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INF5410 Array signal processing. Chapter 2.3 Dispersion

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Chapters in Johnson & Dungeon

- Ch. 1: Introduction.
- Ch. 2: Signals in Space and Time.
  - Physics: Waves and wave equation.
    - » c, λ, f, ω, k vector,...
    - » Ideal and ”real" conditions
- Ch. 3: Apertures and Arrays.
- Ch. 4: Beamforming.
  - Classical, time and frequency domain algorithms.
- Ch. 7: Adaptive Array Processing.

Norsk terminologi

- Bølgeligningen
- Planbølger, sfæriske bølger
- Propagerende bølger, bølgetall
- Sinking/sakking: $\vec{\alpha}$
- Dispersjon
- Attenuasjon eller demping
- Refraksjon
- Ikke-linearitet
- Diffraksjon; nærfelt, fjernfelt
- Gruppeantenne (= array)

Kilde: Bl.a. J. M. Hovem: ``Marin akustikk'', NTNU, 1999

Non-real wavenumber and c

- Let $k = k_{\Re} + jk_{\Im}$
- 1D solution to wave equation:

$$s(x,t) = A\,\mathrm{exp}\{j(\omega t - k\cdot x)\} = A\,\mathrm{exp}\{k_{\Im}\cdot x\}\cdot\mathrm{exp}\{j(\omega t - k_{\Re}\cdot x)\}$$

  - Real part is propagation
  - Imaginary part is attenuation

- Let $c = c_{\Re} + jc_{\Im}$ and insert into dispersion equation:

$$k = \frac{\omega}{c} = \frac{\omega}{c_{\Re} + jc_{\Im}} = \frac{\omega}{c_{\Re}^2 + c_{\Im}^2}(c_{\Re} - jc_{\Im}) = \frac{\omega}{c^2}(c_{\Re} - jc_{\Im})$$

- Thus the real and imaginary parts of c and k correspond to each other

Deviations from simple media

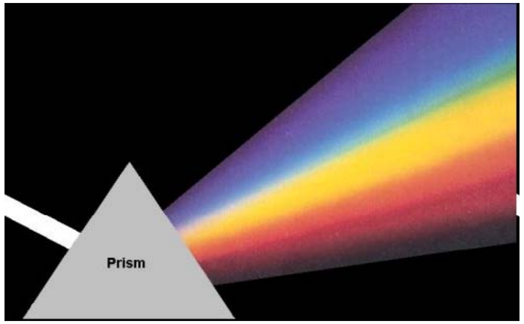
1. Dispersion: $c = c(\omega)$
   - Group and phase velocity, dispersion equation: $\omega = f(k) \neq c \cdot k$
   - Evanescent (= non-propagating) waves: purely imaginary k
2. Loss: $c = c_{\Re} + jc_{\Im}$
   - Wavenumber is no longer real, imaginary part gives attenuation.
   - Waveform changes with distance
3. Non-linearity: $c = c(s(t))$
   - Generation of harmonics, shock waves
4. Refraction, non-homogenoeus medium: $c=c(x,y,z)$
   - Snell's law

Dispersion and Attenuation

- Ideal medium: Transfer function is a delay only
- Attenuation: Transfer function contains resistors
- Dispersion: Transfer function is made from capacitors and inductors (and resistors) => phase varies with frequency

1. Dispersion

- Different propagation speeds for components with different wavelengths
- Light in a glass prism
- Ocean Waves: Large wavelengths travel faster



$$c = \sqrt{(\frac{g\lambda}{2\pi}\tanh(2\pi\frac{H}{\lambda}))} \approx \sqrt{(gH)}$$

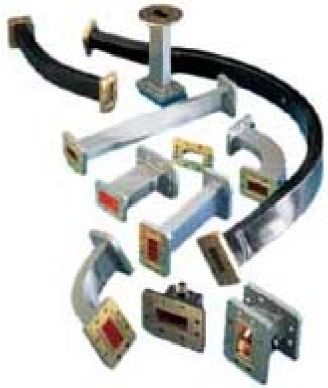
(Shallow water approximation)

