CHAPTER 3

Further properties of splines and B-splines

In Chapter 2 we established some of the most elementary properties of B-splines. In this chapter our focus is on the question "What kind of functions can be represented as linear combinations of B-splines?" This may seem like a rather theoretical and non interesting issue from a practical point of view, but if our spline spaces contain sufficiently many interesting functions we will gain the flexibility that is required for practical applications.

The answer to the question above is that our spline space contain a large class of piecewise polynomials and this ensures that splines are reasonably flexible, much more flexible than polynomials. To prove this we start by showing that polynomials of degree d can be represented in terms of splines of degree d in Section 3.1. This is proved by making use of some simple properties of the B-spline matrices. As a bonus we also prove that B-splines are linearly independent and therefore provide a basis for spline spaces, a result that is crucial for practical computations. In Section 3.2 we investigate the smoothness of splines and B-splines in detail, and this allows us to conclude in Section 3.3 that spline spaces contain a large class of piecewise polynomials.

3.1 Linear independence and representation of polynomials

Our aim in this section is to show that any polynomial can be represented as a linear combination of B-splines, and also that B-splines are linearly independent. To do this we first need some simple properties of the B-spline matrices defined in Theorem 2.18.

3.1.1 Some properties of the B-spline matrices

To study the B-spline matrices we associate a certain polynomial with each B-spline. We start by associating the polynomial $\rho_{j,0}(y) = 1$ with $B_{j,0}$ and, more generally, the polynomial in y given by

$$\rho_{j,d}(y) = (y - t_{j+1})(y - t_{j+2}) \cdots (y - t_{j+d}), \tag{3.1}$$

is associated with the B-spline $B_{j,d}$ for $d \geq 1$. This polynomial is called the *dual polynomial* of the B-spline $B_{j,d}$. On the interval $[t_\mu, t_{\mu+1})$ we have the $d+1$ nonzero B-splines $\boldsymbol{B}_d =$

$(B_{\mu-d,d}, \ldots, B_{\mu,d})^T$. We collect the corresponding dual polynomials in the vector

$$\boldsymbol{\rho}_d = \boldsymbol{\rho}_d(y) = (\rho_{\mu-d,d}(y), \ldots, \rho_{\mu,d}(y))^T. \tag{3.2}$$

The following lemma shows the effect of applying the matrix $\boldsymbol{R}_d$ to $\boldsymbol{\rho}_d$.

Lemma 3.1. *Let μ be an integer such that $t_\mu < t_{\mu+1}$ and let $\boldsymbol{\rho}_d(y)$ be the dual polynomials defined by* (3.2)*. For $d \geq 1$ the relation*

$$\boldsymbol{R}_d(x)\boldsymbol{\rho}_d(y) = (y-x)\boldsymbol{\rho}_{d-1}(y). \tag{3.3}$$

holds for all $x, y \in \mathbb{R}$.

Proof. Writing out (3.3) in component form we see that what we need to prove is

$$\frac{(x-t_j)\rho_{j,d}(y) + (t_{j+d}-x)\rho_{j-1,d}(y)}{t_{j+d}-t_j} = (y-x)\rho_{j,d-1}(y), \tag{3.4}$$

for $j = \mu-d+1, \ldots, \mu$. Since $\rho_{j,d}(y) = (y-t_{j+d})\rho_{j,d-1}(y)$ and $\rho_{j-1,d}(y) = (y-t_j)\rho_{j,d-1}(y)$, the numerator on the left hand side of (3.4) can be written

$$\big((x-t_j)(y-t_{j+d}) + (t_{j+d}-x)(y-t_j)\big)\rho_{j,d-1}(y).$$

A simple calculation reveals that

$$(x-t_j)(y-t_{j+d}) + (t_{j+d}-x)(y-t_j) = (y-x)(t_{j+d}-t_j). \tag{3.5}$$

Inserting this on the left in (3.4) and simplifying, we obtain the right-hand side. ∎

The crucial relation (3.5) is an example of linear interpolation. For if we define the linear function g by $g(x) = y - x$ for a fixed number y, then linear interpolation at t_j and t_{j+d} gives the relation

$$\frac{t_{j+d}-x}{t_{j+d}-t_j}g(t_j) + \frac{x-t_j}{t_{j+d}-t_j}g(t_{j+d}) = g(x),$$

see Section 1.3 in Chapter 1. If we multiply both sides of this equation by $t_{j+d} - t_j$, we obtain equation (3.5).

In equation 3.3, the $d+1$-vector $\boldsymbol{\rho}_d$ is transformed to a vector with d components. We can reduce the number of components further by applying more $\boldsymbol{R}$'s. By making use of all the matrices $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_d$ we end up with a scalar.

Corollary 3.2. *Let μ be an integer such that $t_\mu < t_{\mu+1}$ and let $\boldsymbol{\rho}_d(y)$ be the dual polynomials defined by* (3.2)*. Then the relation*

$$\boldsymbol{R}_1(x_1)\boldsymbol{R}_2(x_2)\cdots\boldsymbol{R}_d(x_d)\boldsymbol{\rho}_d(y) = (y-x_1)(y-x_2)\cdots(y-x_d). \tag{3.6}$$

holds for all real numbers $x_1, x_2, \ldots, x_d$ and y.

We need one more property of the B-spline matrices. This property cannot be established completely until we have proved that the dual polynomials are linearly independent.

Lemma 3.3. *For $d \geq 2$ and for any x and z in $\mathbb{R}$, the matrices $\boldsymbol{R}_{d-1}$ and $\boldsymbol{R}_d$ satisfy the relation*

$$\boldsymbol{R}_{d-1}(z)\boldsymbol{R}_d(x) = \boldsymbol{R}_{d-1}(x)\boldsymbol{R}_d(z). \tag{3.7}$$

Proof. Applying (3.3) twice we obtain

$$\boldsymbol{R}_{d-1}(x)\boldsymbol{R}_d(z)\boldsymbol{\rho}_d(y) = (y-x)(y-z)\boldsymbol{\rho}_{d-2}(y),$$

and by symmetry we also have

$$\boldsymbol{R}_{d-1}(z)\boldsymbol{R}_d(x)\boldsymbol{\rho}_d(y) = (y-x)(y-z)\boldsymbol{\rho}_{d-2}(y),$$

Equivalently,

$$\boldsymbol{B}\boldsymbol{\rho}_d(y) = \boldsymbol{0} \tag{3.8}$$

for all y, where the $(d-1)\times(d+1)$ matrix $\boldsymbol{B}$ is defined by

$$\boldsymbol{B} = \boldsymbol{R}_{d-1}(x)\boldsymbol{R}_d(z) - \boldsymbol{R}_{d-1}(z)\boldsymbol{R}_d(x).$$

