INF5410 Array signal processing.
Ch. 3: Apertures and Arrays, part I

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Outline

Aperture and Arrays

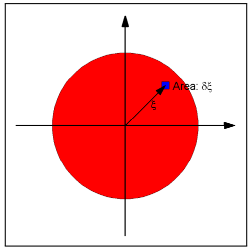
- Study apertures: Examine the effect of sensors that gather signal energy over finite areas.
- Arrays: Group of sensors combined to produce single output.
- At m'th sensor position, $\vec{x}_m$:
  - Fields value: $f(\vec{x}_m, t)$.
  - Sensors output: $y_m(t)$.
  - If sensor is *perfect* (i.e. linear transf., infinite bandwidth, omni-directional):

$$y_m(t) = \kappa \cdot f(\vec{x}_m, t), \ \kappa \in \Re \text{ (or } \mathcal{C}).$$

Aperture function and aperture smoothing function

- Directional $\leftrightarrow$ omni-directional
  - If sensor has (significant) spatial extent, it will spatially integrate energy, i.e. it focus in a particular propagation direction.
  - Example: Parabolic dish.
- Apertures ($\neq$ point sources) are described by the aperture function, $w(\vec{x})$.
  - Spatial extend reflects size and shape
  - Aperture weighting; relative weighting of the field within the aperture (also known as shading, tapering, apodization).
- Aperture smoothing function:
  $W(\vec{k}) = \int_{-\infty}^{\infty} w(\vec{x}) \exp(\jmath \vec{k} \cdot \vec{x}) d\vec{x}$

Aperture smoothing function ($\neq$ Sec. 3.1.1)



- Given
  - Aperture: $w(\vec{x})$
  - Field: $f(\vec{x}, t)$
  - linear sensor
  - contribution from area $\delta\vec{\xi}$ at $\vec{\xi}$: $w(\vec{\xi})f(\vec{\xi}, t)\delta\vec{\xi}$
  - contribution from sensor $z(t) = \int_{aperture} w(\vec{\xi})f(\vec{\xi}, t)\delta\xi$.

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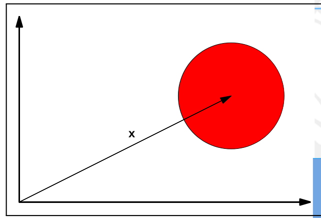
Aperture smoothing function ...



- Assume aperture in $\vec{x}$
- $z(\vec{x}, t) = \int_{\xi} w(\vec{\xi} - \vec{x})f(\vec{\xi})\delta\xi = w(-\vec{x}) * f(\vec{x}, t)$,
  (i.e. spatial correlation)
  should have been spatial convolution

$\Updownarrow$ (space-time F.T.)

$$Z(\vec{k}, w) = \int_{\vec{x}} \int_t z(\vec{x}, t)e^{\jmath wt}e^{\jmath\vec{k}\cdot\vec{x}}\,dt\,d\vec{x}$$
$$= W(-\vec{k})\,F(\vec{k}, w)$$

Must use symmetry assumption ...

$$= W(\vec{k})\,F(\vec{k}, w)$$

This differs from Eq. (3.1)!

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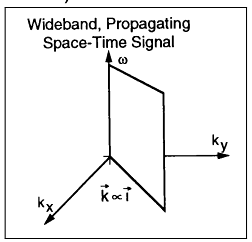
Aperture smoothing function ...

- Assume a single plane wave, propagating in direction $\vec{\zeta}^0, \vec{\zeta}^0 = \vec{k}^0/k$
  $\Rightarrow\ f(\vec{x}, t) = s(t - \vec{\alpha}^0 \cdot \vec{x}),\ \vec{\alpha}^0 = \vec{\zeta}^0/c$
  $\Rightarrow\ F(\vec{k}, w) = S(w)\delta(\vec{k} - w\vec{\alpha}^0)$ (Sec. 2.5.1)

This prop. wave contains energy only along the line $\vec{k} = w\vec{\alpha}^0$ in wavenumber-frequency space.



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Aperture smoothing function ...

