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INF5410 Array signal processing. Chapter 2.3 Attenuation

Sverre Holm

MEDT8007 Simulation Methods in Ultrasound Imaging - NTNU

Sverre Holm

Deviations from simple media

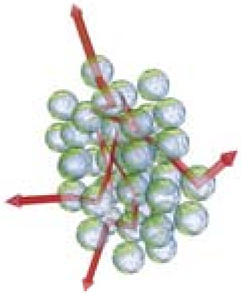
1. Dispersion: $c = c(\omega)$
   - Group and phase velocity, dispersion equation: $\omega = f(k) \neq c \cdot k$
   - Evanescent (= non-propagating) waves: purely imaginary k
2. Loss: $c = c_{\Re} + jc_{\Im}$
   - Wavenumber is no longer real, imaginary part gives attenuation.
   - Waveform changes with distance
3. Non-linearity: $c = c(s(t))$
   - Generation of harmonics, shock waves
4. Refraction, non-homogenoeus medium: $c=c(x,y,z)$
   - Snell's law

Dispersion and Attenuation

- Ideal medium: Transfer function is a delay only
- Attenuation: Transfer function contains resistors
- Dispersion: Transfer function is made from capacitors and inductors (and resistors) => phase & phase velocity vary with frequency

Two mechanisms for attenuation

1. Absorption (emphasis in this lecture)
   - Classical losses (viscous term)
   - Relaxation losses – change of kinetic or translational energy of molecules into internal energy
   - Dominant in ultrasound, at least at low frequency (<10-15 MHz) + sonar + ultrasound in air
2. Multiple scattering
   - Apparent loss as energy is only redirected not converted to heat
   - Seismics: probably a mix of absorption and scattering losses, depending on rock type and frequency
   - Believed to be major cause of losses in elastography
     - » Our research: scattering loss in media with fractal properties – relationship with cancer tumor growth



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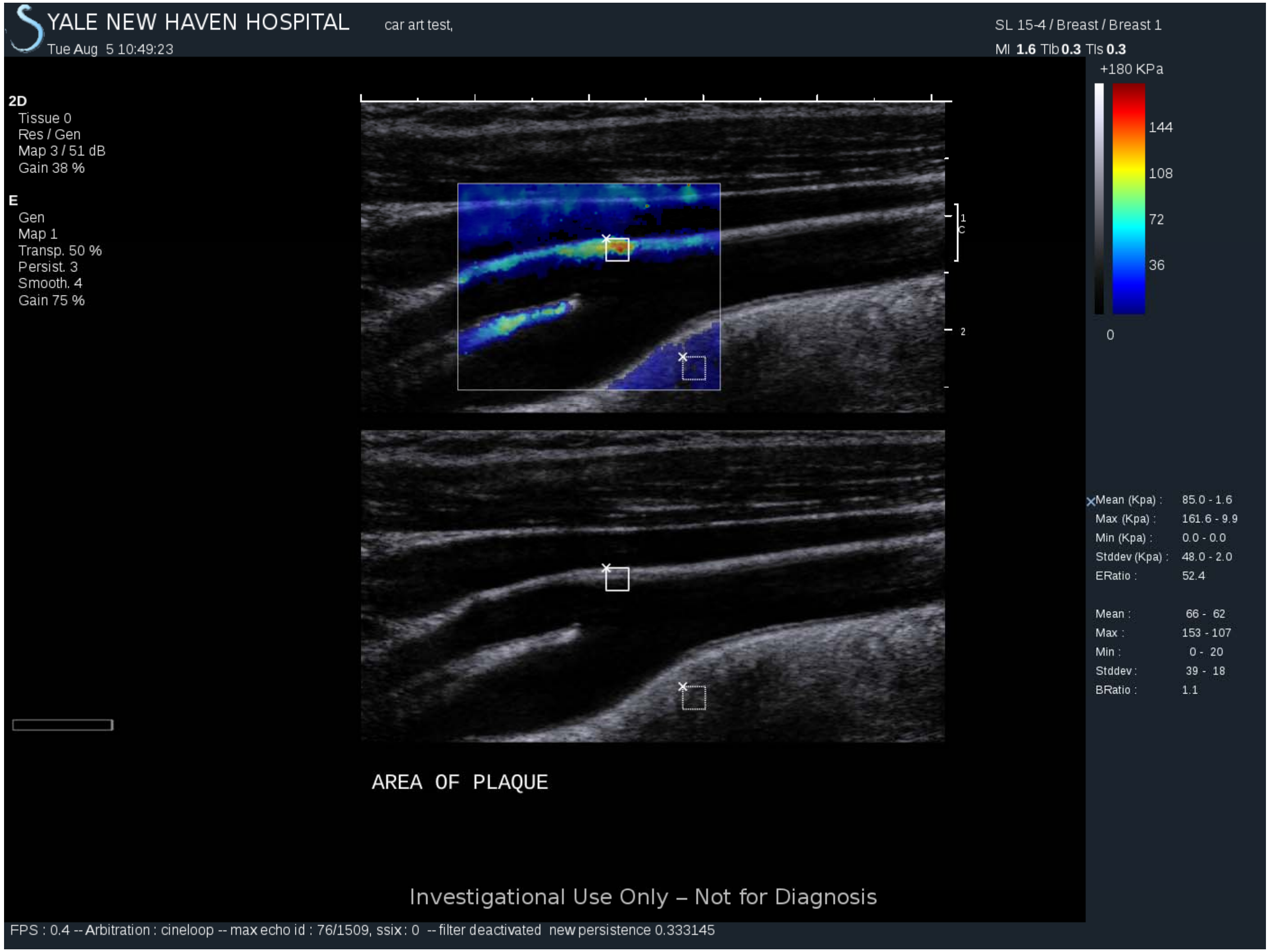
Ultrasound + elastography: *In vivo* assessment of carotid plaque elasticity



Aixplorer,
Mickael Tanter,
ESPCI

Topics

- Viscous attenuation
- Multiple relaxation attenuation
- Power law attenuation: Fractional wave equations

Attenuation/absorption & PDEs

1. Absorption in air and water: $\propto f^2$
   - Viscous differential equation, multiple relaxation
2. Also differential equation for $\propto f^0$
3. Medical ultrasound $\propto f^y$, where $y \approx 1$
4. General differential equation for $0 \leq y \leq 2$?

Viscous wave equation

Additional loss term

- Sound in a viscous fluid, augmented wave eq.:

$$\nabla^2 \vec{s} = \frac{1}{c^2}\frac{\partial^2 \vec{s}}{\partial t^2} - \frac{4\mu}{3\rho_0 c^2}\frac{\partial}{\partial t}\nabla^2 \vec{s} = \frac{1}{c^2}\frac{\partial^2 \vec{s}}{\partial t^2} - \tau\frac{\partial}{\partial t}\nabla^2 \vec{s}$$

  - μ is shear bulk viscocity coefficient
  - τ is a relaxation time
  - Johnson & Dudgeon, problem 2.7
- Approximate solution (low frequency, low loss):

$$k_{\Im} \approx -\frac{\tau}{2c}\omega^2$$

- Attenuation that increases with ω^2

Dispersion relation

- Viscoelastic wave equation: $\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \tau\frac{\partial}{\partial t}(\nabla^2 u) = 0$

- Assume 1-D, and u(x,t)=exp(j(ωt-kx)):

$$(-jk)^2 u(x,t) - \frac{(j\omega)^2}{c_0^2} u(x,t) + \tau(j\omega(-jk)^2)u(x,t) = 0$$

$$k^2 - \frac{\omega^2}{c_0^2} + j\omega\tau k^2 = 0$$

- $k=k_{\Re}+jk_{\Im}=\beta-j\alpha \Rightarrow$ u=exp($-\alpha$x)·exp(j(ωt-βx))
- Let $\omega\tau \ll 1$ and solve for k: $\alpha \approx \frac{\tau}{2c_0}\omega^2$

See solution of problem 2.7 for more details

Deriving the viscous wave equation

1. Equation of state: $$\sigma(t) = E_0 \left(\epsilon(t) + \tau \frac{\partial \epsilon(t)}{\partial t} \right)$$
   - Spring + damper, Kelvin-Voigt, E_0 = Young's modulus.
   - Stress: σ(t) (~pressure)
   - Strain: ϵ(t) (deformation = displacement, *u*, rel. to ref. length).
2. Mass conservation: $$\epsilon(t) = \frac{\partial u}{\partial x}$$
3. Momentum conservation (Newton's second law): $$\nabla \sigma(t) = \rho \frac{\partial^2 u}{\partial t^2}$$

Substitute ϵ from Eq. (2) into Eq. (1). Then differentiate with respect to the spatial variable. This will give $\nabla\sigma$(t) on the left-hand side which is then replaced by the expression from Eq. (3).