To complete the proof, we must show that $\boldsymbol{B} = \boldsymbol{0}$. Let $\boldsymbol{a}$ be any vector in $\mathbb{R}^{d-1}$. Then we know from (3.8) that $\boldsymbol{a}^T\boldsymbol{B}\boldsymbol{\rho}_d(y) = 0$ for all y. Since the $d+1$ polynomials in $\boldsymbol{\rho}_d$ are linearly independent, see Lemma 3.7, this means that $\boldsymbol{a}^T\boldsymbol{B} = \boldsymbol{0}$. But $\boldsymbol{a}$ was arbitrary, so $\boldsymbol{B}$ maps all vectors to $\boldsymbol{0}$, in other words $\boldsymbol{B} = \boldsymbol{0}$. ∎

3.1.2 Marsden's identity and representation of polynomials

The relation (3.6) is a key in finding the B-spline representation of polynomials. If we set $x_1 = \cdots = x_d = x$ and remember that $R_1(x)\cdots R_d(x) = \boldsymbol{B}_d(x)$, the relation becomes

$$(y-x)^d = \boldsymbol{B}_d(x)^T\boldsymbol{\rho}_d(y) = \sum_{j=\mu-d}^{\mu} B_{j,d}(x)\rho_{j,d}(y), \tag{3.9}$$

provided x is in the interval $[t_\mu, t_{\mu+1})$. The interpretation of this is that if for fixed y, we use the sequence of numbers $(\rho_{j,d}(y))_{j=\mu-d}^{\mu}$ as B-spline coefficients, the resulting spline is the polynomial $(y-x)^d$, as long as we restrict our attention to the interval $[t_\mu, t_{\mu+1})$. But since the coefficients $(\rho_{j,d}(y))_{j=\mu-d}^{\mu}$ are independent of μ and therefore of the knot interval, the polynomial formula (3.9) can be generalised to a statement about how the polynomial $(y-x)^d$ is represented in terms of B-splines.

Theorem 3.4 (Marsden's identity)**.** *Let the knot vector $\boldsymbol{t} = (t_j)_{j=1}^{n+d+1}$ be given. Then the relation*

$$(y-x)^d = \sum_{j=1}^{n} \rho_{j,d}(y)B_{j,d}(x) \tag{3.10}$$

holds for all real numbers y, and all real numbers x in the interval $[t_{d+1}, t_{n+1})$.

The power of Theorem 3.4 lies in the fact that the coefficients $\boldsymbol{\rho}_d$ depend on y. Using this result we can show explicitly how the powers $1, x, \ldots, x^d$ can be written in terms of B-splines.

Corollary 3.5. *For any x in the interval $[t_{d+1}, t_{n+1})$, the power basis $\{x^i\}_{i=0}^d$ can be expressed in terms of B-splines through the relations*

$$1 = \sum_{j=1}^{n} B_{j,d}(x), \quad \text{for } d \geq 0, \tag{3.11}$$

$$x = \sum_{j=1}^{n} t^*_{j,d} B_{j,d}(x), \quad \text{for } d \geq 1, \tag{3.12}$$

$$x^2 = \sum_{j=1}^{n} t^{**}_{j,d} B_{j,d}(x), \quad \text{for } d \geq 2, \tag{3.13}$$

where

$$t^*_{j,d} = (t_{j+1} + \cdots + t_{j+d})/d \tag{3.14}$$

$$t^{**}_{j,d} = \sum_{i=j+1}^{j+d-1} \sum_{k=i+1}^{j+d} t_i t_k / \binom{d}{2}. \tag{3.15}$$

In general, for $r = 0, 1, \ldots, d$, the relation

$$x^r = \sum_{j=1}^{n} \sigma^r_{j,d} B_{j,d}(x) \tag{3.16}$$

holds for any x in the interval $[t_{d+1}, t_{n+1})$. Here $\sigma^r_{j,d}$ are the symmetric polynomials given by

$$\sigma^r_{j,d} = \Big(\sum t_{j_1} t_{j_2} \cdots t_{j_r}\Big) / \binom{d}{r}, \qquad \text{for } r = 0, 1, \ldots, d, \tag{3.17}$$

where the sum is over all integers $j_1, \ldots, j_r$ with $j+1 \leq j_1 < \cdots < j_r \leq j+d$, a total of $\binom{d}{r}$ terms.

Proof. If we differentiate both sides of equation (3.10) a total of $d-r$ times with respect to y, set $y = 0$, and rearrange constants we end up with

$$x^r = (-1)^r \frac{r!}{d!} \boldsymbol{B}_d(x)^T D^{d-r} \boldsymbol{\rho}_d(0) = (-1)^r \frac{r!}{d!} \sum_j B_{j,d}(x) D^{d-r} \rho_{j,d}(0). \tag{3.18}$$

Multiplying together the factors of $\rho_{j,d}$ we find

$$\rho_{j,d}(y) = y^d - t^*_{j,d} y^{d-1} + t^{**}_{j,d} y^{d-2} + \text{ lower order terms.} \tag{3.19}$$

From this it follows that

$$D^d \rho_{j,d}(0) = d!, \quad D^{d-1} \rho_{j,d}(0) = -(d-1)! t^*_{j,d}, \quad D^{d-2} \rho_{j,d}(0) = (d-2)! t^{**}_{j,d}. \tag{3.20}$$

Setting $r = 0$, 1 and 2 in (3.18) and inserting the appropriate formula in (3.20) leads to equations (3.11), (3.12), and (3.13). In general we have the formula

$$\rho_{j,d}(y) = \sum_{r=0}^{d} (-1)^r \binom{d}{r} \sigma^r_{j,d} y^{d-r}.$$

Using the same reasoning as above, we therefore find that

$$(-1)^r \frac{r!}{d!} D^{d-r} \rho_{j,d}(0) = \frac{r!(d-r)!}{d!} \binom{d}{r} \sigma^r_{j,d} = \sigma^r_{j,d},$$

so (3.16) follows from (3.18). ∎

The coefficients $\sigma^r_{j,d}$ are scaled versions of the *elementary symmetric polynomials* of degree d. They play an important role in the study of polynomial rings.

Example 3.6. In the cubic case, the relations (3.11)–(3.13) are

$$1 = \sum_{j=1}^{n} B_{j,3}(x), \tag{3.21}$$

$$x = \sum_{j=1}^{n} \frac{t_{j+1} + t_{j+2} + t_{j+3}}{3} B_{j,3}(x), \tag{3.22}$$

$$x^2 = \sum_{j=1}^{n} \frac{t_{j+1}t_{j+2} + t_{j+1}t_{j+3} + t_{j+2}t_{j+3}}{3} B_{j,3}(x), \tag{3.23}$$

$$x^3 = \sum_{j=1}^{n} t_{j+1}t_{j+2}t_{j+3} B_{j,3}(x), \tag{3.24}$$

valid for any x in $[t_{d+1}, t_{n+1})$.