- Subst. of $F(\vec{k}, w) = S(w)\delta(\vec{k} - w\vec{\alpha}^0)$ into $Z(\vec{k}, w) = W(\vec{k})F(\vec{k}, w)$ gives

$$Z(\vec{k}, w) = W(\vec{k})S(w)\delta(\vec{k} - w\vec{\alpha}^0)$$

- i.e. the spectrum of the output signal ($Z(\vec{k}, w)$) is multiplied by a wavenumber-frequency-dependent gain $W(\vec{k})$.

Aperture smoothing function ...

- Linear aperture:
  $b(x) = 1,\ |x| \le D/2$
  $\Rightarrow W(\vec{k}) = \frac{\sin k_x D/2}{k_x/2}$
- Rectangular aperture:
  $w(x,y) = b_1(x) b_2(y)$
  $\Rightarrow W(k_x, k_y) = W(x)W(y)$
  $W(k_x, k_y) = \frac{\sin k_x D_x/2}{k_x/2} \frac{\sin k_y D_y/2}{k_y/2}$
- Circular aperture:
  $o(x,y) = 1,\ \sqrt{x^2+y^2} \le R$
  $\Rightarrow O(k_{xy}) = \frac{2\pi R}{k_{xy}} J_1(k_{xy} R)$

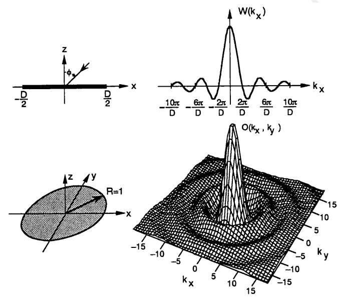
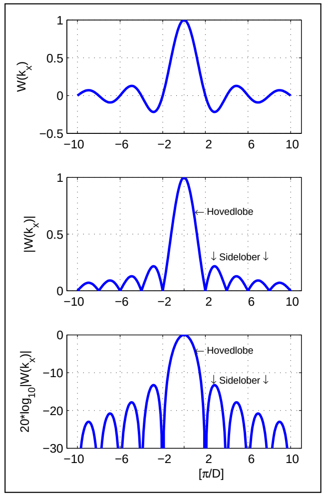


Figure 3.3 The commonest apertures, the linear and circular, are illustrated on the left, and their aperture smoothing functions on the right. A linear aperture lies along the x axis and spans D spatial units; the z axis intersects the midpoint of the aperture. The angle ϕ is formed by the z axis and the direction of propagation of a plane wave. Positive values of ϕ are taken clockwise from the z axis. Because the linear aperture lies along the x axis, its aperture smoothing function depends only on k_x. The circular aperture's aperture smoothing function $O(\cdot)$ is plotted in two-dimensional wavenumber space (k_x, k_y). Because of the aperture's circular symmetry, the aperture smoothing function depends only on the radius k_{xy}. First derived by G. B. Airy, $O(k_{xy})$ is often termed the "Airy disk" in the optical literature [61: 64].

Characterizing $W(\vec{k})$

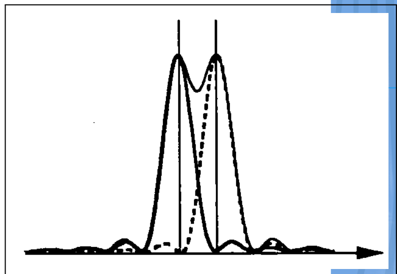
- Linear aperture:
  1. sidelobe at $k_{x_0} \approx 2.86\pi/D$
  $|W(k_{x_0}| \approx 0.2172D \Rightarrow$
  $\frac{\text{ML}}{\text{SL}} \approx \frac{D}{0.2172} = 4.603 \propto 13.3\text{dB}$
- Circular aperture:
  1. SL at $k_{xy_0} \approx 5.14/R$
  $\frac{\text{ML}}{\text{SL}} \approx 7.56 \propto -17.57\text{dB}.$
- Projection-slice theorem



Classical resolution

- Spatial extent of $w(\vec{x})$ determines the resolution with which two plane waves can be separated.
- Ideally, $W(\vec{k}) = \delta(\vec{k})$, i.e. infinite spatial extent!

Rayleigh criterion:
Two incoherent plane waves, propagating in two slightly different directions, are resolved if the mainlobe peak of one aperture smoothing function replica falls on the first zero of the other aperture smoothing function replica, i.e. half the mainlobe width.



