Fractional derivatives

Sverre Holm

UNIVERSITY OF OSLO

n-th order derivative

- Sequence of n-fold integrals and n-fold derivatives:

$$\cdots, \int_a^t d\tau_2 \int_a^{\tau_2} f(\tau_1)d\tau_1, \int_a^t f(\tau_1)d\tau_1, f(t), \frac{df(t)}{dt}, \frac{d^2 f(t)}{dt^2}, \cdots$$

- Fourier transform: $FT\left(\frac{d^n f(t)}{dt^n}\right) = (i\omega)^n F(\omega)$

$$\cdots, (i\omega)^{-2}F(\omega), (i\omega)^{-1}F(\omega), F(\omega), i\omega F(\omega), (i\omega)^2 F(\omega), \cdots$$

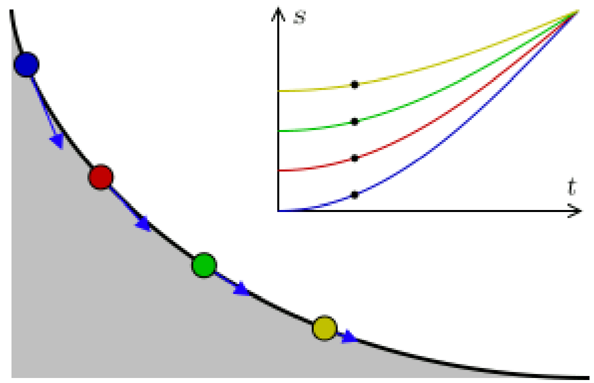
- I Podlubny, Fractional Differential Equations, Academic Press, 1999

Abel's Integral Equation - $D^{0.5}$

- Tautochrone curve, total time for the particle to fall:

$$T(y_0) = \frac{1}{\sqrt{2g}} \int_0^{y_0} \frac{1}{(y_0 - y)^{0.5}} dy$$

- Related to $D^{0.5}$
  - Abel, Auflösung einer mechanischen Aufgabe, J. Reine u. Angew. Math, 1826,
  - B. Holmboe, ”Abel: Œuvres complètes”, 1839, IV Résolution d'un problème mécanique.



Derivative of arbitrary order

- Derivative of order α:

$$\frac{d^\alpha f(t)}{dt^\alpha} = {}_a D_t^\alpha f(t)$$

  - α <0 ⇔ integration
  - a and t: limits in defining integral

- Fourier transform (neglecting initial cond's):

$$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega)$$

Fourier approach to fractional operator (1)

- Integer (m>α) + fraction (α-m<0):

$$FT\left(\frac{d^{\alpha}f(t)}{dt^{\alpha}}\right) = (i\omega)^{\alpha}F(\omega) = (i\omega)^{m}F(\omega)(i\omega)^{\alpha-m}$$

- First part: ordinary derivative
- Second part: fractional part
  - What is its inverse Fourier transform?

Fourier transform

$$h(t) = \frac{1}{\Gamma(\beta)} \frac{1}{t^{1-\beta}}, t > 0 \Leftrightarrow H(\omega) = (i\omega)^{-\beta}$$

Podlubny, 1999, pp. 110-

- 0<β<1
- Γ(.) is the gamma function
  - Generalization of the factorial: Γ(n+1)=n!
- Let β=m-α and rewrite:

$$h(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, t > 0 \Leftrightarrow H(\omega) = (i\omega)^{\alpha-m}$$

Fourier approach to fractional operator (2)

- Fractional Fourier transform as a convolution of derivative of order first integer m larger than α and a memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{d^m f(t)}{dt^m} * \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}$$

Fractional derivative: Two flavors

- Riemann-Liouville: order $\alpha \in$ R, m-1$\leq\alpha$<m:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau $$

  – First convolution, then integer order derivation

- Caputo: order m-1 $\leq \alpha$ <m:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau $$

  – First integer order derivative, then convolution

Riemann-Liouville vs Caputo

- Riemann-Liouville requires initialization of derivatives of non-integer orders:

$$\lim_{t\to a} {}_aD_t^{\alpha-1} f(t), \lim_{t\to a} {}_aD_t^{\alpha-2} f(t), \cdots$$

- Caputo requires initialization of integer order derivatives: $f^{(k)}(0)$, k=0, 1, ..., m-1
  - Usually have physical meaning
  - Simpler to use in numerical solutions