More physical + more complex: Viscous + multiple relaxation

$$\alpha = -k_{\Im} = A_0\omega^2 + \sum_{n=1}^{N} A_n \frac{\omega_n}{\omega^2 + \omega_n^2}\omega^2$$

- Term 1: Classical losses – exchange of energy into heat, primarily viscous losses + heat conduction, diffusion, and radiation
- Sum: Relaxation losses – change of kinetic or translational energy of molecules into internal energy
- Each term in sum rises with ω^2 then levels off
- Nachman et al: An equation for acoustic propagation in inhomogeneous media with relaxation losses, JASA 1990
- Builds on Zener eq. of state:

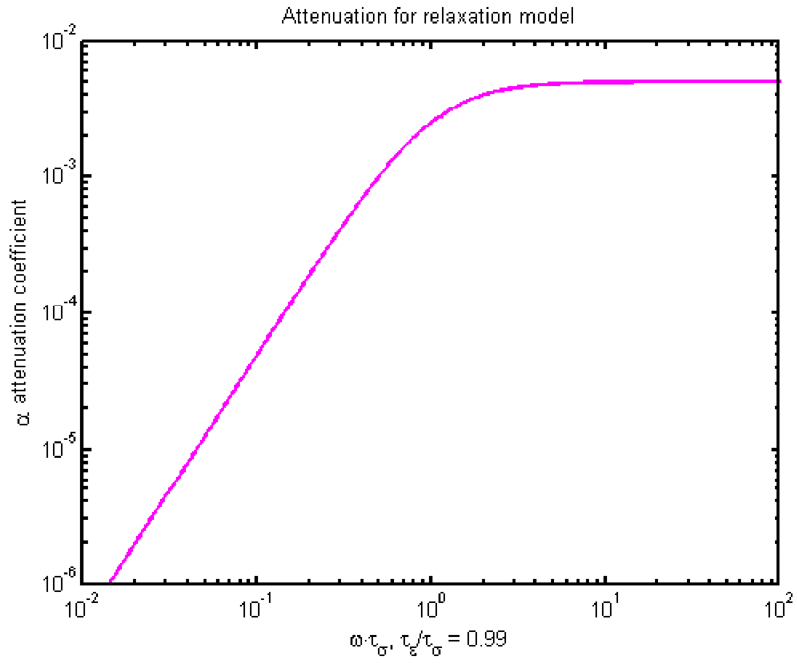
$$\sigma(t) + \tau_\epsilon \frac{\partial \sigma(t)}{\partial t} = E_0 \left(\epsilon(t) + \tau_\sigma \frac{\partial \epsilon(t)}{\partial t} \right)$$

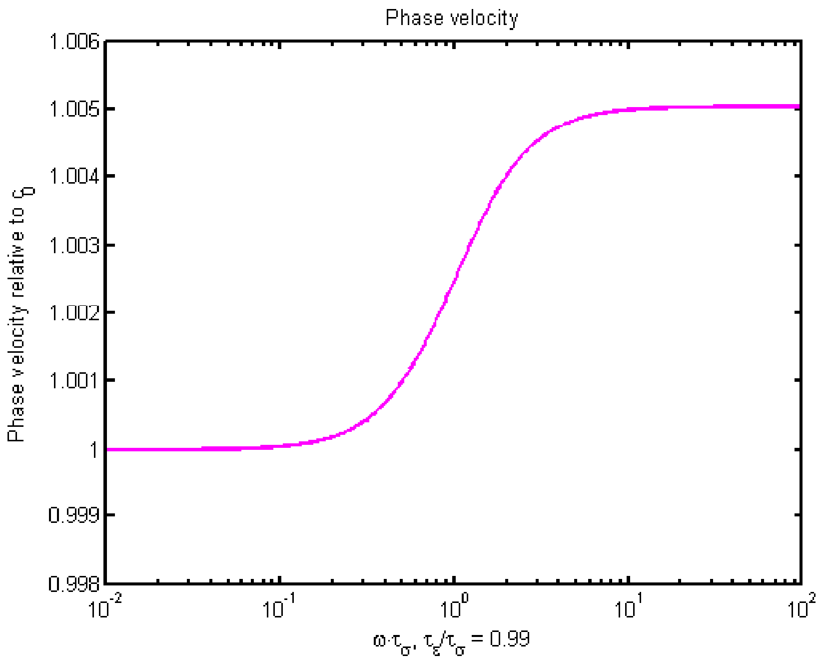
Parameter values, relaxation

- Time constants in Zener model are very close:
  - Air (nitrogen and oxygen):
    - » Oxygen at 0 % RH, 20 C and 1 atm
      $\Rightarrow$ c from 343.23 to 343.35 m/s between 10-100 Hz
      $\Rightarrow \tau_\sigma/\tau_\epsilon \approx 1.0007$
  - Fluorine
    - » at 102 C and 1 atm
      $\Rightarrow$c from 332 to 339 m/s between 5-200 kHz
      $\Rightarrow \tau_\sigma/\tau_\epsilon \approx 1.043$.

A single relaxation term

$$k_{\Im} = A_n \frac{\omega_n}{\omega^2 + \omega_n^2} \omega^2$$





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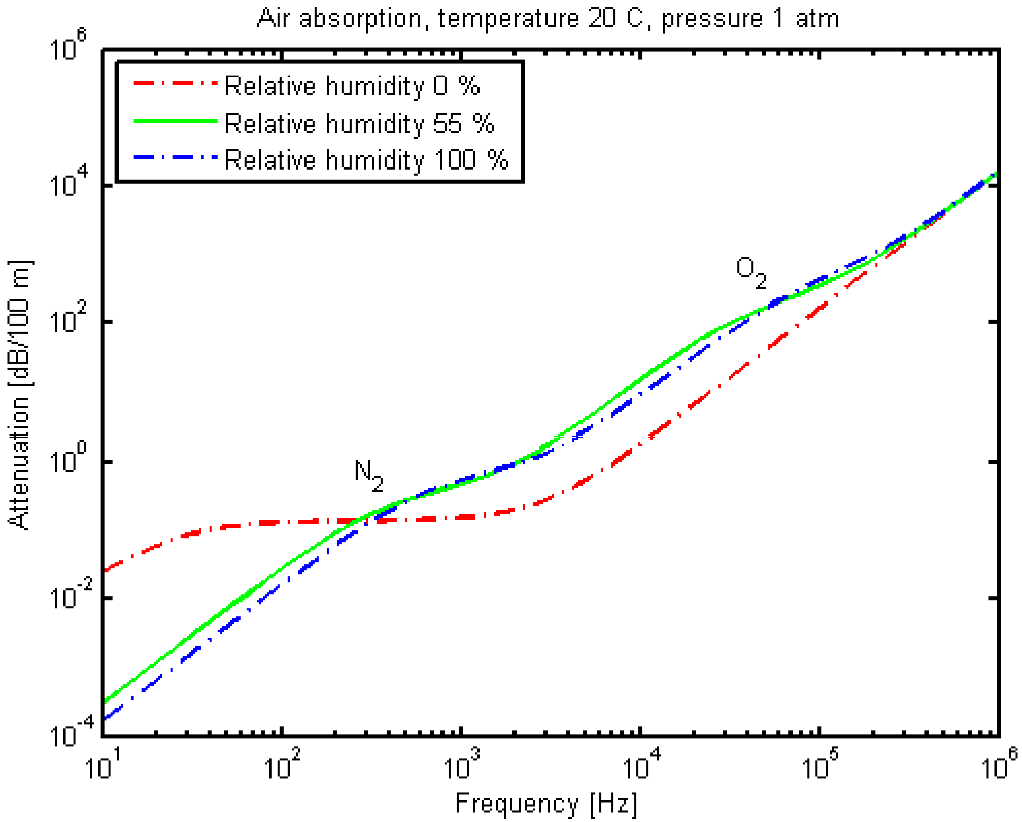
Relaxation + viscous: Air

- Viscous losses dominate the first term (A_0)
- N=2:
  - Nitrogen (78.1%): f1<650 Hz
  - Oxygen (20.9%): f2<80 kHz
- Dispersion: Δc_{phase} < 0.14 m/s

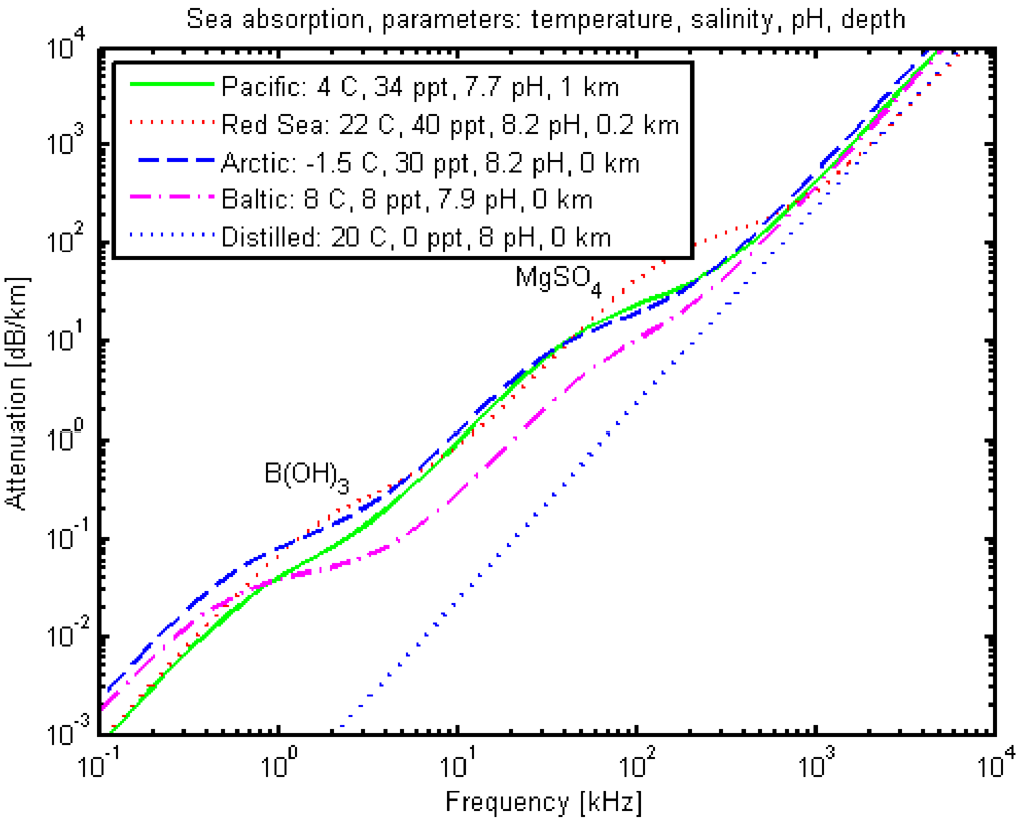


Evans, Bass, Sutherland: Atmospheric absorption of sound: Theoretical predictions, JASA 1972