3.1.3 Linear independence of B-splines

Recall from Appendix A that a set of functions $\{\phi_j\}_{j=1}^n$ are linearly independent on an interval I if $\sum_{j=1}^n c_j \phi_j(x) = 0$ for all $x \in I$ implies that $c_j = 0$ for all j. In other words, the only way to represent the 0-function on I is by letting all the coefficients be zero. A consequence of this is that any function that can be represented by $(\phi_j)_{j=1}^n$ has a unique representation.

To prove that B-splines are linearly independent, we start by showing that the B-splines that are nonzero on a single knot interval are linearly independent.

Lemma 3.7. *The B-splines $\{B_{j,d}\}_{j=\mu-d}^{\mu}$ and the dual polynomials $\{\rho_{j,d}\}_{j=\mu-d}^{\mu}$ are both linearly independent on the interval $[t_\mu, t_{\mu+1})$.*

Proof. From Corollary 3.5 we know that the power basis 1, x, …, x^d, and therefore any polynomial of degree d, can be represented by linear combinations of B-splines. On the interval $[t_\mu, t_{\mu+1})$ the only nonzero B-splines are $\{B_{j,d}\}_{j=\mu-d}^{\mu}$. These B-splines therefore form a basis for polynomials of degree d on $[t_\mu, t_{\mu+1})$, and in particular they are linearly independent on this interval since there are only $d+1$ of them. The symmetry of x and y in (3.10) leads to the same conclusion for the dual polynomials. ∎

From this local result we shall obtain a global linear independence result for B-splines. But first we need to be more precise about the type of knot vectors we consider.

Definition 3.8. *A knot vector $\boldsymbol{t} = (t_j)_{j=1}^{n+d+1}$ is said to be $d+1$-extended if*

1. $n \geq d+1$,

2. $t_{d+1} < t_{d+2}$ and $t_n < t_{n+1}$,

3. $t_j < t_{j+d+1}$ *for* $j = 1, 2, \ldots, n$.

A $d+1$*-extended knot vector for which* $t_1 = t_{d+1}$ *and* $t_{n+1} = t_{n+d+1}$ *is said to be* $d+1$*-regular.*

In most applications $d+1$-regular knot vectors are used, but linear independence can be proved in the more general situation of a $d+1$-extended knot vector.

Theorem 3.9. *Suppose that* $\boldsymbol{t}$ *is a* $d+1$*-extended knot vector. Then the B-splines in* $\mathbb{S}_{d,\boldsymbol{t}}$ *are linearly independent on the interval* $[t_{d+1}, t_{n+1})$.

Proof. We prove the result in the case where $\boldsymbol{t}$ is $d+1$-regular and leave the proof in the general case to the reader, see Exercise 2. Suppose that the spline $f = \sum_{j=1}^{n} c_j B_{j,d}$ is identically zero on $[t_{d+1}, t_{n+1})$; we must prove that $c_j = 0$ for $j = 1, \ldots, n$. Let j be an arbitrary integer in the range $[1, n]$. Since no knot occurs more than $d+1$ times there is a nonempty interval $[t_\mu, t_{\mu+1})$ contained in $[t_j, t_{j+d+1}]$, the support of $B_{j,d}$. But all the nonzero B-splines on $[t_\mu, t_{\mu+1})$ are linearly independent, so $f(x) = 0$ on this interval implies that $c_k = 0$ for $k = \mu - d, \ldots, \mu$. Since $B_{j,d}$ is one of the nonzero B-splines, we have in particular that $c_j = 0$. ∎

The condition that no knot must occur with multiplicity higher than $d+1$ is essential, for otherwise one of the B-splines will be identically zero and the B-splines will certainly be linearly dependent. The other conditions are not essential for the linear independence, see exercise 3.

3.2 Differentiation and smoothness of B-splines

In this section we study differentiation of splines with the matrix representation of B-splines as a starting point. We start by defining jumps and derivatives for piecewise continuous functions.

Definition 3.10. *A function* f *defined on some interval* $[a, b]$ *is* piecewise continuous *on* $[a, b]$ *provided* f *is continuous on* $[a, b]$ *except at a finite number of points* (x_i) *where the* one-sided limits

$$f(z+) = \lim_{\substack{x \to z \\ x > z}} f(x), \quad f(z-) = \lim_{\substack{x \to z \\ x < z}} f(x). \tag{3.25}$$

exist for $z = x_i$*, and* $i = 1, 2, \ldots, n$*. The number*

$$J_z f = f(z+) - f(z-) \tag{3.26}$$

is called the jump *of* f *at* z.

Definition 3.11. *If the function* f *has piecewise continuous* r*th derivative* $f^{(r)}$ *on* $[a, b]$ *for some integer* $r \geq 0$*, it is said to be* piecewise C^r. *If* $J_z(f^{(k)}) = 0$ *for* $k = 0, \ldots, r$ *at some* $z \in (a, b)$ *then* f *is said to be* C^r *at* z*. Differentiation for functions that are piecewise* C^r *is defined by*

$$D^r f(x) = \begin{cases} D_+^r f(x), & x \in [a, b), \\ D_-^r f(x), & x = b, \end{cases}$$

where the right derivative D_+^r *and the* left derivative D_-^r *are defined by*

$$D_+^r f(x) = f^{(r)}(x+), \quad x \in [a, b),$$
$$D_-^r f(x) = f^{(r)}(x-), \quad x \in (a, b].$$

At a point where the rth derivative of f is continuous this definition of differentiation agrees with the standard one since the two one-sided derivatives $D^r_+ f$ and $D^r_- f$ are the same at such a point.

Example 3.12. It is easy to check that the quadratic B-spline

$$B(x|0,0,1,2) = (2x - \frac{3}{2}x^2)B(x|0,1) + \frac{1}{2}(2-x)^2 B(x|1,2)$$

is continuous on $\mathbb{R}$. The first derivative

$$DB(x|0,0,1,2) = (2-3x)B(x|0,1) - (2-x)B(x|1,2)$$

is piecewise continuous on $\mathbb{R}$ with a discontinuity at $x = 0$, and the second derivative

$$D^2B(x|0,0,1,2) = -3B(x|0,1) + B(x|1,2)$$

is piecewise continuous on $\mathbb{R}$ with discontinuities at 0, 1, and 2. The third derivative is identically zero and continuous everywhere. This B-spline is therefore C^0 at $x = 0$, it is C^1 at $x = 1$ and $x = 2$ and at all other real numbers it has infinitely many continuous derivatives.

3.2.1 Derivatives of B-splines

Our next aim is to study differentiation of B-splines. We will start by working with polynomials and considering what happens on one knot interval, and then generalise to splines.

