\right), \\ \text{constraint:} \\ \mathbf{w}^{\mathrm{H}}\mathbf{e} = 1 \end{array}} \Leftrightarrow \boxed{\mathbf{w} = \frac{\mathbf{R}^{-1}\mathbf{e}}{\mathbf{e}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{e}}} \Rightarrow \boxed{P(\mathbf{e}) = \frac{1}{\mathbf{e}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{e}}}$$

- [beampattern] $\neq$ [steered response]

Slide 11: Minimum variance beamforming, *continued*

Example

Uniform linear array, $M = 10$

Comparing:

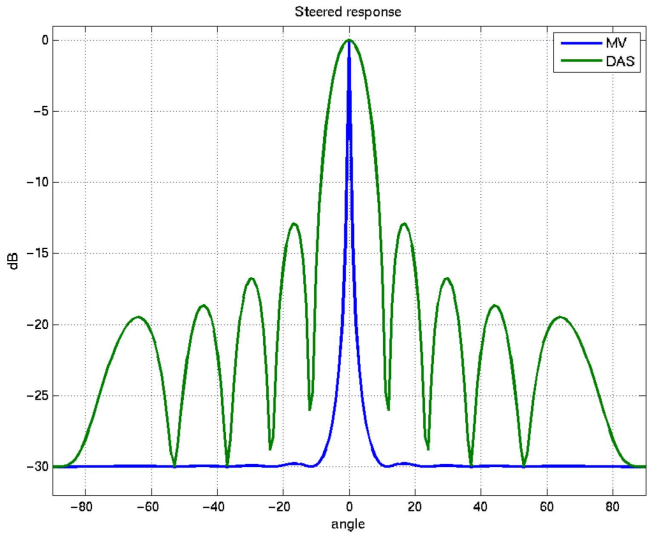
- Delay-and-sum
- Minimum-variance

Investigating:

- Steered responses
- Beampatterns

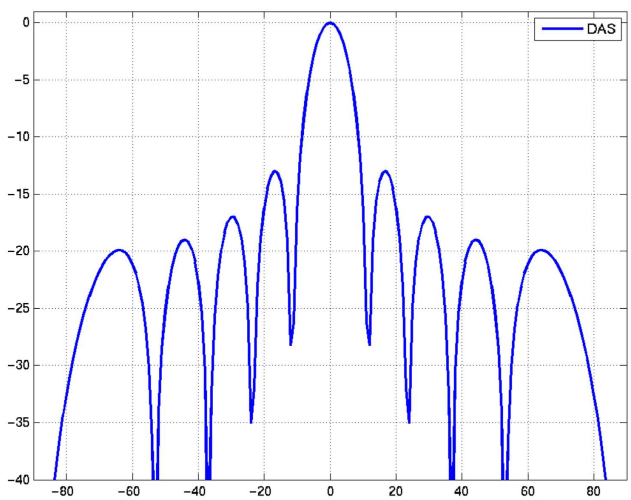
Slide 12: Comparison: delay-and-sum / minimum-variance