Two kinds of dispersion

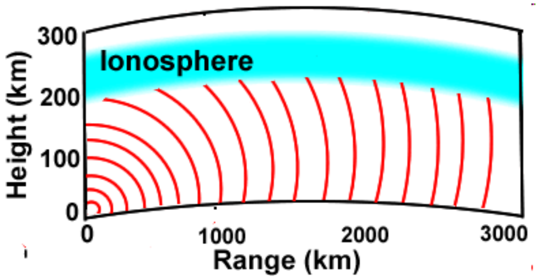
- Intrinsic or material dispersion
  - Prism
  - Ocean waves
- Geometric dispersion
  - Constructive interference of waves in bounded or heterogeneous media
  - Waveguides

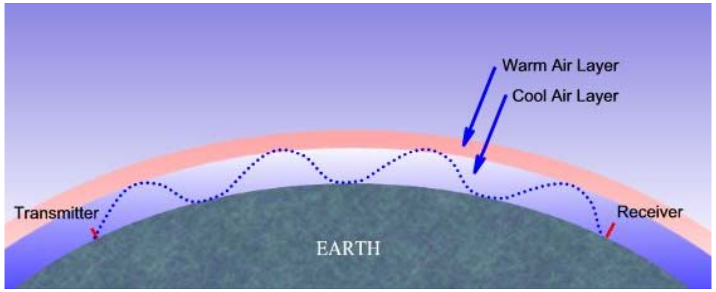
Dispersion – waveguide (geometric)

- Electromagnetic waves
  - Waveguide made from metal = microwave component
  - Ionosphere
    - » E.g. DFC77 – clock signals from Frankfurt, Germany
    - » λ = 3e8/77.5e3 = 3.87 km
  - Troposphere
    - » Lowest ~10 km
    - » FM radio from Denmark to the South of Norway in summer



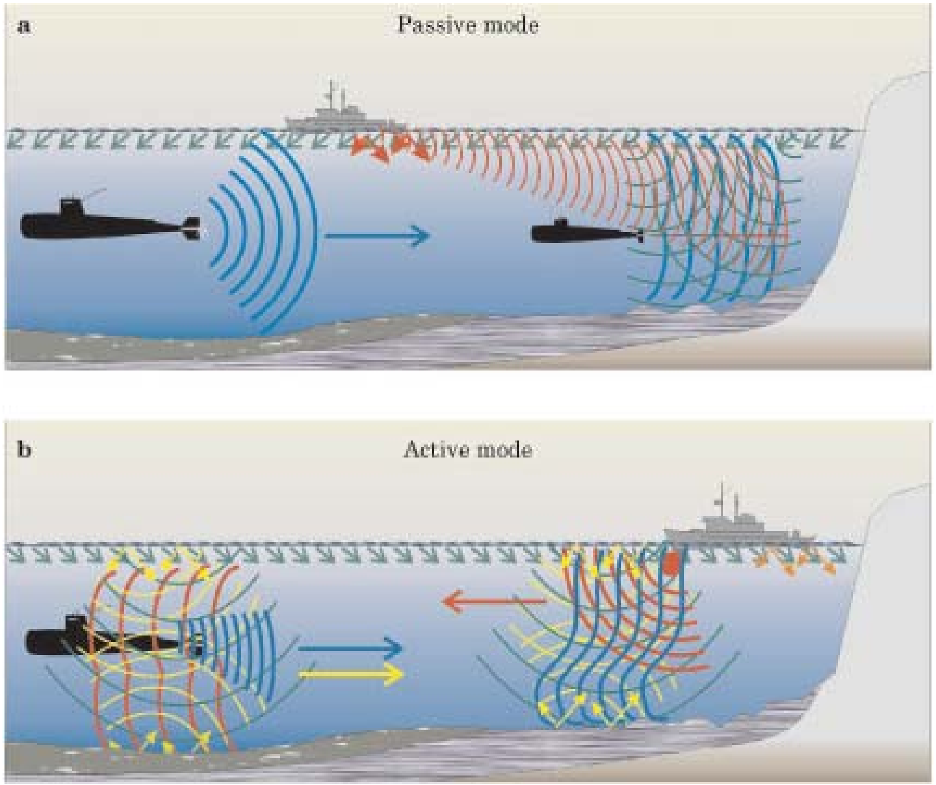
Bølgeleder





Acoustic waveguide

- Between sea surface and sea bottom
- Shallow depths
  - Desired signal (blue)
  - Ocean surface noise (green)
  - Shipping noise (red).
  - Backscattered reverberation (yellow)