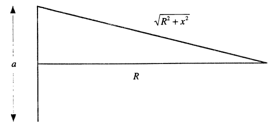
Classical resolution ...

- Linear aperture of size D
  $$W(k_x) = \frac{\sin(k_x D/2)}{k_x/2} (= D\text{sinc}(k_x D/2)) = \frac{\sin(\pi \sin\theta D/\lambda)}{\pi \sin\theta/\lambda}$$
  - -3 dB width: $\theta_{-3dB} \approx 0.89\lambda/D$
  - -6 dB width: $\theta_{-6dB} \approx 1.21\lambda/D$
  - Zero-to-zero distance: $\theta_{0-0} = 2\lambda/D$
- Circular aperture of diameter D
  $$W(k_{xy}) = \frac{2\pi D/2}{k_{xy}} J_1(k_{xy} D/2)$$
  - -3 dB width: $\theta_{-3dB} \approx 1.02\lambda/D$
  - -6 dB width: $\theta_{-6dB} \approx 1.41\lambda/D$
  - Zero-to-zero distance: $\theta_{0-0} \approx 2.44\lambda/D$
- Rule-of-thumb; Angular resolution: $\theta = \lambda/D$

Geometrical optics

- Validity: down to about a wavelength
- Near field-far field transition
  - $d_R = D^2/\lambda$ for a maximum phase error of $\lambda/8$ over aperture
- f-number
  - Ratio of range and aperture: $f_{\#} = R/D$
- Resolution
  - Angular resolution: $\theta = \lambda/D$
  - Azimuth resolution: $u = R\theta = f_{\#}\lambda$
- Depth of focus
  - Aperture is focused at range R. Phase error of $\lambda/8$ yields $r = \pm f_{\#}^2\lambda$ or DOF=$2f_{\#}^2\lambda$ (proportional to phase error)

Geom.Opt: Near field/Far field crossover



(2.1)

The differential path length Δ associated with a point x on the aperture and a range R can be evaluated using simple geometry.

$$\begin{aligned}\Delta &= \sqrt{R^2+x^2} - R \\ &= R\sqrt{1+\left(\frac{x}{R}\right)^2} - R \\ &\cong \frac{x^2}{2R}\end{aligned} \quad (2.2)$$

This differential error across the aperture is thus essentially quadratic, and can be reduced arbitrarily be increasing R. That is, in the far field the radiation from each point on the aperture arrives (essentially) coherently, adding constructively. As we move the point target closer to the aperture, the delay error increases inversely with R until, at some crossover range R_C between the near field and far field, it becomes non-negligible. We define this range $R = R_C$ (rather arbitrarily) as that for which the maximum error is

$$\Delta = \lambda/8 \quad (2.3)$$

As the maximum error will always be associated with the ends of the aperture, we substitute $x = a/2$ and use $\Delta = \lambda/8$ in eq. (2.2) to obtain

$$\boxed{\frac{R_C}{a} = \frac{a}{\lambda}} \quad (2.4)$$

That is, *the crossover range, measured in apertures, is equal to the aperture, measured in wavelengths.*

The far field is often called the *Fraunhofer region*, where we can ignore the differential phase terms associated with different propagation lengths. The near field is often called the *Fresnel region*, characterized by the (approximately) quadratic phase attributable to different propagation lengths from different aperture points.

Let us calculate the near field/far field crossover of a practical ultrasonic aperture operating at a center frequency of 3.5 MHz. Let

$$a = 28mm$$
$$\lambda = .44mm$$

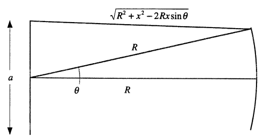
This aperture might have 128 elements spaced at $\lambda/2$. The transition between the Fraunhofer region and the Fresnel region occurs at

$$R_C = 1782mm$$

or almost two meters! *All modern diagnostic ultrasonic imaging occurs in the extreme near field.* This distinguishes the discipline from many other imaging technologies and presents a notable engineering challenge.