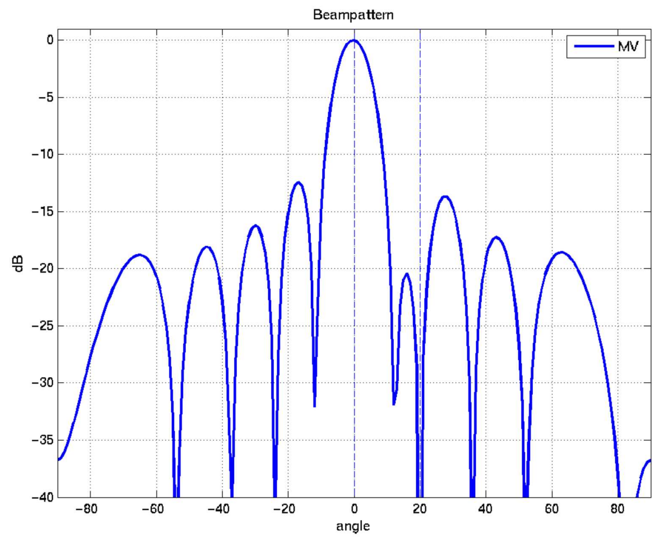
1 source

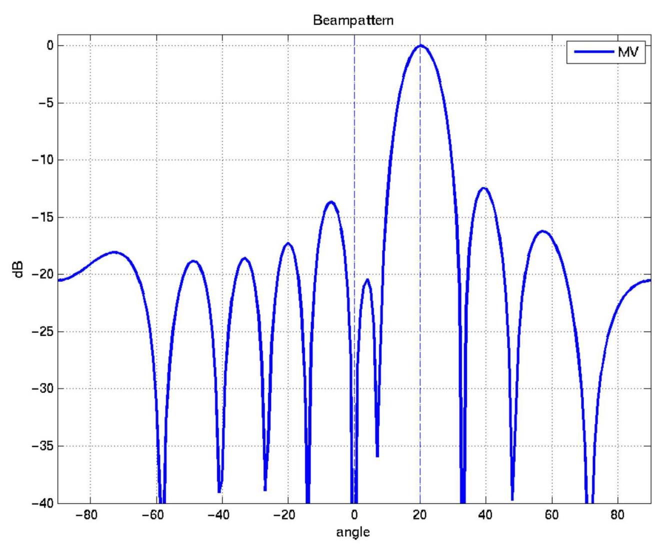


Slide 13: Comparison: delay-and-sum / minimum-variance

2 sources




Beampattern
MV
dB
angle
0
−5
−10
−15
−20
−25
−30
−35
−40
−80
−60
−40
−20
0
20
40
60
80


Beampattern
MV
dB
angle
0
−5
−10
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−20
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−80
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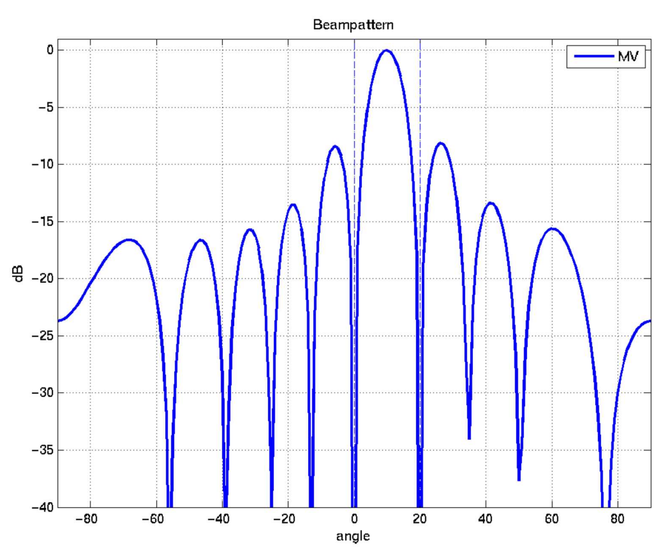
−20
0
20
40
60
80


Beampattern
MV
dB
angle
0
−5
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−80
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−20
0
20
40
60
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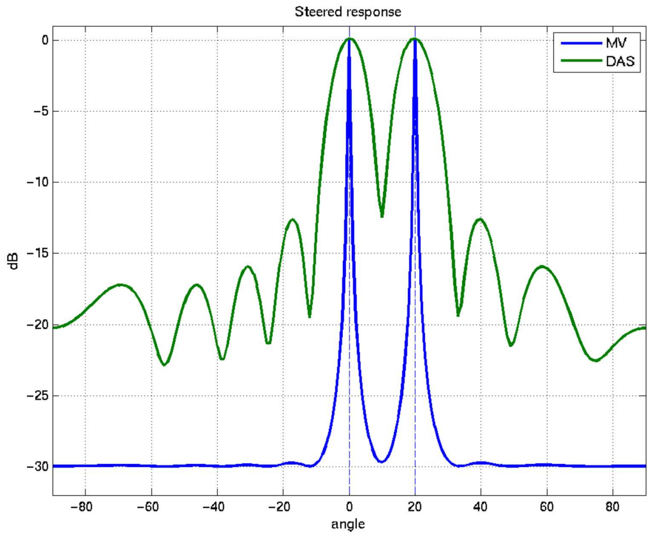
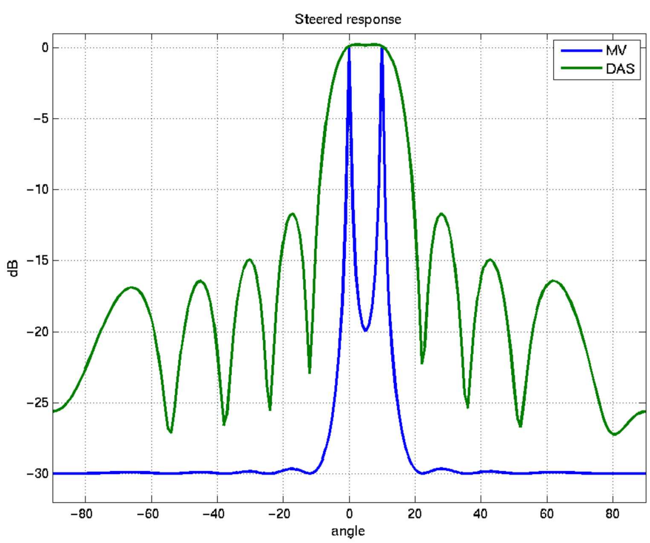