- Kuperman and Lynch, “Shallow-water acoustics”, Physics Today, 2004

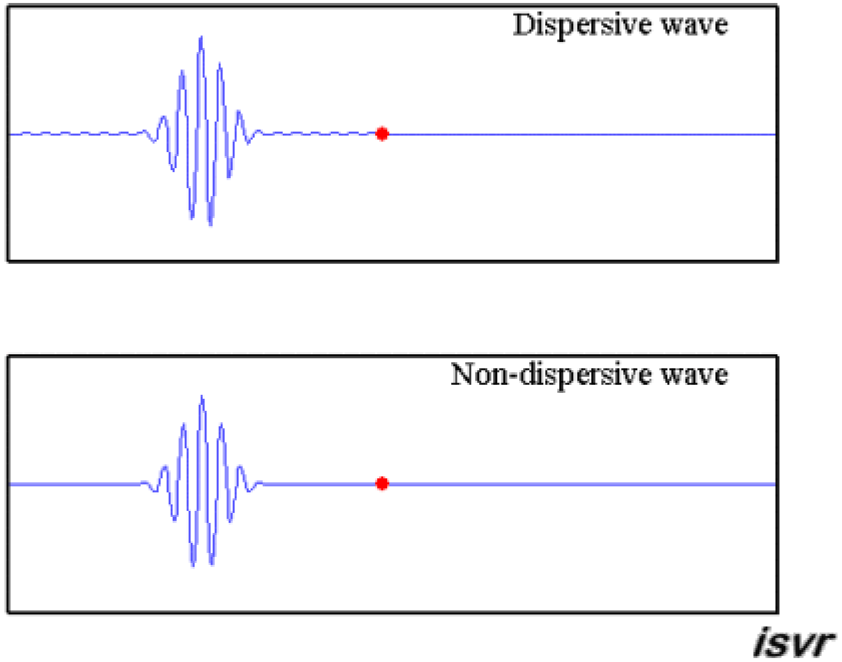
Dispersion

- Wave equation in simple medium:

$$\nabla^2 \overrightarrow{s} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$

- Dispersion relation $\omega = c \cdot k$

- Dispersive medium: $c = c(\omega)$ varies with frequency



http://www.isvr.soton.ac.uk/SPCG/Tutorial/Tutorial/Tutorial_files/Web-further-dispersive.htm

Dispersion – string-like (intrinsic)

New

- String-like medium: $\nabla^2 \overrightarrow{s} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} + \frac{\omega_c^2}{c^2}s$
- Solution as before: $s(\overrightarrow{x}, t) = A\,\text{exp}\{j(\omega t - \overrightarrow{k} \cdot \overrightarrow{x})\}$
- Assume 1-D and insert: $-k^2 s = \frac{1}{c^2}(-\omega^2)s + \frac{\omega_c^2}{c^2}s$
- Dispersion relation:

$$k^2 = \frac{1}{c^2}(\omega^2 - \omega_c^2) => \omega = c\sqrt{(}k^2 + \frac{\omega_c^2}{c^2})$$