Bass, Sutherland, Zuckerwar, Blackstock, Hester, Atmospheric absorption of sound: Further developments, JASA, 1995

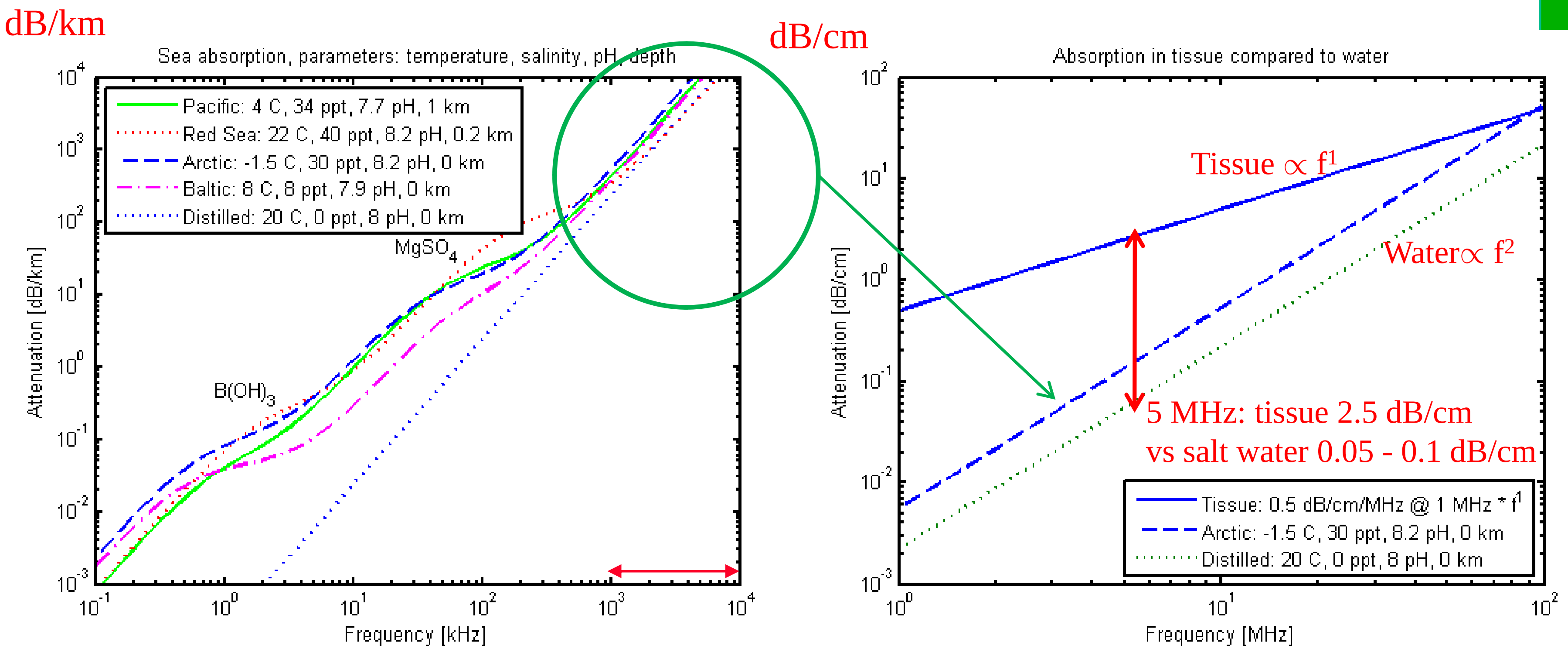
Relaxation + viscous: Sea water

- A_0: Viscous absorption of water molecule = distilled water
- N=2:
  - Boron acid: f_1<2 kHz
  - Magnesium sulphate: f_2<150 kHz
- Dispersion: $\Delta c_{phase} < 0.05$ m/s



Ainslie & McColm, A simplified formula for viscous and chemical absorption in sea water, JASA, 1998

Absorption: water vs tissue

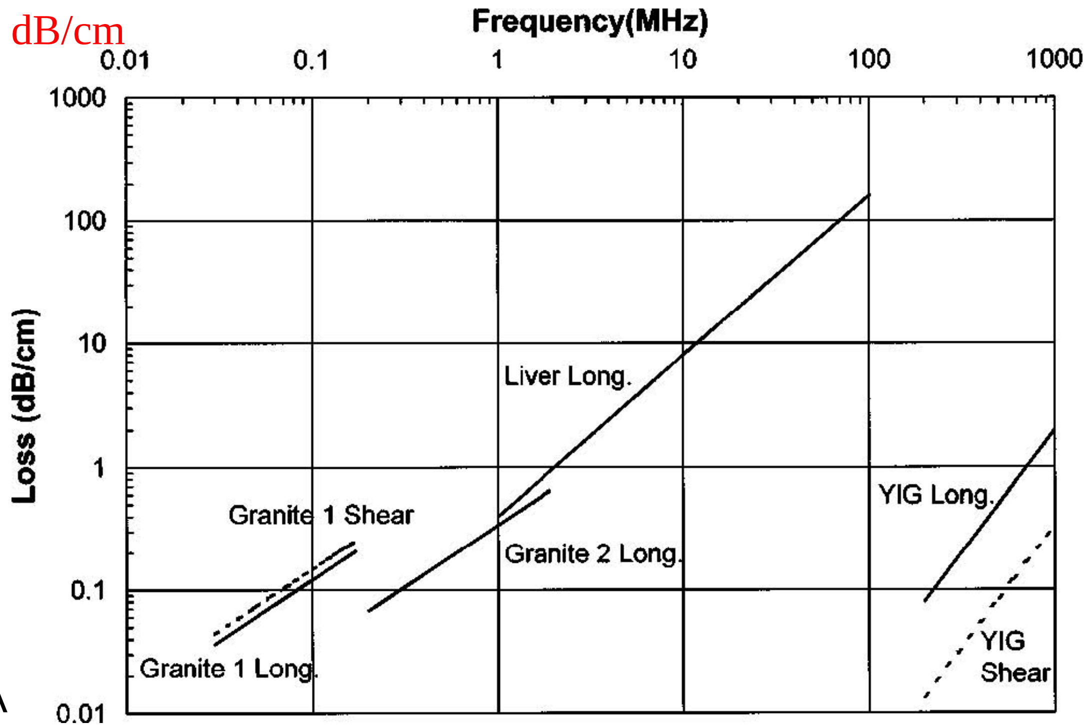


Medicine, geophysics, ...

- Similar power laws
- Pressure waves:
  - Granite: y ≈ 1
  - Liver: y ≈ 1.3
- Shear waves:
  - YIG: y=2
    (Yttrium indium garnet)
  - Granite: y ≈ 1

- Szabo and Wu, “A model for longitudinal and shear wave propagation in viscoelastic media”, JASA (2000).

dB/cm



Data for shear and longitudinal wave loss which show power-law dependence over four decades of frequency.

Medical ultrasound: absorption

$$\alpha = \alpha_0 \cdot f^y$$

- Rule-of-thumb:
  $\alpha_0 = 0.5$ dB/MHz/cm, y=1
- Ex: 5 MHz, 10 cm depth
  - 5 · 20 · 0.5 = 50 dB
  - Absorption loss dominates over spherical spreading loss

- Liver:
  - Exponent: y= 1,...,1.3
  - α_0=0.35,..., 0.9 dB/MHz/cm at 1 MHz.
- Breast up to y=1.5

Kadaba, Bhagat, Wu, “Attenuation and backscattering of ultrasound in freshly excised animal tissues, IEEE Trans. Biomedical Eng., 1980,

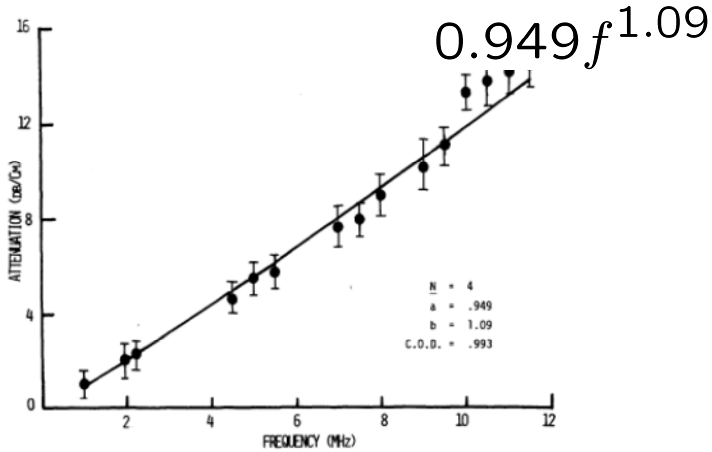


Fig. 6. Attenuation coefficient versus frequency for freshly excised LV muscle tissue.