From Definition 3.11 we see that the rth derivative of a B-spline $B_{j,d}$ is given by

$$D^r B_{j,d} = \sum_{k=j}^{j+d} D^r B^k_{j,d} B_{k,0}, \quad r \geq 0, \tag{3.27}$$

where $D^r B^k_{j,d}$ is the ordinary rth derivative of the polynomial representing $B_{j,d}$ on the interval $[t_k, t_{k+1})$. This explicit formula is of little interest because it is difficult to compute. What we want is something similar to the recurrence relation (2.1).

Recall from Theorem 2.18 that on a knot interval $[t_\mu, t_{\mu+1})$ the row vector of the nonzero B-splines $\boldsymbol{B}_d$ is given by

$$\boldsymbol{B}_d(x) = R_1(x) \cdots R_d(x). \tag{3.28}$$

We can differentiate this product of matrices as if the factors were numbers. Indeed, let $\boldsymbol{A}$ be a matrix where each entry is a function of x. The derivative $D\boldsymbol{A}$ of $\boldsymbol{A}$ is defined as the matrix obtained by differentiating each entry of $\boldsymbol{A}$ with respect to x. We have the following rule for differentiating a product of two matrices

Lemma 3.13. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two matrices with entries that are functions of x and with dimensions such that the matrix product $\boldsymbol{AB}$ makes sense. Then*

$$D(\boldsymbol{AB}) = (D\boldsymbol{A})\boldsymbol{B} + \boldsymbol{A}(D\boldsymbol{B}).$$

Proof. Let $(\boldsymbol{AB})_{ij}$ be an arbitrary element in the product $\boldsymbol{AB}$. Then

$$\begin{aligned}D(\boldsymbol{AB})_{ij} &= D\left(\sum_k a_{ik}b_{kj}\right) = \sum_k D\left(a_{ik}b_{kj}\right)\\ &= \sum_k \left((Da_{ik})b_{kj} + a_{ik}(Db_{kj})\right)\\ &= \sum_k (Da_{ik})b_{kj} + \sum_k a_{ik}(Db_{kj})\\ &= ((D\boldsymbol{A})\boldsymbol{B})_{ij} + (\boldsymbol{A}(D\boldsymbol{B}))_{ij}\end{aligned}$$

which proves the lemma. ∎

Applying this rule successively to the product (3.28) we get

$$D\boldsymbol{B}_d(x) = \sum_{k=1}^{d} \boldsymbol{R}_1(x)\cdots\boldsymbol{R}_{k-1}(x)D\boldsymbol{R}_k(x)\boldsymbol{R}_{k+1}(x)\ldots\boldsymbol{R}_d(x), \tag{3.29}$$

where $D\boldsymbol{R}_k$ denotes the matrix obtained by differentiating each entry in $\boldsymbol{R}_k(x)$ with respect to x,

$$D\boldsymbol{R}_k(x) = \begin{pmatrix} \frac{-1}{t_{\mu+1}-t_{\mu+1-k}} & \frac{1}{t_{\mu+1}-t_{\mu+1-k}} & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \ldots & \frac{-1}{t_{\mu+k}-t_{\mu}} & \frac{1}{t_{\mu+k}-t_{\mu}} \end{pmatrix}. \tag{3.30}$$

The dimensions of the matrix $D\boldsymbol{R}_k$ are the same as those of $\boldsymbol{R}_k$, so both are transformations from $\mathbb{R}^{k+1}$ to $\mathbb{R}^k$.

To simplify equation 3.29, we need the following lemma.

Lemma 3.14. *For $k \geq 2$ and any real number x, the matrices $\boldsymbol{R}_k$ and $\boldsymbol{R}_{k+1}$ satisfy the relation*

$$D\boldsymbol{R}_k\boldsymbol{R}_{k+1}(x) = \boldsymbol{R}_k(x)D\boldsymbol{R}_{k+1}. \tag{3.31}$$

Proof. Equation 3.31 follows by differentiating both sides of 3.7 with respect to z and letting $d = k+1$. ∎

Using equation 3.31 we can move the D in (3.29) from $\boldsymbol{R}_k$ to $\boldsymbol{R}_d$ for each k. The end result is

$$D\boldsymbol{B}_d(x) = d\,\boldsymbol{R}_1(x)\cdots\boldsymbol{R}_{d-1}(x)D\boldsymbol{R}_d = d\,\boldsymbol{B}_{d-1}(x)D\boldsymbol{R}_d. \tag{3.32}$$

Let us now see how higher derivatives of B-splines can be determined. To find the second derivative we differentiate (3.32). Since $D(D\boldsymbol{R}_d) = 0$ we obtain

$$D^2\boldsymbol{B}_d(x)^T = d\,D\boldsymbol{B}_{d-1}(x)^T D\boldsymbol{R}_d.$$

If we apply (3.32)) to $D\boldsymbol{B}_{d-1}$ we find

$$D^2\boldsymbol{B}_d(x)^T = d(d-1)\boldsymbol{B}_{d-2}(x)^T D\boldsymbol{R}_{d-1}D\boldsymbol{R}_d.$$

In general, for the rth derivative we find

$$D^r \boldsymbol{B}_d(x)^T = \frac{d!}{(d-r)!} \boldsymbol{B}_{d-r}(x)^T D\boldsymbol{R}_{d-r+1} \cdots D\boldsymbol{R}_d.$$

Since in addition $\boldsymbol{B}_{d-r}(x)^T = \boldsymbol{R}_1(x) \cdots \boldsymbol{R}_{d-r}(x)$ the following theorem has been proved.

Theorem 3.15. *Let x be a number in $[t_\mu, t_{\mu+1})$. Then the rth derivative of the vector of B-splines $\boldsymbol{B}_d(x) = (B_{\mu-d,d}(x), \ldots, B_{\mu,d}(x))^T$ is given by*

$$D^r \boldsymbol{B}_d(x)^T = \frac{d!}{(d-r)!} \boldsymbol{B}_{d-r}(x)^T D\boldsymbol{R}_{d-r+1} \cdots D\boldsymbol{R}_d. \tag{3.33}$$

Suppose that $f(x) = \sum_{j=1}^n c_j B_{j,d}(x)$. The r'th derivative of f at x is given by

$$D^r f(x) = \frac{d!}{(d-r)!} \boldsymbol{R}_1(x) \cdots \boldsymbol{R}_{d-r}(x) D\boldsymbol{R}_{d-r+1} \cdots D\boldsymbol{R}_d \boldsymbol{c}_d, \tag{3.34}$$

for any integer r such that $0 \leq r \leq d$.

Note that the symmetry property (3.31) gives us a curious freedom in how to represent the rth derivative: It does not matter which of the d matrices $\boldsymbol{R}_k$ we differentiate as long as we differentiate r of them. In Theorem 3.15 it is the r matrices of largest dimension that have been differentiated.