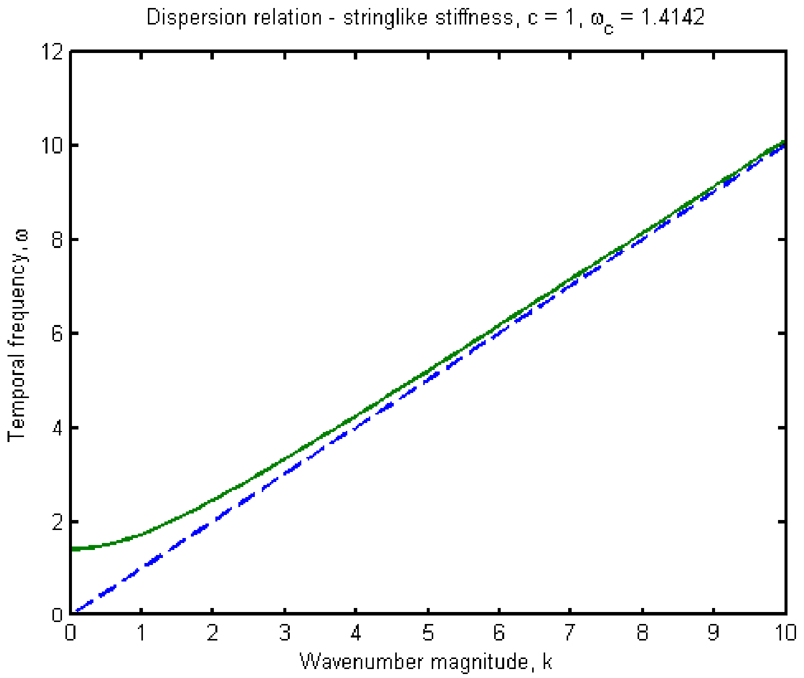
- Klein-Gordon equation for relativistic electrons

Dispersion – stringlike stiffness

- Dispersion relation:

$$\omega = c\sqrt{(k^2 + \frac{\omega_c^2}{c^2})}$$

- Cut-off frequency ω_c :
  - $\omega < \omega_c$ => imaginary k
  - no propagation possible



Waveguide (geometric)

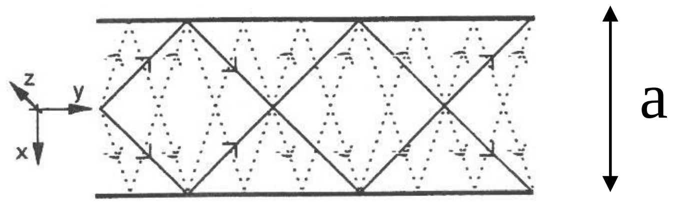


Figure 2.9 A simple metal waveguide can be used to control the direction of propagation of electromagnetic waves. The $TE_{m,0}$ solution to the wave equation may be interpreted as a plane wave bouncing from side to side down the waveguide.

- Metal & electromagnetic waves or acoustic with hard walls
  - Conducting walls: E-field is normal to wall, E(x)=0, x=0,a
  - Acoustic: Zero pressure on walls
- Many propagating fields: $TE_{m,0}$ family, E-field parallel to z-axis:

$$E_z(x,y,z,t) = E(x)\cdot E(y)\cdot E(t) = A\sin\frac{m\pi x}{a}e^{j(\omega t - k_y y)}, k_x = \frac{m\pi}{a}$$

- m=1: one half period = a, m=2: two half periods = a, ...

Waveguide

- Waveguide k_x inserted in wave equation:

$$k_x = \frac{m\pi}{a}, k_x^2 + k_y^2 = \frac{\omega^2}{c^2} => |k_y| = \sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2}}$$

(similar to previous example if $\omega_c = m^2\pi^2/a^2$, each *m* is a mode)

- Angle of propagation rel. to just bouncing up and down:

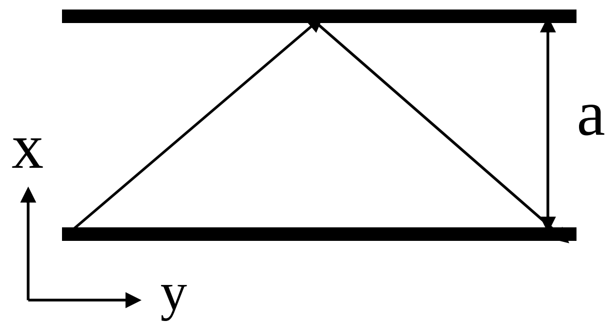
$$\tan\theta = |\frac{k_y}{k_x}| = \sqrt{\frac{\omega^2}{m^2}\frac{a^2}{\pi^2 c^2} - 1} \approx \frac{\omega a}{m\pi c}$$

(High frequency approx)

- Like a sailboat tacking (zigzag across a headwind)
- The larger the ω, the more parallel to the waveguide