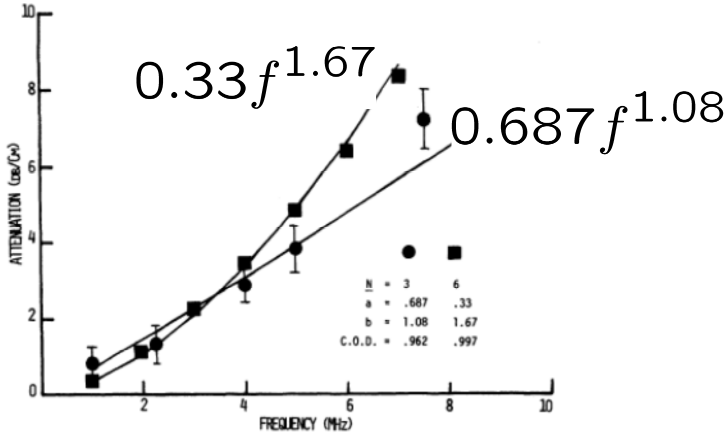
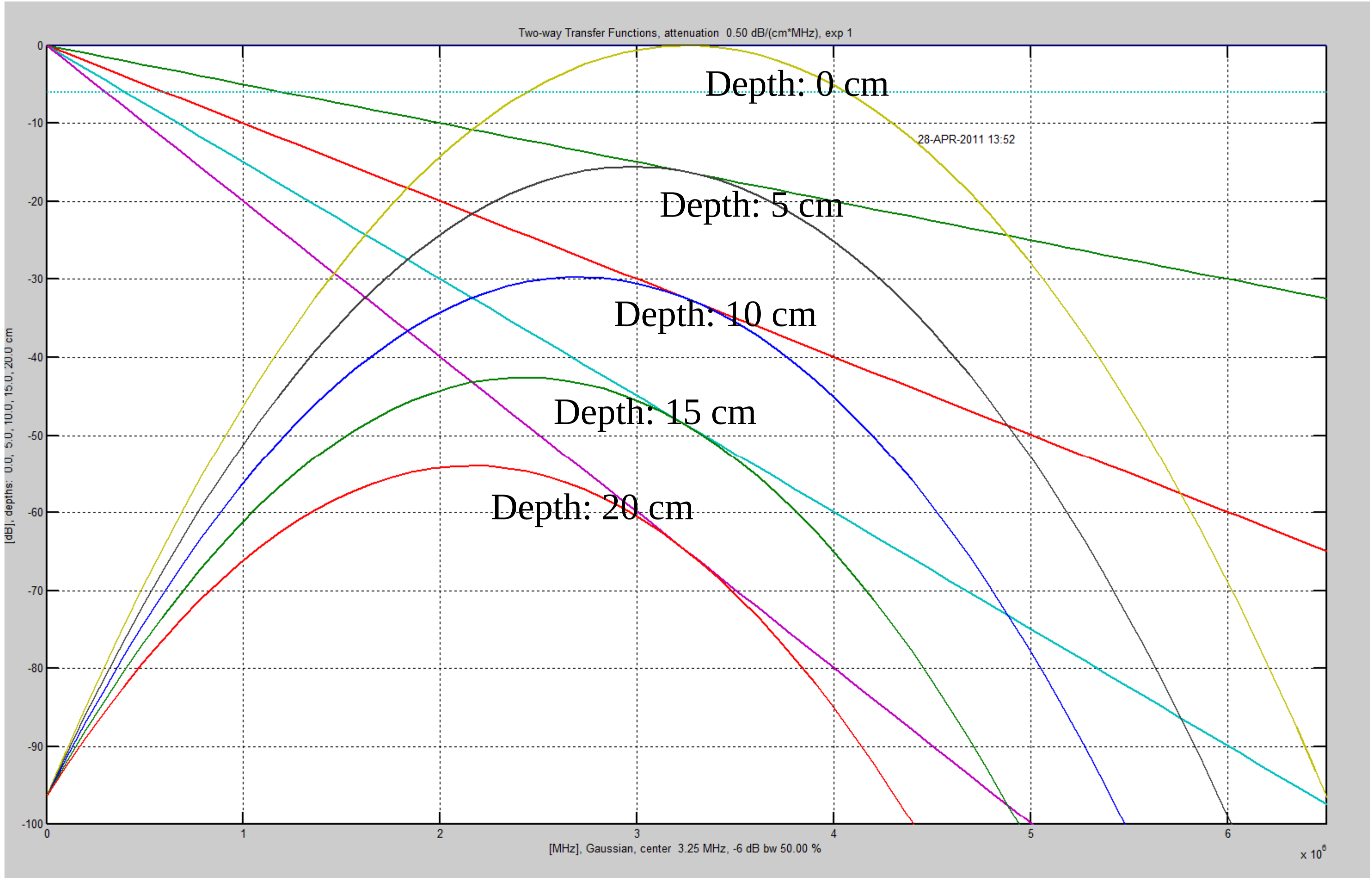


Fig. 7. Attenuation coefficient versus frequency for freshly excised (●) and fixed (■) spleen tissue.

Medical: Spectrum vs depth



Can show that for a Gaussian spectrum, the center frequency will fall linearly with depth for f^y attenuation (Kuc, 1984, IEEE ASSP)

Simulated in Ultrasim

Effect of attenuation/loss

- Fall in amplitude due to spherical spreading: $20\log(R/R_0)$
- Additional losses
  - In water for underwater acoustics
  - Can usually neglect it for audible sound
- Combined: $20\log(R/R_0) + \alpha R$
- Plays a role in estimating level i.e. range for
  - long-range sonar
  - Ultrasound in air positioning (40 - 80 kHz)

Attenuation - Dispersion

- Attenuation and dispersion are linked to guarantee causality
- O'Donnell, Jaynes, Miller, `Kramers-Kronig relationship between ultrasonic attenuation and phase velocity,' J. Acoust. Soc. Am., 1981
  - Predicted dispersion in dog myocardium
  - Very small => distortion of pulse form in medical ultrasound is negligible

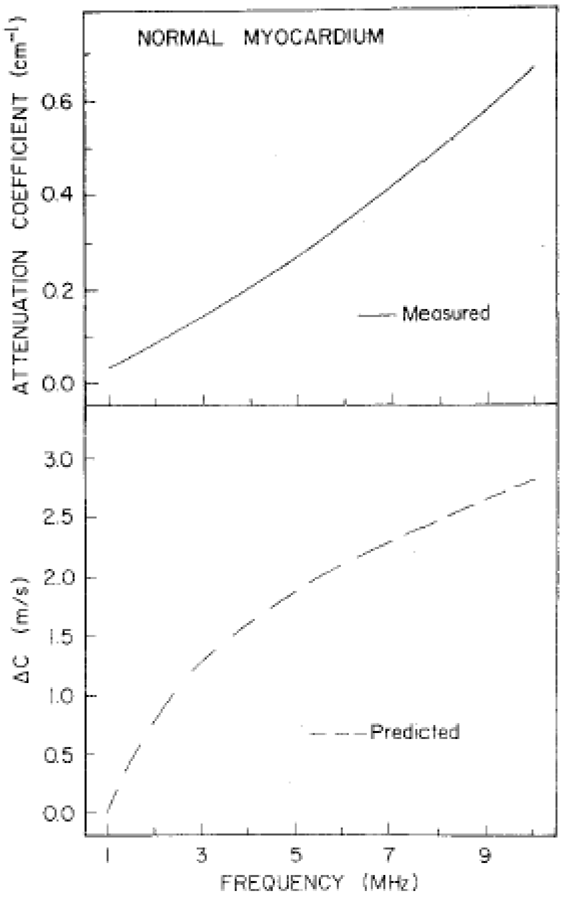
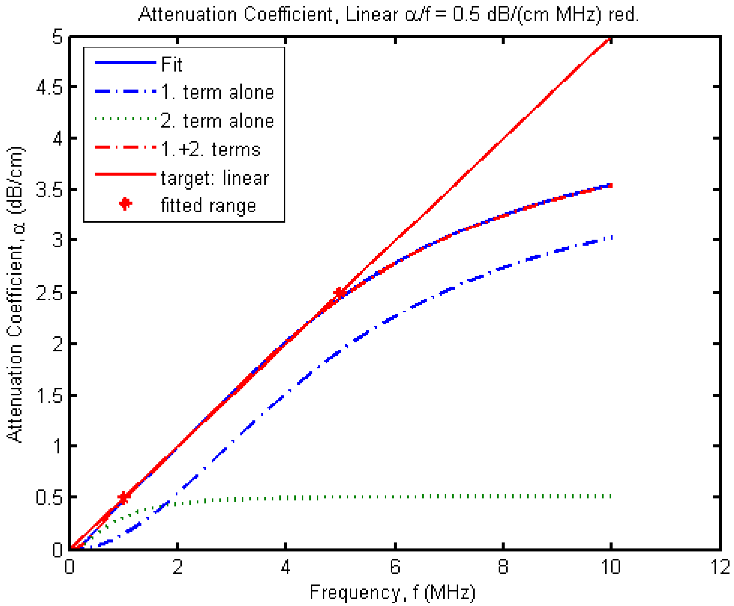


