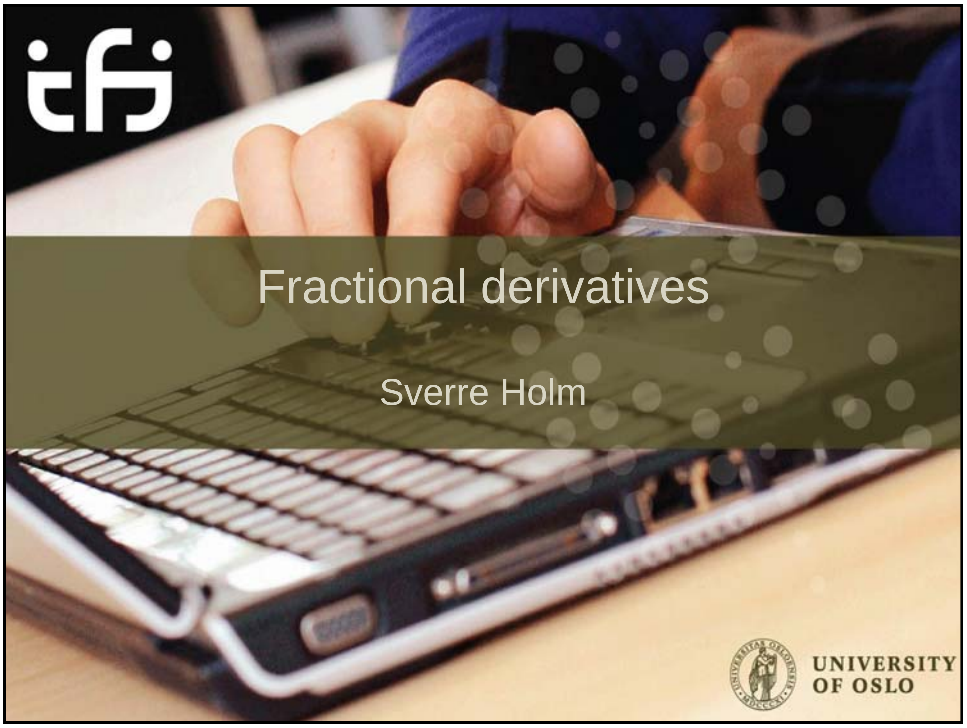


Fractional derivatives

Sverre Holm

n-th order derivative

- Sequence of n-fold integrals and n-fold derivatives:

$$\cdots, \int_a^t d\tau_2 \int_a^{\tau_2} f(\tau_1) d\tau_1, \int_a^t f(\tau_1) d\tau_1, f(t), \frac{df(t)}{dt}, \frac{d^2 f(t)}{dt^2}, \cdots$$

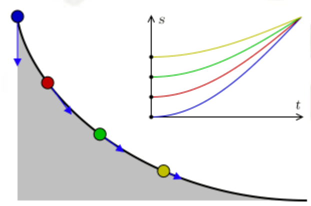
- Fourier transform: $FT\left(\frac{d^n f(t)}{dt^n}\right) = (i\omega)^n F(\omega)$

$$\cdots, (i\omega)^{-2} F(\omega), (i\omega)^{-1} F(\omega), F(\omega), i\omega F(\omega), (i\omega)^2 F(\omega), \cdots$$

- I Podlubny, Fractional Differential Equations, Academic Press, 1999

First physical problem

- Niels Henrik Abel (1802 – 1829): a generalized version of the tautochrone problem (Abel's mechanical problem)
  - Requires derivative of order 0.5
- Tautochrone/isochrone curve: the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of the starting point.
- The curve is a cycloid, and the time is equal to π times the square root of the radius over the acceleration of gravity.
  - Abel, Auflösung einer mechanischen Aufgabe, J. Reine u. Angew. Math, 1826,
  - B. Holmboe, "Abel: Ouevres complètes", 1839, IV Résolution d'un probléme mécanique.