(From Wright: *Image Formation ...*)

Geom.Opt: Near field/Far field crossover



(2.5)

The differential path length Δ between an aperture point x and the center, as our point target moves along the circular arc, is

$$\begin{aligned}\Delta &= \sqrt{R^2+x^2-2Rx\sin\theta} - R \\ &= R\sqrt{1+\left(\frac{x}{R}\right)^2 - 2\left(\frac{x}{R}\right)\sin\theta} - R\end{aligned} \quad (2.6)$$

Expanding this in terms of x/R and θ $\quad \sqrt{1+b} \approx 1+\frac{b}{2}, \; b \ll 1$; $\sin\theta \approx \theta, \; \theta \ll 1$

$$\begin{aligned}\Delta &\cong -x\theta + \frac{x^2}{2R} \\ &\cong -x\theta\end{aligned} \quad (2.7)$$

As in the last section, we ignore the quadratic term on the basis of far field operation.

Destructive interference occurs when the magnitude of any differential path length error exceeds $\lambda/4$, so we want to calculate the angular extent for which

$$-\lambda/4 \le \Delta \le \lambda/4 \quad (2.8)$$

As the maximum error will always be associated with the ends of the aperture, we substitute $x = \pm a/2$ and use $\Delta = \pm\lambda/4$ in eq. (2.7) to get the angles associated with this maximum error.

$$\theta\big|_{\Delta=\pm\lambda/4} = \pm\frac{\lambda}{2a} \quad (2.9)$$

We define this angular extent as the angular resolution θ_R.

$$\boxed{\theta_R = \frac{\lambda}{a}} \quad (2.10)$$

That is, *the angular resolution, measured in radians, is the inverse of the aperture, measured in wavelengths.*

It is also convenient to define the ratio of the range and the aperture, which re-occurs in different contexts, as the f-number.

$$\boxed{f_\# = \frac{R}{a}} \quad (2.11)$$

The distance around the circular arc associated with θ_R is simply $R\theta_R$, yielding the azimuthal resolution u_R.

$$\boxed{u_R = f_\#\lambda} \quad (2.12)$$

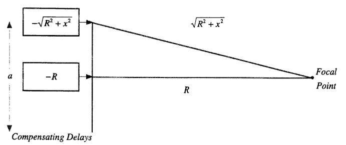
That is, *the azimuthal resolution, measured in wavelengths, is the f-number.*

(From Wright: *Image Formation ...*)

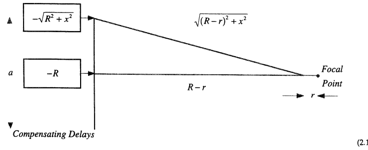
Geom.Opt: Near field/Far field crossover

Depth of Focus of a Focused Linear Aperture

We can bring the far field diffraction pattern into the near field by *focusing* the aperture. This is done by applying compensating delays to incremental portions of the aperture. (In the figure below, we show the compensating delays as corrections to path length.)



(2.15)



(2.16)

The differential delay between the center of the aperture and any other aperture point x (including the compensating delays) is

$$\begin{aligned}\Delta &= \left[\sqrt{(R-r)^2+x^2} - \sqrt{R^2+x^2}\right] - \left[(R-r)-R\right] \\ &= R\left[\sqrt{\left(1-\frac{r}{R}\right)^2+\left(\frac{x}{R}\right)^2} - \sqrt{1+\left(\frac{x}{R}\right)^2} + \left(\frac{r}{R}\right)\right]\end{aligned} \quad (2.17)$$

Expanding this in a two dimensional Taylor's series in terms of r/R and x/R and keeping the lowest order term yields

$$\Delta = \frac{x^2 r}{2R^2} \quad (2.18)$$

The delay error across the aperture is thus seen to be essentially linear in r and quadratic in x. As we move closer to the aperture, the sign of the error is positive, and further away yields a negative error. Similar to our analysis of the near field/far field crossover, we define the depth of focus as the extent of the incremental range r for which

$$-\lambda/8 \le \Delta \le \lambda/8 \quad (2.19)$$

over the aperture. Substituting $x = a/2$ and $\Delta = \pm\lambda/8$ into eq. (2.18), and using $f_\# = R/a$, we see that the depth of focus is bounded by

$$r\big|_{\Delta=\pm\lambda/8} = \pm f_\#^2\lambda \quad (2.20)$$

so the total depth of focus r_D is

$$\boxed{r_D = \pm f_\#^2\lambda = 2f_\#^2\lambda} \quad (2.21)$$

That is, *the one-sided depth of focus, measured in wavelengths, is the square of the f-number.* Other definitions for r_D can be found in the literature, based on criteria other than the $\lambda/8$ differential error we have employed here. Let us look at the effects of moving our point target to the edge of the depth of focus and beyond.