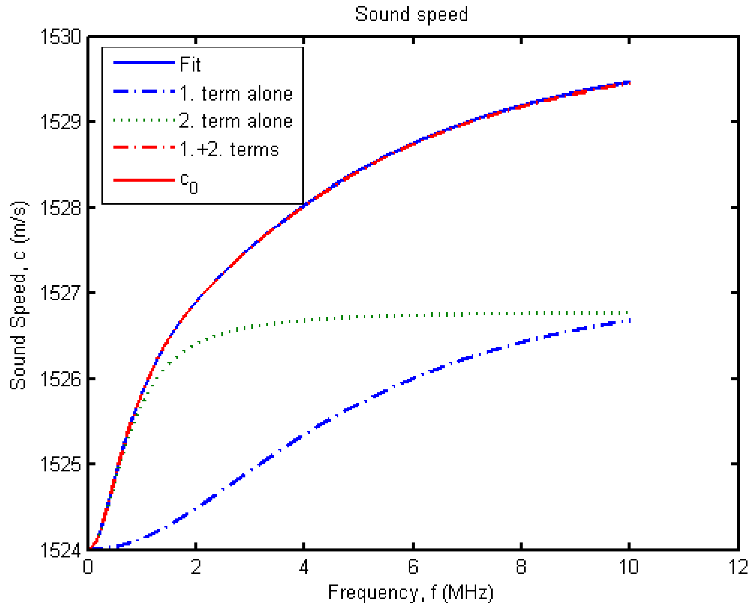
FIG. 5. The attenuation measured in normal dog myocardium is illustrated in the top panel. The lower panel presents the dispersion predicted by applying Eq. (30) to the attenuation data of the top panel.

$$\alpha = -k_{\Im} = A_0\omega^2 + \sum_{n=1}^{N} A_n \frac{\omega_n}{\omega^2 + \omega_n^2}\omega^2$$

Relaxation model - medical ultrasound

- Two relaxation terms, fit to f^1 for $f \in 1..5$ MHz
  - Tabei, Mast, Waag: Simulation of ultrasonic focus aberration and correction through human tissue, JASA, 2003

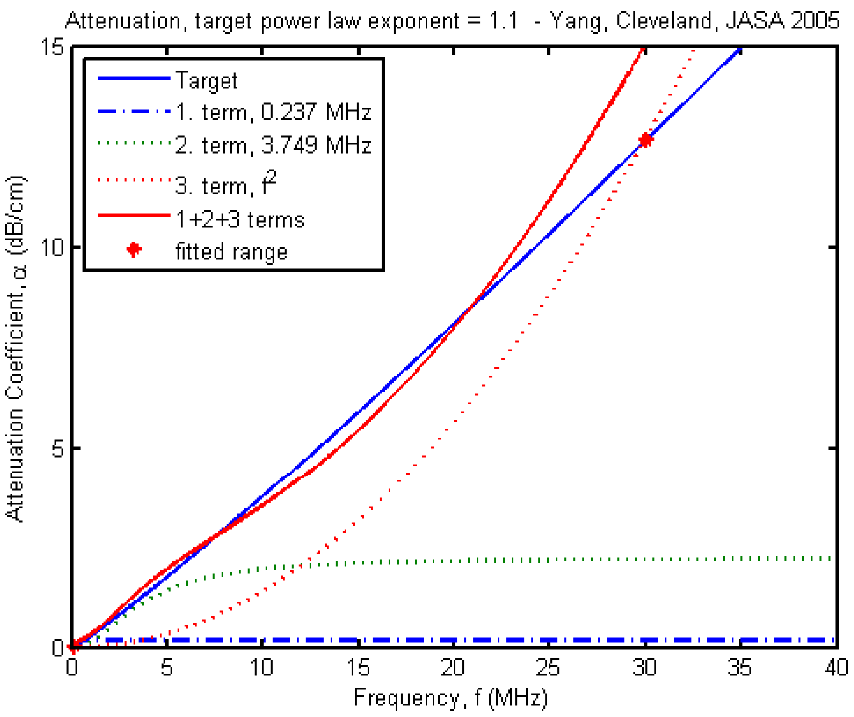




$$\alpha = -k_{\Im} = A_0\omega^2 + \sum_{n=1}^{N} A_n \frac{\omega_n}{\omega^2+\omega_n^2}\omega^2$$

Relaxation model - medical ultrasound

- f^2 + two relaxation models, fit to $f^{1.1}$ for $f \in 0.1..30$ MHz
- Yang and Cleveland, Time domain simulation of nonlinear acoustic beams generated by rectangular pistons with application to harmonic imaging, JASA, 2005
- Note: Parameter values are found from optimization, they don't really correspond to actual physical relaxation processes in the medium as in the air and water models.



Lossy wave equations

$$\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + Lu = 0$$

- Physics-based:
  - Viscoelastic, y=2:

$$Lu = \tau \frac{\partial}{\partial t}(\nabla^2 u) \approx \frac{\tau}{c_0^2}\partial^3 u/\partial t^3$$

  - Solution $\propto \omega^2$ for $\omega\tau \ll 1$

- Fractional derivatives:
  - Szabo94 (Chen,Holm03):

$$Lu \propto -\frac{\partial^{y+1} u}{\partial t^{y+1}}$$

  - Chen, Holm 04:

$$Lu \propto -\frac{\partial}{\partial t}(-\nabla^2)^{y/2} u$$

  - Wismer 06, Caputo 67:

$$Lu = \tau^{y-1}\frac{\partial^{y-1}}{\partial t^{y-1}}(\nabla^2 u)$$

More physical than the others

  - Solution $\propto \omega^y$ for $\omega\tau \ll 1$

Derivative of arbitrary order

Two interpretations of fractional derivative:

1. Fourier: $$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega)$$

2. Convolution of ordinary derivative of integer order m>α and causal memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} \propto \frac{d^m f(t)}{dt^m} * \frac{1}{t^{1+\alpha-m}}$$

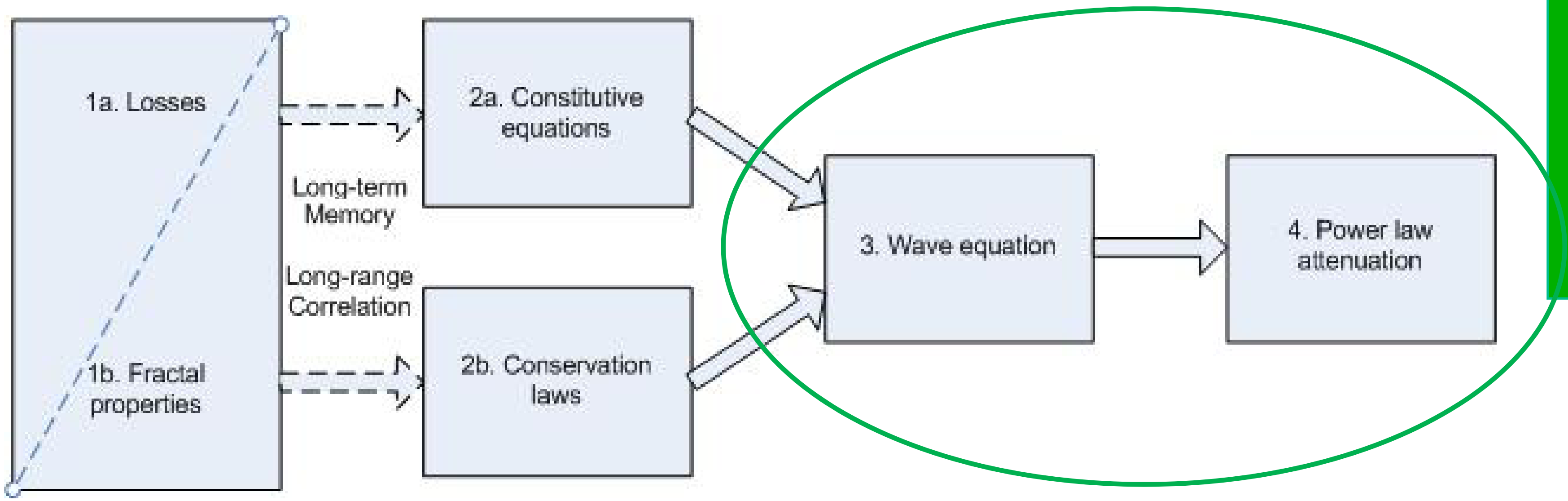
Fractional derivative (Caputo)