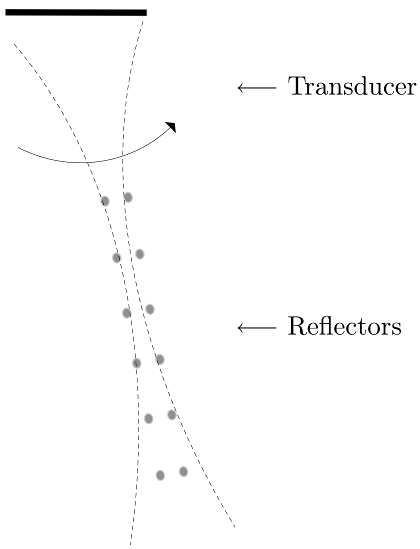
Slide 14: Comparison: delay-and-sum / minimum-variance

2 closely spaced sources



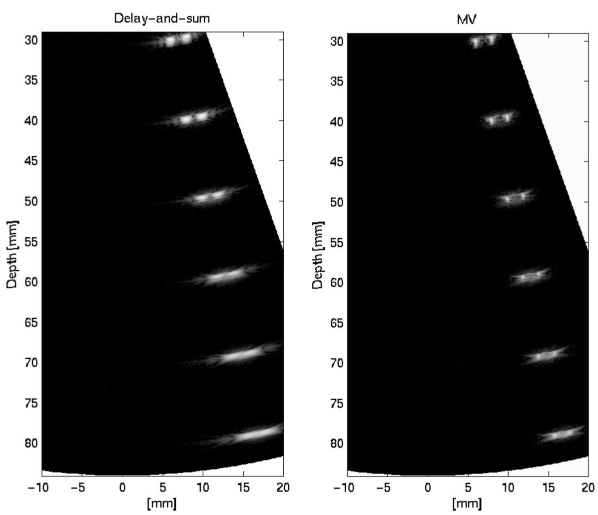
Slide 15: Example: Ultrasound imaging

Experimental setup



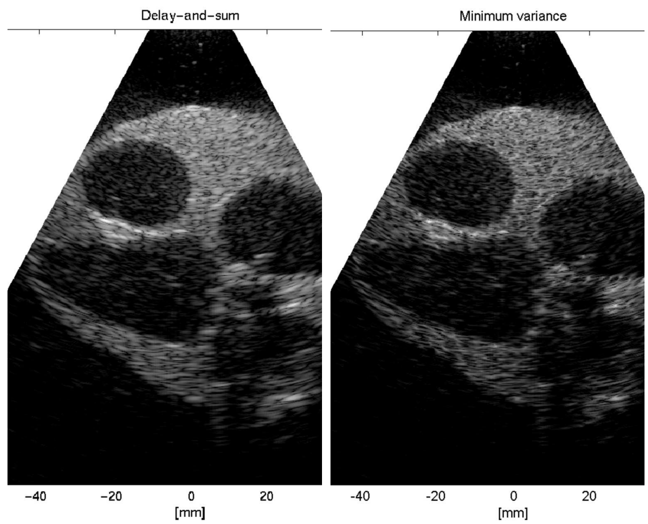
Slide 16: Example: Ultrasound imaging, *continued*

