Linear Algebra Cheatsheet

UiO Language Technology Group

1 Basics of vectors and matrices

1.1 Matrices

- **Matrix** is a rectangular 2-dimensional array of numbers (scalars).
- $\boldsymbol{M} \in \mathbb{R}^{m \times n}$ is a matrix $\boldsymbol{M}$ with m rows and n columns.
- For example:

$$\boldsymbol{M} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

- Here, $\boldsymbol{M}$ is a 3×4 matrix: it has 3 rows and 4 columns. 3 and 4 are the **dimensions** of $\boldsymbol{M}$.

1.2 Entries

- Matrices consist of **entries**.
- $\boldsymbol{M}_{i,j}$ or $\boldsymbol{M}_{[i,j]}$ is the entry in the i^{th} row and j^{th} column of $\boldsymbol{M}$.
- For example:

$$\boldsymbol{M}_{0,0} = 1$$

- NB: we use 0-indexed notation, following *Python* conventions.

1.3 Vectors

- **Vector** is a $1 \times n$ matrix (NB: we use *row vectors*).
- $\boldsymbol{v} \in \mathbb{R}^n$ is a vector $\boldsymbol{v}$ with n entries or components (n-dimensional vector).
- For example:

$$\boldsymbol{v} = [4, 3, 2, 1]$$

- Here, $\boldsymbol{v}$ is a 4-dimensional vector.
- $\boldsymbol{v}_i$ or $\boldsymbol{v}_{[i]}$ is the i^{th} entry of the vector.
- For example:

$$\boldsymbol{v}_1 = 3$$

2 Addition and scalar multiplication

2.1 Matrix addition

- **Matrix addition** is simply adding the entries of two or more matrices one by one.
- This summation results in another matrix:
- $\boldsymbol{M_0} + \boldsymbol{M_1} = \boldsymbol{M_2}$
- For example:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 0 & 0 & 0 \\ 4 & 3 & 2 \end{bmatrix}$$

- NB: we can add only matrices of *the same dimensionality*!
- The resulting matrix retains the same dimensions (3×3 in the example above).
- One can *subtract* matrices in the same way.

2.2 Multiplication by scalar

- To **multiply a matrix by scalar** (a raw number), one also simply multiplies all its entries by this scalar.
- It results in another matrix of the same dimensionality.
- For example:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} \times 2 = \begin{bmatrix} 4 & 6 & 8 \\ 0 & 0 & 0 \\ 6 & 4 & 2 \end{bmatrix}$$

- Note that the multiplication of a matrix by a scalar and the multiplication of a scalar by a matrix are equal:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} \times 2 = 2 \times \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 & 8 \\ 0 & 0 & 0 \\ 6 & 4 & 2 \end{bmatrix}$$

- One can *divide* a matrix by a scalar in the same way:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} /2 = \begin{bmatrix} 1 & 1.5 & 2 \\ 0 & 0 & 0 \\ 1.5 & 1 & 0.5 \end{bmatrix}$$

- This essentially amounts to the scalar multiplication by fraction:

$$\begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} /2 = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{bmatrix} \times \frac{1}{2} = \begin{bmatrix} 1 & 1.5 & 2 \\ 0 & 0 & 0 \\ 1.5 & 1 & 0.5 \end{bmatrix}$$

2.3 Miscellaneous

- All these *operations can be combined and sequenced together* as any other mathematical operations.
- Remember that a vector is simply a special kind of a matrix: thus, *vectors can be added and multiplied by scalars in exactly the same way.*

3 Vector to vector multiplication

- **Vector to vector multiplication** ($\boldsymbol{v} \cdot \boldsymbol{x}$) is a special case of matrix-matrix multiplication.
- It is defined *only* if the dimensionalities of both vectors match:

  $\boldsymbol{v}, \boldsymbol{x} \in \mathbb{R}^n$
- The result of this multiplication is called the *inner product* or *dot product* and is a scalar:

  $\boldsymbol{v} \cdot \boldsymbol{x} = z$
- It is calculated as a sum of one-by-one multiplications of the corresponding entries of $\boldsymbol{v}$ and $\boldsymbol{x}$:

$$z = \sum_{i=0}^{n} \boldsymbol{v}_i \times \boldsymbol{x}_i$$

- For example:

  $[2, 0, 2] \cdot [1, 3, 1] = 2 \times 1 + 0 \times 3 + 2 \times 1 = 2 + 0 + 2 = 4$
- As simple as that!