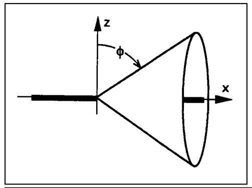
(From Wright: *Image Formation ...*)

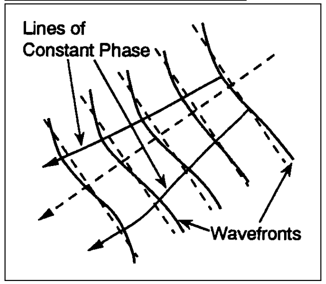
Ultrasound imaging

- Near field/far field transition, D=28mm, f=3.5MHz $\Rightarrow$
  - $\lambda = 1540/3.5 \cdot 10^6 = 0.44$mm and $d_R = D^2/R = 1782$mm
  - All diagnostic ultrasound imaging occurs in the extreme near field!
- Azimuth resolution, D=28mm, f=7MHz $\Rightarrow$
  - $\lambda = 0.22$mm and $\theta = \lambda/D = 0.45°$,
  - i.e. about 200 lines are required to scan $\pm 45°$
- Depth of focus, $f_{\#} = 2$, f=5MHz $\Rightarrow$
  - $\lambda = 0.308$mm and DOF = $2f_{\#}^2\lambda \approx 2.5$mm.
  - Ultrasound requires $T = 2 \cdot 2.5 \cdot 10^{-3}/1540 = 3.2\mu$s to travel the DOF. This is the minimum update rate for the delays in a dynamically focused system.

Ambiguities & Aberrations

- Aperture ambiguities
  - Due to symmetries
- Aberrations
  - Deviation in the waveform from its intended form.
  - In optics; due to deviation of a lens from its ideal shape.
  - More generally; Turbulence in the medium, inhomogeneous medium or position errors in the aperture.
  - Ok if small comp. to λ_0.





$\phi \longleftrightarrow {}'\sin\phi'$

- $\vec{k}$ represents two kind of information
  1. $|\vec{k}| = 2\pi/\lambda$: No. waves per meter
  2. $\vec{k}/|\vec{k}|$: the wave's direction of prop.
- If signal have only a narrow band of spectral components, (i.e. all $\approx w$), we can replace $|k|$ with $w_0/c = 2\pi/\lambda_0$.
  - Example: Linear array along x-axis:

$$W(-k\sin\phi) = \frac{\sin\frac{k_x D\sin\phi}{2}}{\frac{k_x\sin\phi}{2}}$$

$$\Updownarrow$$

$$W(-2\pi\sin\phi/\lambda_0) = W''(\phi) = \lambda_0\frac{\sin D'\pi\sin\phi}{\pi\sin\phi}, \quad D' = D/\lambda_0$$

- $W''(\phi) = W''(\phi+\pi)$, i.e. periodic!! $W(k)$ is not!
- Often $W(u,v)$, $u = \sin\phi\ \cos\theta$, $v = \sin\phi\ \sin\theta$

Co-array for continuous apertures

- $c(\vec{\chi}) \equiv \int w(\vec{x})w(\vec{x}+\vec{\chi})d\vec{x}$, $\vec{\chi}$ called lag and its domain *lag space*.
- Important when array processing algorithms employ the wave's spatiotemporal correlation function to characterize the wave's energy.
- Fourier transform of $c(\vec{\chi})(=|W(\vec{k})|^2)$ gives a smoothed estimate of the power spectrum $S_f(\vec{k}, w)$.

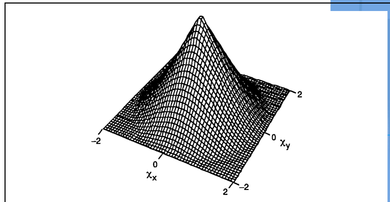


Figure 3.10 The cross-section of the co-array of a circular aperture of radius $R = 1$ is plotted as a function of radial distance χ in lag space.