Resulting images

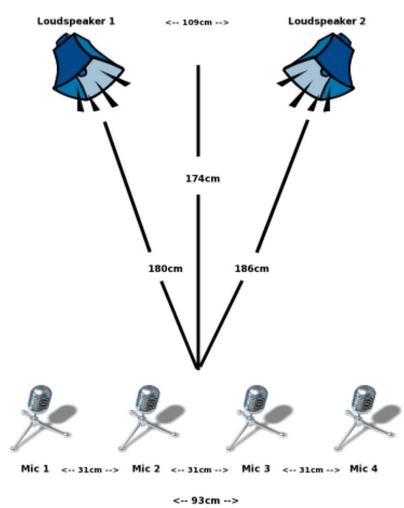


Slide 17: Example: Ultrasound imaging, *continued*

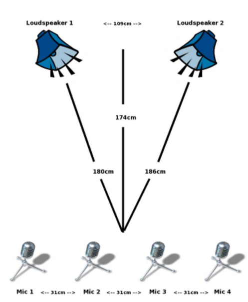
Experimental data, heart phantom



Slide 18: Sound example: Microphone array



Slide 19: Sound example, *continued*



- Single microphone `mix.wav`
- Delay-and-sum `gaute_DAS.wav`

- Minimum-variance `gaute_adaptive.wav`

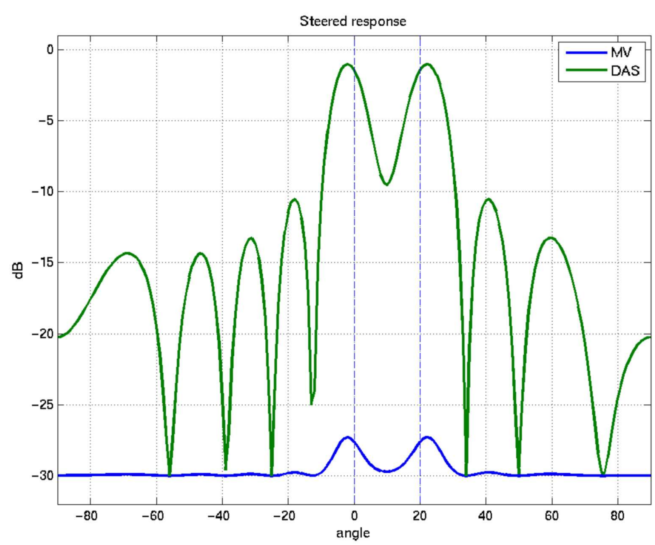
Slide 20: Minimum-variance beamformer:

Uncorrelated signals required for optimum functioning

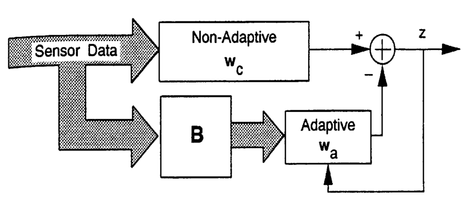
- Correlated signals may give signal cancellation
- Although constraint is fulfilled: signal propagating in the "look-direction" may be canceled by a correlated interferer

Slide 21: Minimum-variance / D-A-S example

2 correlated sources



Slide 22: Generalized sidelobe canceler



- Conventional D-A-S weights $\mathbf{w}_c$: steering in assumed propagation direction
- Adaptive/canceler portion: removal of other signals
- Blocking matrix $\mathbf{B}$, blocks signal from assumed propagation direction from coming into "canceler part"
- $\mathbf{w}_a$: adaptive weights to emphasize what is to be removed
- Minimize total power. Unconstrained minimization:

$$\min_{\mathbf{w}_a} \left(\mathbf{w}_c - \mathbf{B}^{\mathrm{H}}\mathbf{w}_a\right)^{\mathrm{H}} \mathbf{R} \left(\mathbf{w}_c - \mathbf{B}^{\mathrm{H}}\mathbf{w}_a\right) \;\Rightarrow \mathbf{w}_a = \left(\mathbf{B}\mathbf{R}\mathbf{B}^{\mathrm{H}}\right)^{-1} \mathbf{B}\mathbf{R}\mathbf{w}_c$$