Theorem 3.15 is the basis for algorithms for differentiating splines and B-splines, see Section 3.2.2, and leads to the following differentiation formula for a B-spline.

Theorem 3.16. *The derivative of the jth B-spline of degree d on $\boldsymbol{t}$ is given by*

$$DB_{j,d}(x) = d\Big(\frac{B_{j,d-1}(x)}{t_{j+d} - t_j} - \frac{B_{j+1,d-1}(x)}{t_{j+1+d} - t_{j+1}}\Big) \tag{3.35}$$

for $d \geq 1$ and for any real number x. The derivative of $B_{j,d}$ can also be expressed as

$$DB_{j,d}(x) = \frac{d}{d-1}\Big(\frac{x - t_j}{t_{j+d} - t_j} DB_{j,d-1}(x) + \frac{t_{j+1+d} - x}{t_{j+1+d} - t_{j+1}} DB_{j+1,d-1}(x)\Big) \tag{3.36}$$

for $d \geq 2$ and any x in $\mathbb{R}$.

Using the $'0/0 = 0'$ convention the differentiation formula (3.35) can be expressed more explicitly as

$$DB_{j,d} = d \begin{cases} 0, & \text{if } t_j = t_{j+d+1}; \\ \dfrac{B_{j,d-1}}{t_{j+d} - t_j}, & \text{if } t_j < t_{j+d} \text{ and } t_{j+1} = t_{j+1+d}; \\ -\dfrac{B_{j+1,d-1}}{t_{j+1+d} - t_{j+1}}, & \text{if } t_j = t_{j+d} \text{ and } t_{j+1} < t_{j+1+d}; \\ \dfrac{B_{j,d-1}}{t_{j+d} - t_j} - \dfrac{B_{j+1,d-1}}{t_{j+1+d} - t_{j+1}}, & \text{otherwise.} \end{cases}$$

Proof. Clearly (3.35) holds if $x \notin [t_j, t_{j+1+d})$ so suppose $x \in [t_\mu, t_{\mu+1})$ for some $j \le \mu \le j + d$. By (3.33) for $r = 1$ we have

$$(DB_{\mu-d,d}(x), \ldots, DB_{\mu,d}(x)) = d(B_{\mu-d+1,d}(x), \ldots, B_{\mu,d-1}(x))D\boldsymbol{R}_d.$$

Carrying out the matrix multiplication on the right and comparing the jth component on both sides we obtain (3.35). Since (3.35) is independent of μ, it holds for all $x \in [t_j, t_{j+d+1})$.

To prove (3.36) we make use of Lemma 3.14 and differentiate the matrix $\boldsymbol{R}_1$ instead of $\boldsymbol{R}_d$, see exercise 6. ∎

3.2.2 Computing derivatives of splines and B-splines

As for evaluation, see Section 2.4, there are two closely related algorithms for computing the rth derivative of a spline, both based on the matrix representation from Theorem 3.15,

$$D^r f(x) = \frac{d!}{(d-r)!}\boldsymbol{R}_1(x) \cdots \boldsymbol{R}_{d-r}(x) D\boldsymbol{R}_{d-r+1} \cdots D\boldsymbol{R}_d \boldsymbol{c}_d. \tag{3.37}$$

As before, we assume that x lies in the interval $[t_\mu, t_{\mu+1})$ and that the vector $\boldsymbol{c}_d$ contains the B-spline coefficients that multiply the B-splines that are nonzero on $[t_\mu, t_{\mu+1})$ so that $\boldsymbol{c}_d = (c_{\mu-d}, \ldots, c_\mu)^T$. We then have the DL (Derivative Left) Algorithm which computes $D^r f(x)$ by accumulating matrix products from right to left in (3.37), while the DR (Derivative Right) Algorithm computes the rth derivative of all the nonzero B-splines at a point by accumulating matrix products from left to right, then multiplying with the coefficients and summing up.

Algorithm 3.17 (DL)**.** *Let the polynomial degree d, the $2d$ knots $t_{\mu-d+1} \le t_\mu < t_{\mu+1} \le t_{\mu+d}$, the B-spline coefficients $\boldsymbol{c}_d^{(0)} = \boldsymbol{c}_d = (c_j)_{j=\mu-d}^{\mu}$ of a spline f, and a number x in $[t_\mu, t_{\mu+1})$ be given. After evaluation of the products*

$$\begin{aligned} \boldsymbol{c}_{k-1}^{(d-k+1)} &= D\boldsymbol{R}_k \boldsymbol{c}_k^{(d-k)}, \qquad k = d, \ldots, d-r+1, \\ \boldsymbol{c}_{k-1}^{(r)} &= \boldsymbol{R}_k(x)\boldsymbol{c}_k^{(r)}, \qquad k = d-r, \ldots, 1, \end{aligned}$$

the rth derivative of f at x is given by

$$D^r f(x) = d!\, \boldsymbol{c}_0^{(r)}/(d-r)!.$$

Algorithm 3.18 (DR)**.** *Let the polynomial degree d, the knots $t_{\mu-d+1} \le t_\mu < t_{\mu+1} \le t_{\mu+d}$ and a number x in $[t_\mu, t_{\mu+1})$ be given and set $\boldsymbol{B}_0 = 1$. After evaluation of the products*

$$\begin{aligned} \boldsymbol{B}_k(x)^T &= \boldsymbol{B}_{k-1}(x)^T \boldsymbol{R}_k(x), \qquad k = 1, \ldots, d-r, \\ D^{k-d+r}\boldsymbol{B}_k(x)^T &= kD^{k-d+r-1}\boldsymbol{B}_{k-1}(x)^T D\boldsymbol{R}_k, \qquad k = d-r+1, \ldots, d, \end{aligned}$$

the vector $D^r\boldsymbol{B}_d(x)$ will contain the value of the rth derivative of the nonzero B-splines at x,

$$D^r\boldsymbol{B}_d(x) = \big(D^r B_{\mu-d,d}(x), \ldots, D^r B_{\mu,d}(x)\big)^T.$$

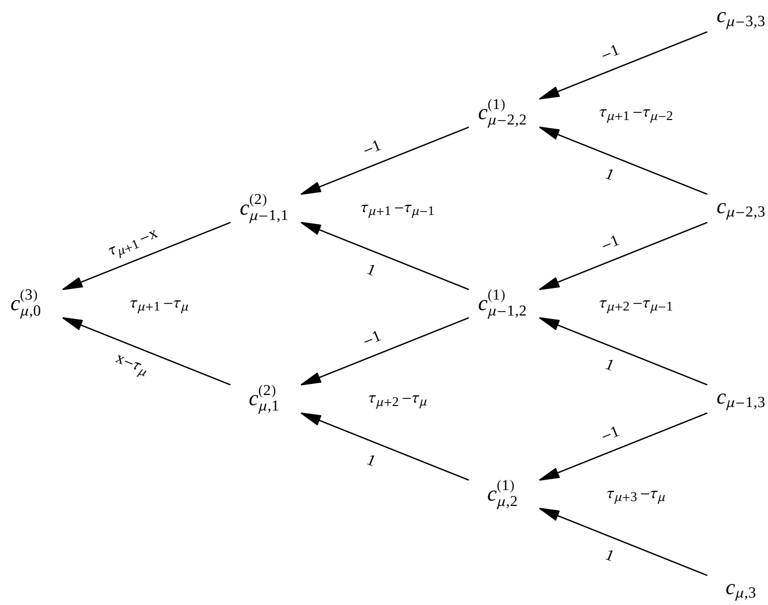

Figure 3.1. A triangular algorithm for computation of the second derivative of a cubic spline at x.