- Caputo: order m-1 $\leq \alpha$ <m:

$$ {}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{-\infty}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau $$

  - First integer order derivative, then convolution
  - Interpretation of e.g. α=0.2 order differentation:
    - » m=1. order differentiation
    - » Integration with forgetting function, order m-α=0.8
  - Memory is introduced

Conceptual model



Many fractional wave equations are descriptive only

Fractional constitutive equation

- Stress, σ vs strain ϵ - Fractional Kelvin-Voigt

$$\sigma(t) = E_0 \left[\epsilon(t) + \tau_\sigma^\alpha \frac{\partial^\alpha \epsilon(t)}{\partial t^\alpha} \right]$$

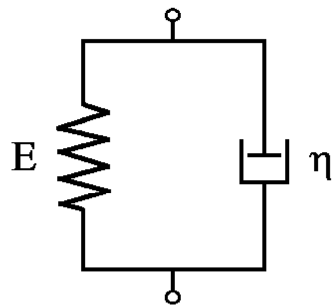
  » E_0 : elastic modulus

  » τ_σ : ratio of viscosity and elasticity

  » Y. Rossikhin and M. Shitikova, “Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids”, Appl. Mech. Rev., 1997

  – α=1 is normal viscoelastic case (figure)

  » Spring (Hooke’s law) + damper like in a car



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Wave equation

Fractional loss operator

- Conservation of mass & momentum =>

$$\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \tau_\sigma^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\nabla^2 u = 0$$

  - M. Caputo, "Linear models of dissipation whose Q is almost frequency independent–II", Geophys. J. Roy. Astr. S. 1967

  – Dispersion equation, *u = exp{i(ωt – kx)}*

    » imaginary part of k is loss:

$$k^2 - \omega^2/c_0^2 + (\tau_\sigma i\omega)^\alpha k^2 = 0.$$

Solve dispersion equation

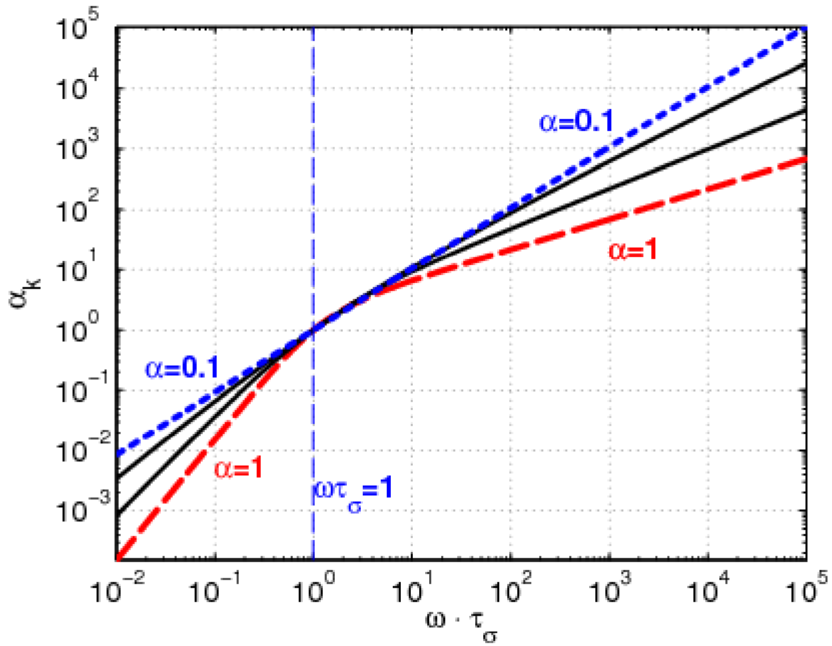
$$k^2 - \omega^2/c_0^2 + (\tau_\sigma i\omega)^\alpha k^2 = 0.$$

- Solution: k = ω/c(ω) – iα_k = ω/c(ω) – i$\alpha_0 \omega^y$
- Low-frequency asymptote ⇔ P-waves in ultrasound
  - y = α+1, y $\in$ (1,2], $\alpha \in$ (0,1]
- High frequency asymptote ⇔ S-waves in elastography
  - y = 1-α/2, y $\in$ [0,1), $\alpha \in$ (0,2]

» Holm, Sinkus, ” A unifying fractional wave equation for compressional and shear waves”, JASA, 2010

Attenuation, fractional Kelvin-Voigt

y = α+1 y = 1-α/2

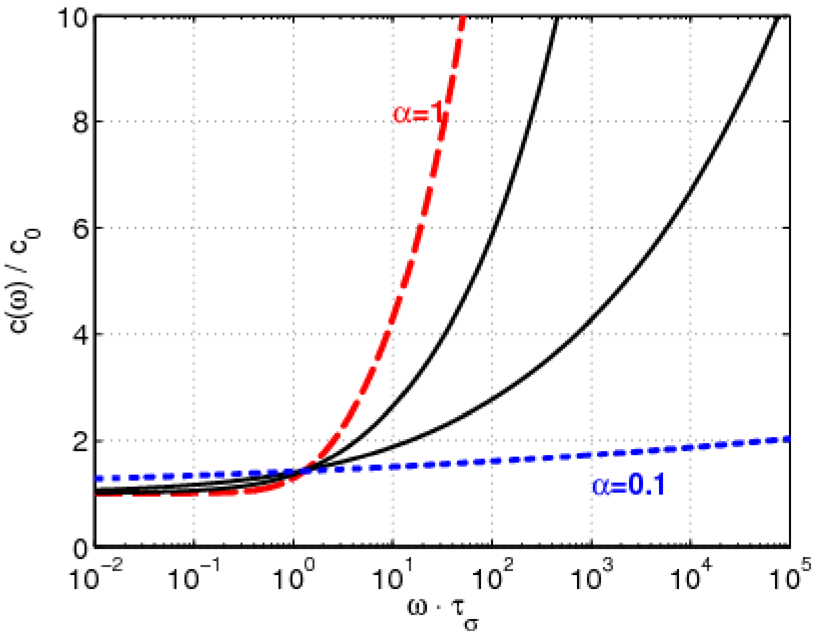


α = 0.1, 0.3, 0.7, and 1

α = 1: viscoelastic case

Phase velocity, fractional Kelvin-Voigt

Ultrasound compression waves: hardly any dispersion



Elastography shear waves: large dispersion

α = 0.1, 0.3, 0.7, and 1

Warning: c $\rightarrow \infty$ for large frequencies!!

α = 1: viscoelastic case

Fractional Zener model instead

- Stress, σ vs strain ϵ - Standard linear solid model

$$\sigma(t) + \tau_\epsilon \frac{\partial \sigma^\beta(t)}{\partial t^\beta} = E_0 \left[\epsilon(t) + \tau_\sigma^\alpha \frac{\partial^\alpha \epsilon(t)}{\partial t^\alpha} \right]$$

  - » ${\tau_\epsilon}^\beta \leq {\tau_\sigma}^\alpha$:Often the two time constants are very close
  - » $\alpha \geq \beta$, we will set them equal
  - » Arterial viscoelasticity, brain, doped corning glass, rock, liver, metals, polymeric materials, rubber
  - » Bagley, Torvik, "On the fractional calculus model of viscoelastic behavior", J. Rheol, 1986

Fractional Zener: Wave equation

- Wave equation:

$$\nabla^2 u - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \tau_\sigma^\alpha \frac{\partial^\alpha}{\partial t^\alpha}\nabla^2 u - \frac{\tau_\epsilon^\beta}{c_0^2}\frac{\partial^{\beta+2} u}{\partial t^{\beta+2}} = 0$$

  - Two fractional loss terms instead of one
  - For α=β=1, this is a single relaxation process, i.e. one term in attenuation for salt water, air, …
    - » Holm, Näsholm,"A causal and fractional all-frequency wave equation for lossy media", Journ. Acoust. Soc. Am, Oct. 2011.