- Full details: D&J pp. 369–371

Slide 23: Eigenanalysis and Fourier analysis

(Chapter 7.3.1)

Eigenvalues / Eigenvectors of $\mathbf{R}$

$$\mathbf{R}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

$$\left.\begin{array}{lll} \mathbf{R}\ \underline{\text{Hermitian}}\ \text{(self-adjoint)} & \Leftrightarrow & \mathbf{R} = \mathbf{R}^{\mathrm{H}} \\ \mathbf{R}\ \underline{\text{Positive semidefinite}} & \Leftrightarrow & \mathbf{x}^{\mathrm{H}}\mathbf{R}\mathbf{x} \geq 0,\ \forall \mathbf{x} \neq \mathbf{0} \end{array}\right\} \Rightarrow$$

- Eigenvalues real & positive: $\lambda_i \geq 0,\ \forall i$
- Eigenvectors $\mathbf{v}_i$: orthogonal

$$\Rightarrow \text{For } \mathbf{V} \triangleq \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_M \\ | & | & & | \end{bmatrix}\text{: } \mathbf{V} \text{ is unitary matrix}$$

$$\Leftrightarrow \mathbf{V}^{\mathrm{H}}\mathbf{V} = \mathbf{I} \Leftrightarrow \boxed{\mathbf{V}^{-1} = \mathbf{V}^{\mathrm{H}}}$$

□

Slide 24: Eigenvalue decomposition

□

Eigendecomposition of $\mathbf{R}$ & $\mathbf{R}^{-1}$: *The spectral theorem*

- $\mathbf{R} = \sum_{i=1}^{M} \lambda_i \mathbf{v}_i \mathbf{v}_i^{\mathrm{H}} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathrm{H}}$
- $\mathbf{R}^{-1} = \sum_{i=1}^{M} \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^{\mathrm{H}} = \mathbf{V}\mathbf{\Lambda}^{-1}\mathbf{V}^{\mathrm{H}}$
- *Note above:* Same eigenvectors, but inverse eigenvalues

Slide 25: [Signal + noise] & [noise] subspaces

$$\mathbf{R} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\mathrm{H}} = \underbrace{\mathbf{V}_{\mathrm{s}}\mathbf{\Lambda}_{\mathrm{s}}\mathbf{V}_{\mathrm{s}}^{\mathrm{H}}}_{\substack{\text{from}\\ \text{signal}}} + \underbrace{\mathbf{V}_{\mathrm{n}}\mathbf{\Lambda}_{\mathrm{n}}\mathbf{V}_{\mathrm{n}}^{\mathrm{H}}}_{\substack{\text{from}\\ \text{noise}}}$$

□

Slide 26: Beamformer output powers

□

Minimum variance, $\hat{\mathbf{R}} = \mathrm{E}\{\mathbf{Y}\mathbf{Y}^{\mathrm{H}}\}$: "full $\mathbf{R}$ estimate"

$$P_{\mathrm{MV}}(\mathbf{e}) = \frac{1}{\underbrace{\mathbf{e}^{\mathrm{H}}\mathbf{V}_{\mathrm{s}}\boldsymbol{\Lambda}_{\mathrm{s}}^{-1}\mathbf{V}_{\mathrm{s}}^{\mathrm{H}}\mathbf{e}}_{\text{signal-plus-noise subspace}} + \underbrace{\mathbf{e}^{\mathrm{H}}\mathbf{V}_{\mathrm{n}}\boldsymbol{\Lambda}_{\mathrm{n}}^{-1}\mathbf{V}_{\mathrm{n}}^{\mathrm{H}}\mathbf{e}}_{\text{noise subspace}}} = \frac{1}{\sum_{i=1}^{N_s}\frac{1}{\lambda_i}\left|\mathbf{e}^{\mathrm{H}}\mathbf{v}_i\right|^2 + \sum_{i=N_s+1}^{M}\frac{1}{\lambda_i}\left|\mathbf{e}^{\mathrm{H}}\mathbf{v}_i\right|^2}$$