Fractional derivative for numerics

- Caputo (lower limit a = -∞):

$$ {}_{-\infty}^{C}D_t^{\alpha} f(t) = \frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \frac{\partial^{m} f(t)}{\partial t^{m}} * g_{m-\alpha}(t) $$

$$ g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}, t > 0 $$

- Convolution with a memory function

Memory function

- Convolution kernel:

$$g_{m-\alpha}(t) = \frac{1}{\Gamma(m-\alpha)} \frac{1}{t^{\alpha+1-m}}$$

  - m-α = ε⁺ : no memory, Γ(ε⁺) → ∞ for ε⁺ → 0 ⇒ kernel → impulse
  - m-α = 1 : infinite memory

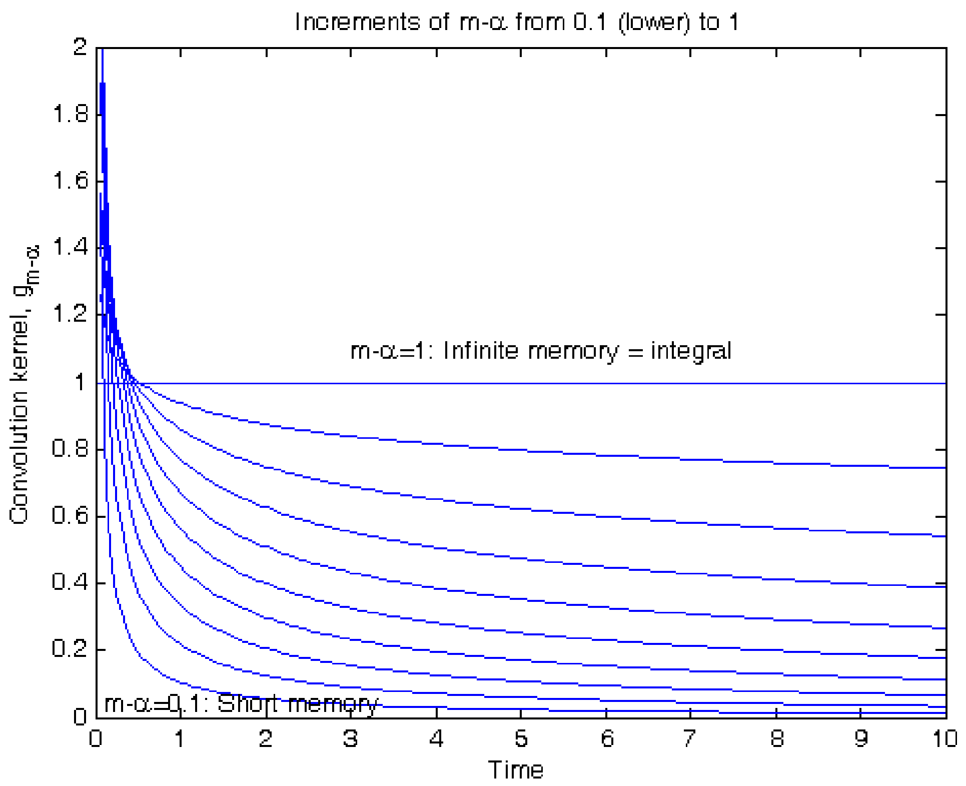


Figure based on Treeby and Cox, "Modeling power law absorption and dispersion for acoustic propagation using the fractional Laplacian", J. Acoust. Soc. Amer, 2010

Fractional derivative of order 0..1

- Example: $0 \leq \alpha < 1$ (Caputo with m=1):

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{\partial f(t)}{\partial t} * g_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{t} \frac{f^{(1)}(t)}{(t-\tau)^\alpha} d\tau$$

- Limits:
  - $\alpha \rightarrow 0 \Rightarrow$ $\frac{\partial^\alpha f(t)}{\partial t^\alpha} \rightarrow \int_{-\infty}^{t} \frac{f^{(1)}(t)}{(t-\tau)^0} d\tau = f(t)$
  - $\alpha \rightarrow 1 \Rightarrow$ $\frac{\partial^\alpha f(t)}{\partial t^\alpha} \rightarrow \int_{-\infty}^{t} f^{(1)}(t)\delta(\tau) d\tau = f^{(1)}(t)$

Conclusion

Two interpretations of fractional derivative:

1. Fourier:

$$FT\left(\frac{d^\alpha f(t)}{dt^\alpha}\right) = (i\omega)^\alpha F(\omega)$$

2. Convolution of ordinary derivative of order m>α and causal memory function:

$$\frac{d^\alpha f(t)}{dt^\alpha} \propto \frac{d^m f(t)}{dt^m} * \frac{1}{t^{1+\alpha-m}}$$