Fractional Zener: Solution

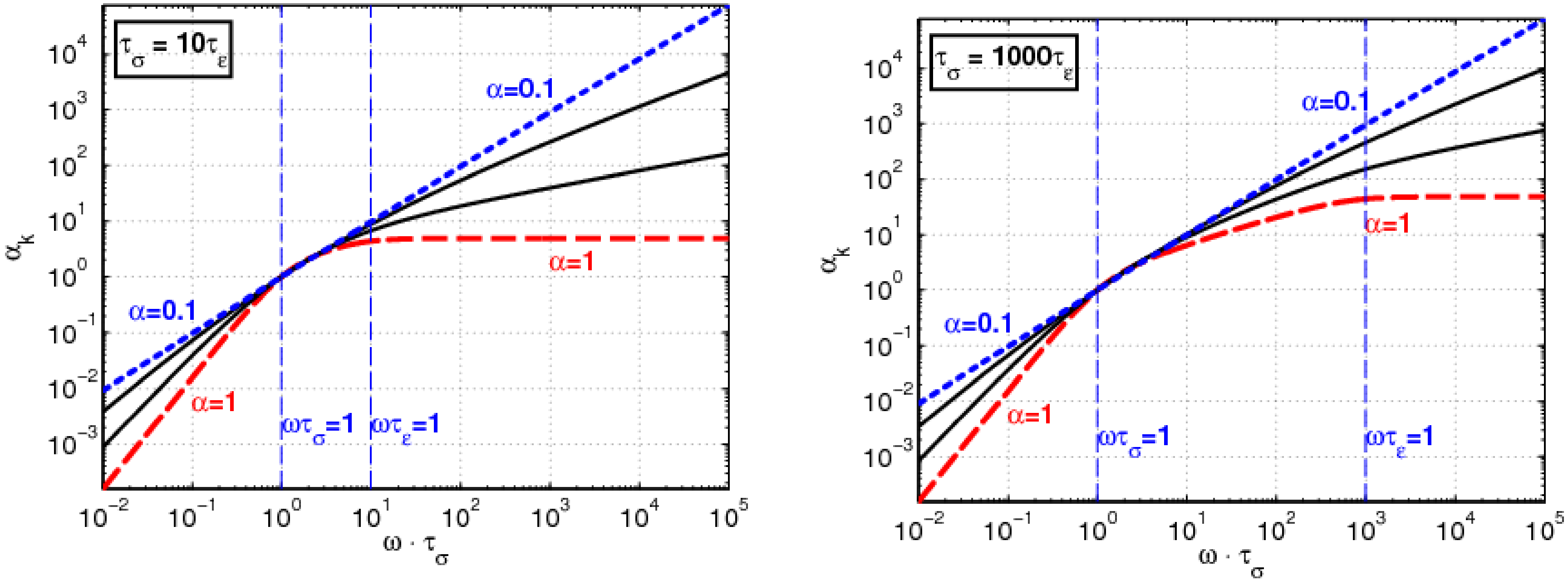
- Dispersion relation $$k^2 = \frac{\omega^2}{c_0^2}\frac{1+(\tau_\epsilon i\omega)^\beta}{1+(\tau_\sigma i\omega)^\alpha}$$
- Attenuation:

$$k = \frac{\omega}{c(\omega)} - i\alpha_k, \quad \alpha_k(\omega) \propto \begin{cases} \omega^{1+\alpha} & \text{low frequencies} \\ \omega^{1-\alpha/2} & \text{intermediate frequencies} \\ \omega^{1-\alpha} & \text{high frequencies} \end{cases}$$

- Standard relaxation, α=1:

$$\alpha_k(\omega) \propto \begin{cases} \omega^2 & \text{low frequencies} \\ \omega^{1/2} & \text{intermediate frequencies} \\ \omega^0 & \text{high frequencies} \end{cases}$$

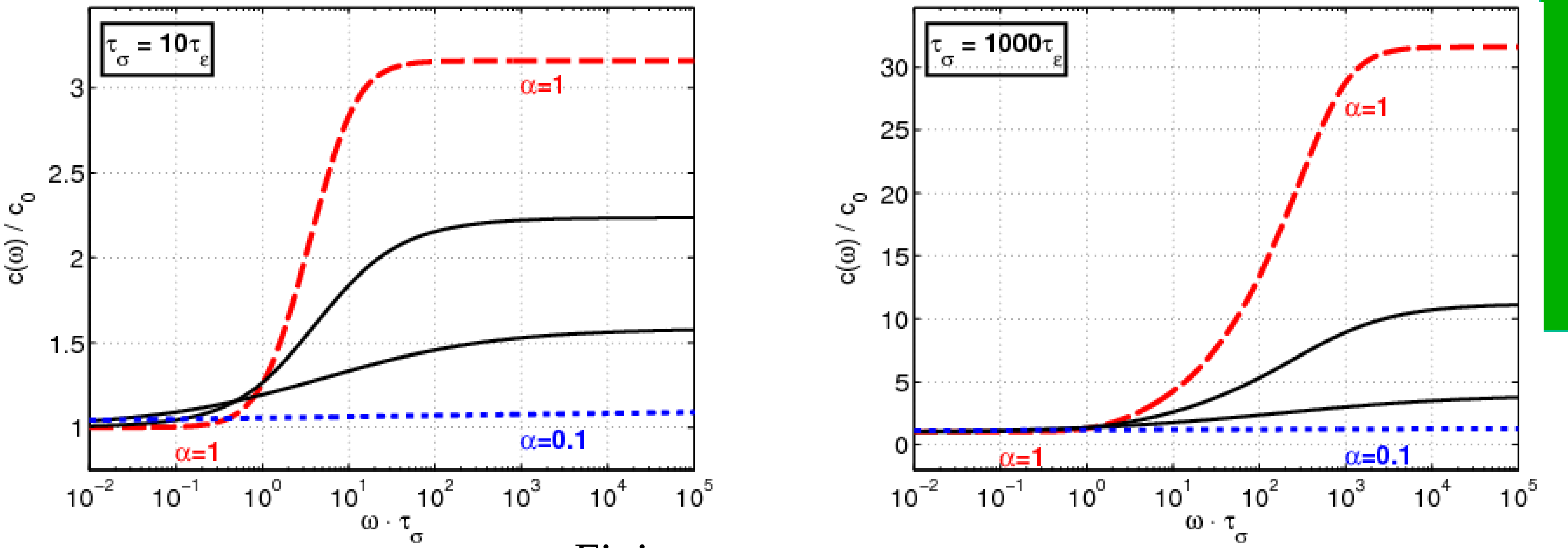
Attenuation, fractional Zener



1. Low frequency
2. Medium frequency = High frequency for Kelvin-Voigt model
3. High frequency

$$\alpha = \beta = 0.1, 0.3, 0.7, \text{ and } 1$$

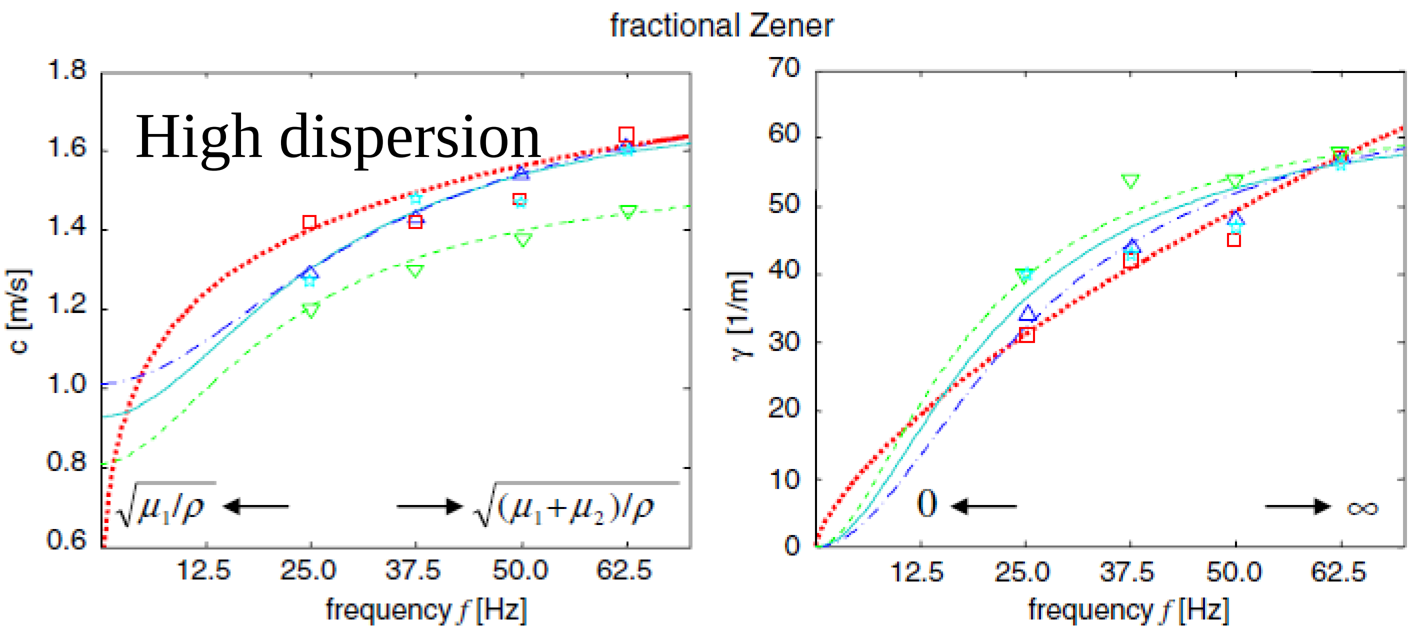
Phase velocity, fractional Zener



Finite c as $\omega \to \infty$

$\alpha = \beta = 0.1, 0.3, 0.7$, and 1

MR Elastography, brain

- 

- Klatt et al, Noninvasive assessment of the rheological behavior of human organs using multifrequency MR elastography: a study of brain and liver viscoelasticity, Phys. Med. Biol 2007

Multiple relaxation vs fractional Zener

- Fractional Zener

$$\kappa_Z(\omega) = \kappa_0 \frac{1+(\tau_\epsilon i\omega)^\beta}{1+(\tau_\sigma i\omega)^\alpha}$$

  - Inverse Fourier transform: Mittag-Leffler function

- Multiple relaxation

$$\kappa_N(\omega) = \kappa_0 - i\omega \sum_{\nu=1}^{N} \frac{\kappa_\nu \tau_\nu}{1+i\omega\tau_\nu} \rightarrow \kappa_0 - i\omega \int_0^\infty \frac{\kappa_\nu(\Omega)}{\Omega + i\omega} d\Omega$$