4 Vector to matrix multiplication

4.1 Requirements

- **Vector to matrix multiplication** ($\boldsymbol{v} \cdot \boldsymbol{W}$) is also a special case of matrix-matrix multiplication.
- It is defined *only* if the dimensionality of the vector and the number of rows in the matrix match:
- $\boldsymbol{v} \in \mathbb{R}^m, \boldsymbol{W} \in \mathbb{R}^{m \times n}$
- ...more explicitly, *the number of columns in the vector and the number of rows in the matrix must be identical*:
- $\boldsymbol{v} \in \mathbb{R}^{1 \times m}, \boldsymbol{W} \in \mathbb{R}^{m \times n}$

4.2 Process

- The result of right-multiplying a vector $\boldsymbol{v} \in \mathbb{R}^m$ (or, equally, $\boldsymbol{v} \in \mathbb{R}^{1\times m}$) by a matrix $\boldsymbol{W} \in \mathbb{R}^{m\times n}$ is a *vector* $\boldsymbol{y} \in \mathbb{R}^n$
  - $\boldsymbol{v} \cdot \boldsymbol{W} = \boldsymbol{y}$
  - note how the matching dimensions m are 'self-destroyed'.
- Each component i of $\boldsymbol{y}$ is a sum of one-by-one multiplying columns of $\boldsymbol{v}$ by the entries of the i^{th} column of $\boldsymbol{W}$.
- For example:

$$\boldsymbol{y} = \boldsymbol{v} \cdot \boldsymbol{W} = [2, 3] \cdot \begin{bmatrix} 4 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = [17, 10, 9]$$

  1. $\boldsymbol{y}_0 = \boldsymbol{v}_0 \times \boldsymbol{W}_{0,0} + \boldsymbol{v}_1 \times \boldsymbol{W}_{1,0} = 2 \times 4 + 3 \times 3 = 8 + 9 = 17$
  2. $\boldsymbol{y}_1 = \boldsymbol{v}_0 \times \boldsymbol{W}_{0,1} + \boldsymbol{v}_1 \times \boldsymbol{W}_{1,1} = 2 \times 2 + 3 \times 2 = 4 + 6 = 10$
  3. $\boldsymbol{y}_2 = \boldsymbol{v}_0 \times \boldsymbol{W}_{0,2} + \boldsymbol{v}_1 \times \boldsymbol{W}_{1,2} = 2 \times 3 + 3 \times 1 = 6 + 3 = 9$

- Here, the result is the 3-dimensional row vector $\boldsymbol{y} \in \mathbb{R}^3$.

5 Matrix to matrix multiplication

5.1 Matrix to matrix is another matrix

- Any row vector is in fact a $1 \times n$ matrix.
- Thus, to **multiply one matrix by another**, is conceptually the same as multiplying a vector by a matrix.
- Again, the number of columns in the left matrix $\boldsymbol{W}^1$ must match the number of rows of the right matrix $\boldsymbol{W}^2$:
  $\boldsymbol{W}^1 \in \mathbb{R}^{m\times n}, \boldsymbol{W}^2 \in \mathbb{R}^{n\times z}$
- But the *result of this multiplication is another matrix*:
  $\boldsymbol{W}^1 \cdot \boldsymbol{W}^2 = \boldsymbol{W}^3 \in \mathbb{R}^{m\times z}$
  - Again, the matching dimensions n are 'self-destroyed'.

5.2 Process

- For example, suppose $\boldsymbol{W}^1 \in \mathbb{R}^{2\times 3}, \boldsymbol{W}^2 \in \mathbb{R}^{3\times 4}$:

$$\boldsymbol{W}^3 = \boldsymbol{W}^1 \cdot \boldsymbol{W}^2 = \begin{bmatrix} 4 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 17 & 19 & 16 \\ 11 & 10 & 11 & 10 \end{bmatrix}$$

- Here, $\boldsymbol{W}^3 \in \mathbb{R}^{2\times 4}$
- It is produced like this:

– Each row i of $\boldsymbol{W^3}$ is a product of multiplying the i^{th} row of $\boldsymbol{W^1}$ (a vector) by $\boldsymbol{W^2}$ (a matrix):

1. $\boldsymbol{W^3}_{[0,:]} = \boldsymbol{W^1}_{[0,:]} \cdot \boldsymbol{W^2} = [4,2,3] \cdot \begin{bmatrix} 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 3 & 2 \end{bmatrix} = [19,17,19,16]$
2. $\boldsymbol{W^3}_{[1,:]} = \boldsymbol{W^1}_{[1,:]} \cdot \boldsymbol{W^2} = [3,2,1] \cdot \begin{bmatrix} 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 \\ 3 & 3 & 3 & 2 \end{bmatrix} = [11,10,11,10]$
3. etc...

5.3 Properties of matrix multiplication

- Matrix multiplication **is not commutative**:
  $\boldsymbol{W}^1 \cdot \boldsymbol{W}^2 \neq \boldsymbol{W}^2 \cdot \boldsymbol{W}^1$
- Matrix multiplication **is associative**:
  $\boldsymbol{W}^1 \cdot \boldsymbol{W}^2 \cdot \boldsymbol{W}^3 = \boldsymbol{W}^1 \cdot (\boldsymbol{W}^2 \cdot \boldsymbol{W}^3) = (\boldsymbol{W}^1 \cdot \boldsymbol{W}^2) \cdot \boldsymbol{W}^3$