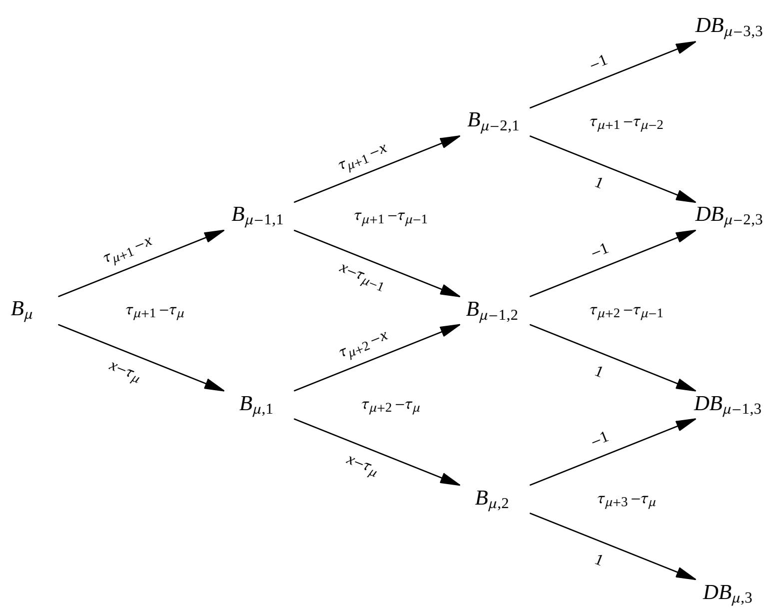

Figure 3.2. A triangular algorithm for computation of the derivative of the nonzero cubic B-splines at x.

Figure 3.1 shows how the second derivative of a cubic spline can be computed, while Figure 3.2 shows the computation of the first derivative of all the nonzero B-splines at a point.

In Algorithm 3.17 we have to compute matrix-vector products on the forms $D\boldsymbol{R}_k\boldsymbol{c}_k$ and $\boldsymbol{R}_k(x)\boldsymbol{c}_k$. The component form of the latter product is given in (2.25), while the component form of the former is obtained by differentiating the linear factors in (2.25) with respect to x. The result is

$$(D\boldsymbol{R}_k\boldsymbol{c}_k)_j = \frac{c_{k,j} - c_{k,j-1}}{t_{j+k} - t_j} \tag{3.38}$$

for $j = \mu - k + 1, \ldots, \mu$.

The alternative algorithm accumulates the matrix products in (2.23) from left to right. The component form of the product $\boldsymbol{B}_{k-1}(x)^T\boldsymbol{R}_k$ is given in (2.26) while the component form of the product $\boldsymbol{B}_{k-1}(x))^T D\boldsymbol{R}_k$ is

$$(\boldsymbol{B}_{k-1}(x))^T D\boldsymbol{R}_k)_j = \frac{B_{j,k-1}(x)}{t_{j+k} - t_j} - \frac{B_{j+1,k-1}(x)}{t_{j+1+k} - t_{j+1}} \tag{3.39}$$

for $j = \mu - k, \ldots, \mu$.

3.2.3 Smoothness of B-splines

A characteristic feature of splines is their smoothness properties as stated in Theorem 1.4 in Chapter 1. In this section we discuss the smoothness properties of splines in detail. We start by characterising the smoothness of B-splines.

Theorem 3.19. *Suppose that the number z occurs m times among the knots $t_j, t_{j+1}, \ldots, t_{j+d+1}$, defining the B-spline $B_{j,d}$ of degree d. If $1 \leq m \leq d+1$ then $D^r B_{j,d}$ is continuous at z for $r = 0, 1, \ldots, d-m$, but $D^{d-m+1}B_{j,d}$ is discontinuous at z.*

This theorem will proved via a sequence of steps. We first note from the explicit formula (2.11) for the Bernstein basis that Theorem 3.19 holds for $m = d+1$. At such a knot the B-spline is discontinuous with a jump of size 1. In the general case the proof is based on the following recurrence relations for jumps.

Lemma 3.20. *The jump in a B-spline at a real number x satisfies the recurrence relation*

$$J_x(B_{j,d}) = \frac{x - t_j}{t_{j+d} - t_j} J_x(B_{j,d-1}) + \frac{t_{j+1+d} - x}{t_{j+1+d} - t_{j+1}} J_x(B_{j+1,d-1}), \tag{3.40}$$

with

$$J_x(B_{j,0}) = \begin{cases} 1, & \text{if } x = t_j, \\ -1, & \text{if } x = t_{j+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.41}$$

For $r \geq 1$ the jump in the rth derivative is given by

$$J_x(D^r B_{j,d}) = d\Big(\frac{J_x(D^{r-1}B_{j,d-1})}{t_{j+d} - t_j} - \frac{J_x(D^{r-1}B_{j+1,d-1})}{t_{j+1+d} - t_{j+1}}\Big), \quad \text{for } x \in \mathbb{R} \text{ and } r \geq 1. \tag{3.42}$$

Proof. Evaluating the recurrence relation (2.1) at $x+$ and $x-$ and subtracting we obtain (3.40), while (3.41) follows directly from the definition of $B_{j,0}$. Differentiating the differentiation formula (3.35) a total of $r-1$ times leads to

$$D^r B_{j,d}(x) = d\Big(\frac{D^{r-1}B_{j,d-1}(x)}{t_{j+d}-t_j} - \frac{D^{r-1}B_{j+1,d-1}(x)}{t_{j+1+d}-t_{j+1}}\Big)$$

for any real number x. The same formula holds if we replace $D = D_+$ by D_-. Taking the difference of the two formulas leads to (3.42). ∎

As usual the $'0/0 = 0'$ convention is used in (3.40) and (3.42).

The first step in the proof of Theorem 3.19 is to that a B-spline is continuous at a knot of multiplicity at most d.