Waveguide and evanescent waves

$$k_y^2 = \frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2}$$



- Critical frequency where $k_y=0$ or $\theta=0$: $\omega=m\pi c/a$
- Below cut-off frequency: $k_y = j\,|k_y|$, i.e. is imaginary:

$$E(x,y,z,t) = A\sin\frac{m\pi x}{a}e^{j(\omega t - k_y y)} = A\sin\frac{m\pi x}{a}e^{|k_y|y}e^{j(\omega t)}$$

- No propagation in y, except an exponentially damped wave
- Non-propagating: **Evanescent wave** (vanish=forsvinne)

Evanescent waves

- Exp. damped: not suited for information transfer, but:

1. Near field microscopy – dist. to object and resolution << λ
   - F. Simonetti, Localization of pointlike scatterers in solids with subwavelength resolution, Applied Physics Letters **89**, 2006
2. Metamaterials (period media) for evanescent -> propagating wave conversion:
   - Li et al, Experimental demonstration of an acoustic magnifying hyperlens, Nature materials, 2009
3. Can evanescent waves be used for effective medium range energy transfer to electronic devices?
   - A. Karalis, J. D. Joannopoulos, M. Soljacic, "Wireless Non-Radiative Energy Transfer," Annals of Physics, 2008.

Phase velocity

- During one period, T, the wave propagates forward by one wavelength, λ.
- Phase velocity: speed at which planes of constant phase, $k \cdot x = C$ propagate
- $|v_p| = \lambda/T = \omega/k$

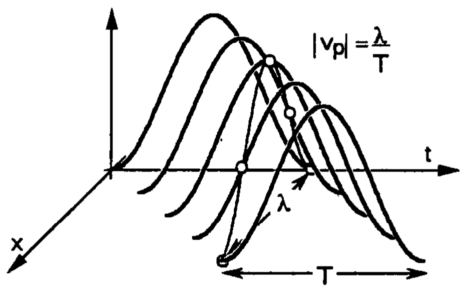


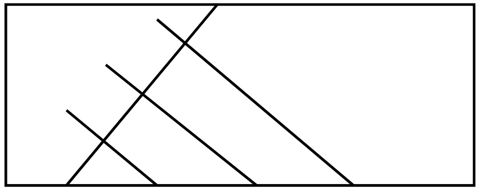
Figure 2.6 The speed of propagation of planes of constant phase is equal to the wavelength divided by temporal period of oscillation. Here, five successive snapshots of the sinusoidal propagating wave were taken at five spatial locations. The time at which the wave reached the last location equaled the period of the sinusoid. Sampling the waveforms at time *T* demonstrates that the spatial waveform is also sinusoidal. The spatial period of the wave equals its wavelength.

Phase velocity in waveguide

- Demo: http://www.physics.ucdavis.edu/Classes/NonclassicalPhysics/phasegroup.html
- In a waveguide, high frequencies travel faster than low frequencies
- The lower frequencies bounce off the walls more often
- Phase velocity: the rate of progress of constant-phase planes down the waveguide:

$$|v_p| = \frac{\omega}{|k_y|} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - k_x^2}} = \frac{c}{\sqrt{1-(\lambda/\lambda_x)^2}} \geq c$$

- Extend phase fronts to y-axis => $v_p = c/\tan\theta > c$



Group velocity (1)

- Consider a group of closely spaced waves in frequency = an information 'package', consisting of two sinusoids:

$$s(t,x) = sin(\omega_1 t - k_1 x) + sin(\omega_2 t - k_2 x)$$

- They are close in k and ω:

$$k_0 = (k_1 + k_2)/2, \Delta k = (k_2 - k_1)/2$$

- Can write signal as

$$s(t,x) = 2sin(\omega_0 t - k_0 x) \cdot cos(\Delta\omega t - \Delta k x)$$

- Interpret as a signal (ω_0, k_0) modulated by ($\Delta\omega$, Δ k)

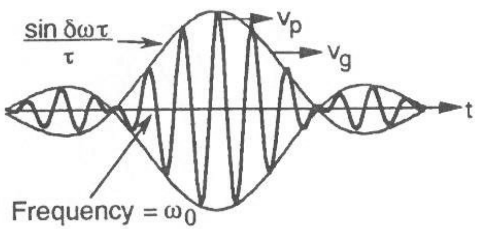


Figure 2.7 The real part of $s(\vec{x}, t)$ for a group of wave components with temporal frequencies in a narrow band between $\omega_0 - \delta\omega$ and $\omega_0 + \delta\omega$.