  - Inverse Fourier: Impulse + causal relaxation functions (= exponential functions)

- Näsholm and Holm, "Linking multiple relaxation, power-law attenuation, and fractional wave equations," Journ. Acoust. Soc. Am, Nov. 2011

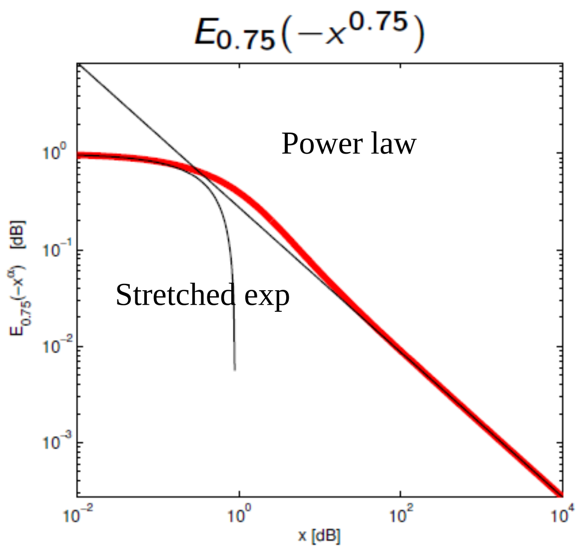
(Deviation) The Mittag-Leffler function $E_{\alpha,1}(x)$

Generalization of the exponential function

Definition

$$E_{\alpha,1}(x) \triangleq \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+1)}$$

$$\left[\text{Compare: } \exp(x) \triangleq \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)}\right] \quad (10)$$



Integral representation, Laplace-transform relation

$$E_{\alpha,1}(-ax^{\alpha}) = \int_0^{\infty} e^{-x\Omega} f_{\alpha}(\Omega,a)\, d\Omega, \quad f_{\alpha}(\Omega,a) = \frac{1}{\pi}\frac{a\Omega^{\alpha-1}\sin(\alpha\pi)}{\Omega^{2\alpha}+2a\Omega^{\alpha}\cos(\alpha\pi)+a^2} \quad (11)$$

$$\text{Summable } f_{\alpha}(\Omega,a): \quad \int_0^{\infty} f_{\alpha}(\Omega,a)\, d\Omega = 1 \quad (12)$$

(Cont.) Linking NSW and fractional Zener wave equations

Wave equations are equal when generalized compressibilities are equal:

$$\boxed{\kappa_{\mathsf{Z}}(t) = \kappa_{\mathsf{N}}(t)} \qquad \Longleftrightarrow$$

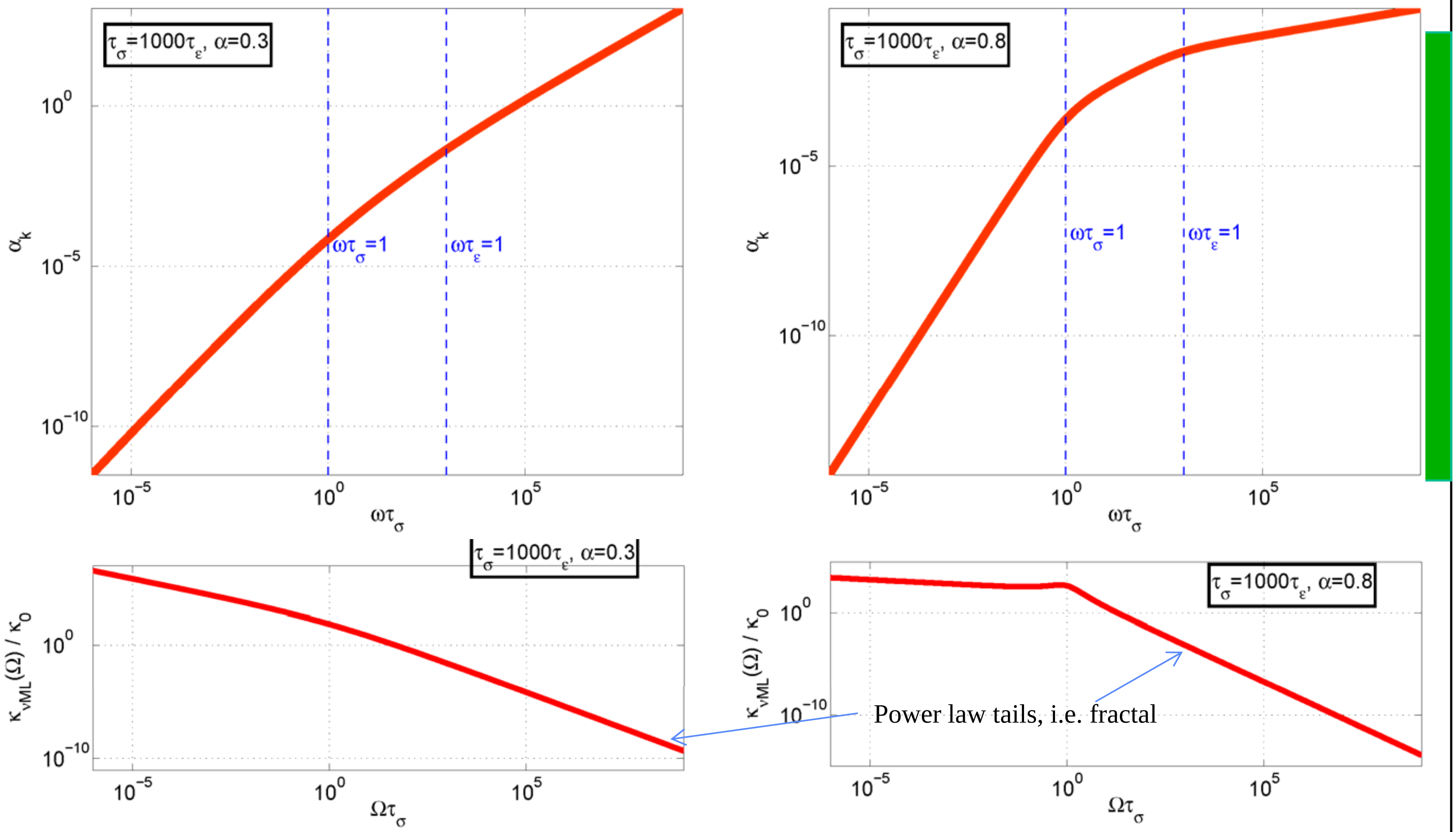
$$\overbrace{\left\{ \delta(t) \left[\frac{\kappa_0 \tau_\epsilon^\alpha}{\tau_\sigma^\alpha} \right] - H(t) \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \kappa_0 (1 - \tau_\epsilon^\alpha / \tau_\sigma^\alpha) E_{\alpha,1}\big(-(t/\tau_\sigma)^\alpha \big) \right\} \right\}}^{\kappa_{\mathsf{Z}}(t)} =$$

$$= \underbrace{\left\{ \delta(t) \left[\kappa_0 - \int_0^\infty \kappa_\nu(\Omega)\, \mathrm{d}\Omega \right] - H(t) \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \int_0^\infty e^{-t\Omega} \kappa_\nu(\Omega)\, \mathrm{d}\Omega \right\} \right\}}_{\kappa_{\mathsf{N}}(t)}$$

What $\kappa_\nu(\Omega)$ to select?

From Mittag-Leffler integral representation (11): $\kappa_{\mathsf{Z}}(t) = \kappa_{\mathsf{N}}(t)$

for $\kappa_\nu(\Omega) = \kappa_0 (1 - \tau_\epsilon^\alpha / \tau_\sigma^\alpha) f_\alpha(\Omega, \tau_\sigma^{-\alpha}) = \dfrac{1}{\pi} \dfrac{\kappa_0 (\tau_\sigma^\alpha - \tau_\epsilon^\alpha) \Omega^{\alpha-1} \sin(\alpha\pi)}{(\tau_\sigma \Omega)^{2\alpha} + 2(\tau_\sigma\Omega)^\alpha \cos(\alpha\pi) + 1}$ (13)


τσ=1000τε, α=0.3
ωτσ=1
ωτε=1
αk
ωτσ
τσ=1000τε, α=0.8
κvML(Ω) / κ0
Ωτσ
Power law tails, i.e. fractal

Conclusions

- Wave equation from fractional stress-strain
  - Causal, unlike many other constructed equations
  - Finite c
- Describes compressional waves and shear waves in 2/3 different frequency regimes
  - power law attenuation ω^y
- Equivalent to a frequency weighted multiple relaxation model
  - In general: Σ exponentials ⇔ power law
- Works in progress: Spatial fractal properties
  - Relation medical elastography - spatial property
  - Simulation, measurement, theoretical modeling

Array Processing Implications

- Lossy media cause signals to decay more rapidly than predicted by ideal wave equation
  - Limits range
  - Ultrasound imaging: low frequency ⇔ deeper penetration, but poorer resolution
- Attenuation and dispersion are coupled
  - Attenuation $\propto f^2 \Rightarrow$ dispersion is zero