Lemma 3.21. *Suppose that no knot among $t_j, t_{j+1}, \dots, t_{j+d+1}$ occurs more than d times. Then the B-spline $B_{j,d}$ is continuous for all real numbers x.*

Proof. The proof is by induction on the degree d. For a B-spline of degree 0 the lemma does not apply so the induction starts with $d = 1$. It is easy to see from the explicit representation in Example 2.2 that a linear B-spline with three distinct knots is continuous. For the induction step we assume that the lemma holds for B-splines of degree $d-1$. To prove that it is also true for B-splines of degree d suppose first that no knots occur more than $d-1$ times. Then the two B-splines $B_{j,d-1}$ and $B_{j+1,d-1}$ are both continuous which means that $B_{j,d}$ is also continuous. Suppose next that x is equal to a knot which occurs exactly d times among $t_j, t_{j+1}, \dots, t_{j+d+1}$. There are three cases. Suppose first that $x = t_j$. Since $t_{j+d-1} < t_{j+d}$ it follows from the induction hypothesis that $J_x(B_{j+1,d-1}) = 0$, while $J_x(B_{j,d-1}) = 1$. From (3.40) we then obtain $J_x(B_{j,d}) = 0$, since $(x - t_j)J_x(B_{j,d-1}) = 0 \cdot 1 = 0$. The proof in the case $x = t_{j+1+d}$ is similar. Finally, if $t_j < x < t_{j+1+d}$ then $x = t_{j+1} = \cdots = t_{j+d}$ so (3.40) yields

$$J_x(B_{j,d}) = \frac{x-t_j}{t_{j+d}-t_j}\cdot 1 + \frac{t_{j+1+d}-x}{t_{j+1+d}-t_{j+1}}(-1) = 0.$$

This completes the proof. ∎

Proof. [The continuity part of Theorem 3.19]
For $r = 0$ the result follows from Lemma 3.21, while for general r it follows from (3.42) and induction on d that $J_z(D^r B_{j,d}) = 0$ for $1 \le r \le d-m$. ∎

To complete the proof of the continuity property we now determine the jump in the first discontinuous derivative of a B-spline.

Lemma 3.22. *Suppose that the number z occurs exactly m times among the knots $t_j, \dots, t_{j+1+d}$. Then the $d-m+1$th derivative of $B_{j,d}$ has a jump at z given by*

$$J_z(D^{d-m+1}B_{j,d}) = \frac{d!}{(m-1)!}(t_{j+1+d}-t_j)\Big/\prod_{\substack{k=j\\ t_k\neq z}}^{j+1+d}(t_k - z) \neq 0. \tag{3.43}$$

Proof. As usual the proof is by induction of the degree d. We first note that (3.43) holds in the case where $m = d + 2$, so we may assume that $m \le d + 1$. It is easy to check that equation (3.43) holds when $d = 0$ and $m = 1$. Suppose that (3.43) holds for B-splines of degree $d - 1$. For a B-spline of degree d we apply (3.42) with $r = d - m + 1$. There are three cases to consider. Suppose first that $z = t_j$. Since z occurs $m - 1$ times among the knots of $B_{j+1,d-1}$ it follows from the continuity property that $J_z(D^{d-m}B_{j+1,d-1}) = 0$. In view of the induction hypothesis, equation (3.42) therefore takes the form

$$J_z(D^{d-m+1}B_{j,d}) = d\frac{J_z(D^{d-m}B_{j,d-1})}{t_{j+d} - t_j} = \frac{d!}{(m-1)!} / \prod_{\substack{k=j \\ t_k \neq t_j}}^{j+d} (t_k - t_j).$$

Multiplying the numerator and denominator by $t_{j+1+d} - t_j$ proves (3.43) in this case. A similar argument is valid when $z = t_{j+1+d}$.

The remaining situation is $t_j < z < t_{j+1+d}$. In this case both $B_{j,d-1}$ and $B_{j+1,d-1}$ have a knot of multiplicity m at z. Applying (3.42) and the induction hypothesis we then obtain

$$\begin{aligned} J_z(D^{d-m+1}B_{j,d}) &= \frac{d!}{(m-1)!}\Big(\prod_{\substack{k=j \\ t_k \neq z}}^{j+d} (t_k - z)^{-1} - \prod_{\substack{k=j+1 \\ t_k \neq z}}^{j+1+d} (t_k - z)^{-1}\Big) \\ &= \frac{d!}{(m-1)!} \prod_{\substack{k=j+1 \\ t_k \neq z}}^{j+d} (t_k - z)^{-1}\Big(\frac{1}{t_j - z} - \frac{1}{t_{j+1+d} - z}\Big) \\ &= \frac{d!}{(m-1)!}(t_{j+1+d} - t_j) / \prod_{\substack{k=j \\ t_k \neq z}}^{j+1+d} (t_k - z) \end{aligned}$$

which completes the proof. ∎

3.3 B-splines as a basis for piecewise polynomials

Our ultimate purpose is to use B-splines as building blocks for constructing and representing functions and data, but what exactly are the functions in a spline space $\mathbb{S}_{d,\boldsymbol{t}}$? We know that they are piecewise polynomials, with different polynomial pieces meeting at the knots. We also know that the exact continuity between two pieces is controlled by the multiplicity of the knot at the join. If the knot z occurs with multiplicity m we know from Theorem 3.19 that there is at least one B-spline with its first $d - m$ derivatives continuous, but with the derivative of order $d - m + 1$ discontinuous. When we take linear combinations of the B-splines and form $\mathbb{S}_{d,\boldsymbol{t}}$, the spline functions will in general inherit this smoothness at z, although there will be some functions that will be even smoother, like for example the function with all coefficients zero, the zero function. In this section we will start by defining piecewise polynomial spaces in terms of the smoothness at the joins and show that $\mathbb{S}_{d,\boldsymbol{t}}$ can be characterised in this way. We start by defining piecewise polynomial spaces.

Definition 3.23. *Let d be a nonnegative integer, let $[a,b]$ be a real interval, let the sequence $\boldsymbol{\Delta} = (\xi_i)_{i=1}^N$ be a partition of $[a,b]$,*

$$a = \xi_1 < \xi_2 < \cdots < \xi_{N-1} < \xi_N = b,$$

and let $\boldsymbol{r} = (r_i)_{i=2}^{N-1}$ be a sequence of integers. By $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ we denote the linear space of piecewise polynomials of degree d on $[a,b]$ with r_i continuous derivatives at ξ_i. In other words $f \in \mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ if and only if the restriction of f to (ξ_{i-1}, ξ_i) is a polynomial of degree d for $i = 2, \ldots, N$, and $D^k f$ is continuous at ξ_i for $k = 0, \ldots, r_i$ and $i = 2, \ldots, N-1$.

It is quite simple to check that linear combinations of functions in $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ are again in $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$; it is therefore a linear space.