Group velocity (2)

- Modulation pattern's maximum moves with a speed which makes the argument constant, i.e. $\Delta\omega t - \Delta k x = C$
- Differentiate with respect to t:
  $$v_g = dx/dt = \Delta\omega/\Delta k \rightarrow d\omega/dk$$
- The group velocity is the important one

Phase and Group velocity

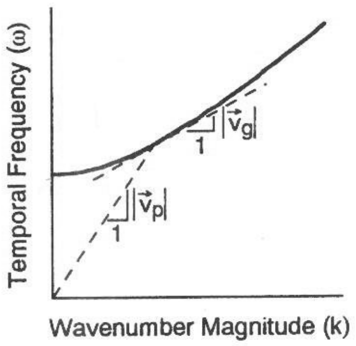


Figure 2.8 The group propagation speed $|\vec{v}_g|$ equals the slope of the dispersion relation at a particular wavelength. The phase propagation speed $|\vec{v}_p|$ equals the slope of the line joining a point on the dispersion relation to the origin.

Waveguide: group velocity

- Group velocity:

$$|v_g| = \frac{1}{|\frac{dk_y}{d\omega}|} = \frac{1}{\frac{d}{d\omega}\sqrt{\frac{\omega^2}{c^2} - k_x^2}} = c\sqrt{1-(\lambda/\lambda_x)^2} \leq c$$

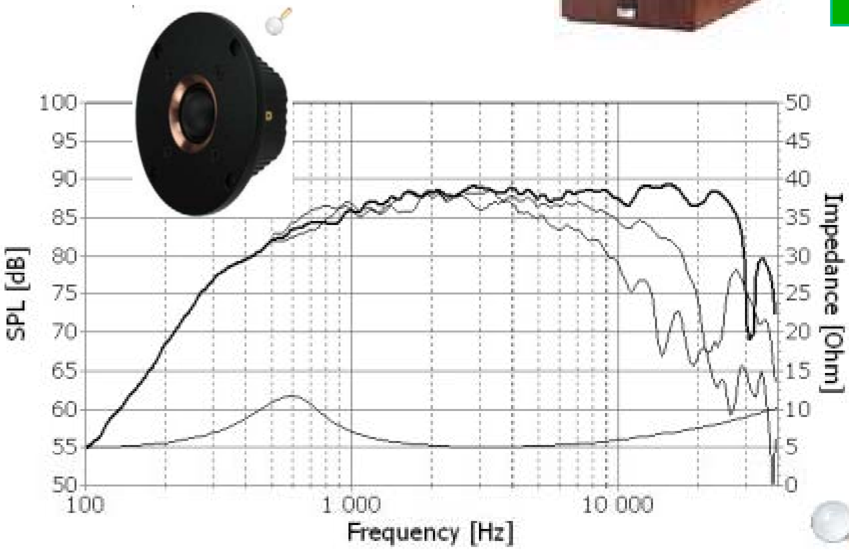
- Phase velocity:

$$|v_p| = \frac{\omega}{|k_y|} = \frac{c}{\sqrt{1-(\lambda/\lambda_x)^2}} \geq c$$

- Geometric average is always c, i.e. $c^2 = v_p \cdot v_g$

Dispersion in loudspeakers means something else

- Physics: Medium spreads ('disperses') pulse in time
- Loudspeakers: Ability to spread ('disperse') sound in space
- Seas T25CF002, "MILLENNIUM": 25mm soft dome tweeter
  - Frequency responses show free field sound pressure at 0, 30, and 60 deg



Array Processing Implications

- In dispersive media, narrowband sources propagate to the array at speeds different from the medium's characteristic speed
  - Group velocity, not the phase velocity or c, must be used for beamforming
- Dispersion causes the received waveform emanating from a broadband source to vary with range
  - May have to compensate in beamforming
- Dispersion can remove some frequency components entirely
  - Evanescent waves