Derivative of arbitrary order

- Derivative of order α:

$$\frac{d^\alpha f(t)}{dt^\alpha} = {}_a D_t^\alpha f(t)$$

  - α <0 ⇔ integration
  - a and t: limits in defining integral
- Fourier transform (neglecting initial cond's):

$$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega)$$

Fourier approach to fractional operator (1)

- Integer (m>α) + fraction (α-m<0):

$$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega) = (i\omega)^m F(\omega)(i\omega)^{\alpha-m}$$

- First part: ordinary derivative
- Second part: fractional part
  - What is its inverse Fourier transform?

Fourier transform

$$h(t) = \frac{1}{\Gamma(\beta)}\frac{1}{t^{1-\beta}}, t > 0 \Leftrightarrow H(\omega) = (i\omega)^{-\beta}$$

Podlubny, 1999, pp. 110-

- 0<β<1
- Γ(.) is the gamma function
  - Generalization of the factorial: Γ(n+1)=n!
- Let β=m-α and rewrite:

$$h(t) = \frac{1}{\Gamma(m-\alpha)}\frac{1}{t^{1+\alpha-m}}, t > 0 \Leftrightarrow H(\omega) = (i\omega)^{\alpha-m}$$

Fourier approach to fractional operator (2)

- Fractional Fourier transform as a convolution of derivative of order first integer m larger than α and a memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{d^m f(t)}{dt^m} * \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{1+\alpha-m}}$$

Fractional derivative: Two flavors

- Riemann-Liouville: order $\alpha \in$ R, m-1$\leq\alpha$<m:

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

  – First convolution, then integer order derivation

- Caputo: order m-1 $\leq \alpha$ <m:

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau$$

  – First integer order derivative, then convolution

Riemann-Liouville vs Caputo

- Riemann-Liouville requires initialization of derivatives of non-integer orders:

$$\lim_{t \to a} {}_aD_t^{\alpha-1} f(t), \lim_{t \to a} {}_aD_t^{\alpha-2} f(t), \cdots$$

- Caputo requires initialization of integer order derivatives: $f^{(k)}(0)$, k=0, 1, ..., m-1
  – Usually have physical meaning
  – Simpler to use in numerical solutions

Fractional derivative for numerics

- Caputo (lower limit a = $-\infty$):

$$ {}_{-\infty}^{\;\;C}D_t^{\alpha} f(t) = \frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \frac{\partial^m f(t)}{\partial t^m} * g_{m-\alpha}(t)$$

$$g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, t > 0$$

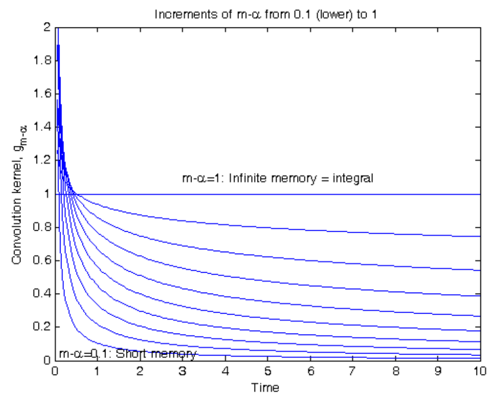
- Convolution with a memory function

Memory function

- Convolution kernel:

$$g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}$$

  – m-α = ε⁺ : no memory, Γ(ε⁺) → ∞ for ε⁺ → 0 ⇒ kernel → impulse
  – m-α = 1 : infinite memory



Fractional derivative of order 0..1

- Example: 0≤α<1 (Caputo with m=1):

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial f(t)}{\partial t} * g_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f^{(1)}(t)}{(t-\tau)^\alpha} d\tau$$

- Limits:
  - $\alpha \to 0 \Rightarrow$ $\frac{\partial^\alpha f(t)}{\partial t^\alpha} \to \int_{-\infty}^{t} \frac{f^{(1)}(t)}{(t-\tau)^0} d\tau = f(t)$
  - $\alpha \to 1 \Rightarrow$ $\frac{\partial^\alpha f(t)}{\partial t^\alpha} \to \int_{-\infty}^{t} f^{(1)}(t)\delta(\tau) d\tau = f^{(1)}(t)$

Conclusion

Two interpretations of fractional derivative:

1. Fourier:

$$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega)$$

2. Convolution of ordinary derivative of order m>α and memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} \propto \frac{d^m f(t)}{dt^m} * \frac{1}{t^{1+\alpha-m}}$$