Eigenvector method, $\hat{\mathbf{R}} = \sum_{i=N_s+1}^{M}\lambda_i\mathbf{v}_i\mathbf{v}_i^{\mathrm{H}}$: "noise-only $\mathbf{R}$ estimate"

$$P_{\mathrm{EV}}(\mathbf{e}) = \frac{1}{\mathbf{e}^{\mathrm{H}}\mathbf{V}_{\mathrm{n}}\boldsymbol{\Lambda}_{\mathrm{n}}^{-1}\mathbf{V}_{\mathrm{n}}^{\mathrm{H}}\mathbf{e}} = \frac{1}{\sum_{i=N_s+1}^{M}\frac{1}{\lambda_i}\left|\mathbf{e}^{\mathrm{H}}\mathbf{v}_i\right|^2}$$

MUSIC, $\hat{\mathbf{R}} = \sum_{i=N_s+1}^{M}\mathbf{v}_i\mathbf{v}_i^{\mathrm{H}}$: "normalized noise-only $\mathbf{R}$ estimate"

$$P_{\mathrm{MUSIC}}(\mathbf{e}) = \frac{1}{\mathbf{e}^{\mathrm{H}}\mathbf{V}_{\mathrm{n}}\mathbf{V}_{\mathrm{n}}^{\mathrm{H}}\mathbf{e}} = \frac{1}{\sum_{i=N_s+1}^{M}\left|\mathbf{e}^{\mathrm{H}}\mathbf{v}_i\right|^2}$$

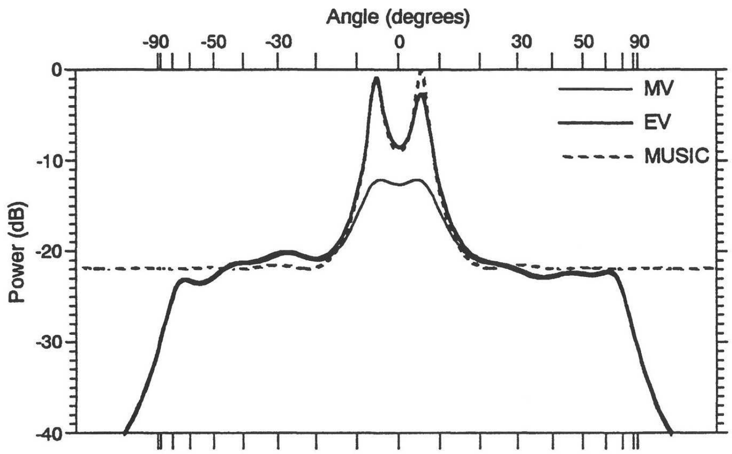
Slide 27: [Signal + noise], & [noise] subspaces, *continued*

- $\mathbf{R}$ always Hermitian $\Rightarrow$ $\forall$ eigenvectors orthogonal
- N_s largest eigenvectors: span the [signal+noise subspace]
- $M - N_s$ smallest span the [noise subspace]
- Signal steering vectors $\in$ [signal+noise subspace], $\perp$ [noise subspace]
- [Largest eigenvectors] $\neq$ [signal vectors] However: signal vectors are linear combinations of [largest eigenvectors]

Slide 28: EV / MUSIC

- (Inverse of) projection of possible steering vectors onto the noise subspace
- MUSIC: Dropping λ_i in $\hat{\mathbf{R}}$ $\Leftrightarrow$ noise whitening
- EV & MUSIC: peaks at Directions-Of-Arrival (DOAs)
  - No amplitude preservation
  - Require knowledge of # signals, N_s
  - How to estimate N_s?

Slide 29: MV / EV / MUSIC comparison



Slide 30: Situation with signal coherence

$$\mathbf{R} = \mathbf{S}\mathbf{C}\mathbf{S}^{\mathrm{H}} + \mathbf{K}_{\mathrm{n}}$$

- signal coherence ⇒ rank deficient signal matrix $\mathbf{S}\mathbf{C}\mathbf{S}^{\mathrm{H}}$
- For minimum-variance: May cause signal cancellation
- Gives signal+noise-subspace consisting of linear combinations of signal vectors
- EV & MUSIC: may fail to produce peaks at DOA locations
- Want to reduce cross-correlation terms in $\mathbf{C}$.
- Cure: Spatial smoothing (subarray averaging)