Lemma 3.24. *The dimension of $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ is $n = (N-1)d + 1 - \sum_{i=2}^{N-1} r_i$.*

To see why Lemma 3.24 is reasonable we can argue as follows. If there were no smoothness conditions ($r_i = -1$ for all i) we would have a space of dimension $(N-1)(d+1)$ (there are $N-1$ subintervals and on each we have a space of polynomials of degree d). All together there are $\sum_{i=2}^{N-1}(r_i+1)$ smoothness conditions so

$$\dim \mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta}) \geq (N-1)(d+1) - \sum_{i=2}^{N-1}(r_i+1) = (N-1)d + 1 - \sum_{i=2}^{N-1} r_i. \tag{3.44}$$

A priori we only get a lower bound since we cannot be sure that each continuity constraint reduces the dimension by one. A more careful investigation reveals that the dimension agrees with this lower bound, see Exercise 7.

There are many ways to represent the piecewise polynomials in $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$. One possibility is to pick one's favourite polynomial basis and represent each piece as a polynomial of degree d and ignore the smoothness conditions. Another possibility is to use the truncated power basis that is employed to prove Lemma 3.24 in Exercise 7. The following theorem shows that $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ can in fact be represented in terms of B-splines on an appropriate knot vector.

Theorem 3.25 (Curry-Schoenberg). *Let $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ be a given space of piecewise polynomials and let the $d+1$-extended knot vector $\boldsymbol{t} = (t_j)_{j=1}^{n+d+1}$ be defined by*

$$\boldsymbol{t} = (t_1, \ldots, t_{d+1}, \overbrace{\xi_2, \ldots, \xi_2}^{d-r_2}, \ldots, \overbrace{\xi_i, \ldots, \xi_i}^{d-r_i}, \ldots, \overbrace{\xi_{N-1}, \ldots, \xi_{N-1}}^{d-r_{N-1}}, t_{n+1}, \ldots, t_{n+d+1})$$

where n is given in Lemma 3.24 and the end knots satisfy $t_1 \leq \cdots \leq t_{d+1} \leq a$ and $b \leq t_{n+1} \leq \cdots \leq t_{n+d+1}$. Then

$$\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta}) = \mathbb{S}_{d,\boldsymbol{t}}|_{[a,b]},$$

where $\mathbb{S}_{d,\boldsymbol{t}}|_{[a,b]}$ is the space obtained by restricting the functions in $\mathbb{S}_{d,\boldsymbol{t}}$ to the interval $[a,b]$.

Proof. Let $\mathbb{S} = \mathbb{S}_{d,\boldsymbol{t}}|_{[a,b]}$. We note that by the construction of the knot vector, the B-splines in $\mathbb{S}$ satisfy the smoothness conditions of $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ so $\mathbb{S} \subseteq \mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$. On the other hand, the length of the knot vector $\boldsymbol{t}$ is $n+d+1$ so $\dim \mathbb{S} = \dim \mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$. But a subspace that has the same dimension as the full space must agree with the full space so $\mathbb{S} = \mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$. ∎

Exercises for Chapter 3

3.1 Suppose that $d = 3$ and that $\hat{\boldsymbol{t}} = (0, 0, 1, 3, 4, 5)$ so we can associate two cubic B-splines $\hat{B}_{1,3}$ and $\hat{B}_{2,3}$ with $\hat{\boldsymbol{t}}$. We want to prove that these two B-splines are linearly independent on $[1, 3]$.

a) Let $\boldsymbol{t}$ denote the augmented knot vector $\boldsymbol{t} = (0, 0, 0, 1, 3, 4, 5, 5)$. Show that we can associate 4 B-splines $\{B_{i,3}\}_{i=1}^{4}$ with $\boldsymbol{t}$ and that these are linearly independent on $[1, 3]$.

b) Show that the two B-splines $\hat{B}_{1,3}$ and $\hat{B}_{2,3}$ are linearly independent.

3.2 Prove Theorem 3.9.

3.3 Let $\boldsymbol{t} = (t_j)_{j=1}^{n+d+1}$ be knot vector with $n \geq 1$ and such that no knot occurs more than $d+1$ times. Show that the B-splines $\{B_{j,d}\}_{j=1}^{n}$ are linearly independent on the interval $[t_1, t_{n+d+1})$.

3.4 Let $\boldsymbol{A}$ be matrix where each entry is a function of x and let α be a scalar function of x. Prove the formula

$$D(\alpha \boldsymbol{A}) = (D\alpha)\boldsymbol{A} + \alpha(D\boldsymbol{A}).$$

3.5 a) Count the number of operations (additions/subtractions, multiplications, divisions) involved in computing the matrix $\boldsymbol{R}_k(x)$ defined in (2.20). Do the same for the matrix $D\boldsymbol{R}_k$ defined in (3.30).

b) Recall that in the formula (3.34) for the rth derivative of f, we have the freedom to differentiate any r of the d matrices $\{\boldsymbol{R}_k(x)\}_{k=1}^{d}$. Based on the count in (a), show that the choice made in (3.34) is the most efficient.

3.6 In this exercise we are going to prove the differentiation formula (3.36).

a) Show that

$$(DB_{\mu-d,d}(x), \ldots, DB_{\mu,d}(x)) = dD\boldsymbol{R}_1\boldsymbol{R}_2(x)\cdots\boldsymbol{R}_d(x) \tag{3.45}$$

for any x in $[t_\mu, t_{\mu+1})$.

b) Show that (3.45) leads to (3.36) and that the latter equation is valid for any x. Why do we need the restriction $d \geq 2$?

3.7 In this exercise we will provide a proof of Lemma 3.24. Let π_d denote the linear space of polynomials of degree at most d. Recall that the powers $1, x, \ldots, x^d$ is a basis for π_d on any interval $[a, b]$ with $a < b$ and that the dimension of π_d is $d + 1$.

a) Let $\boldsymbol{\Delta} = (\xi_i)_{i=}^{N}$ be a partition of some interval $[a, b]$,

$$a = \xi_1 < \xi_2 < \cdots < \xi_{N-1} < \xi_N = b$$

and let $\mathbb{S}_d^{-1}(\boldsymbol{\Delta})$ denote the set of functions that are polynomials of degree $d+1$ on each subinterval (ξ_{i-1}, ξ_i) for $i = 2, \ldots, N$ (no continuity is assumed between the different pieces). Show that the dimension of $\mathbb{S}_d^{-1}(\boldsymbol{\Delta})$ is $(N-1)(d+1)$. Hint: Show that the functions $\{\eta_{i,k}\}_{i=1,k=0}^{N-1,d}$ defined by

$$\eta_{i,k}(x) = \begin{cases} (x-\xi_i)^k, & \text{if } \xi_i \le x < \xi_{i-1}; \\ 0, & \text{otherwise}; \end{cases}$$

form a basis for $\mathbb{S}_d^{-1}(\boldsymbol{\Delta})$.

b) Show that a different basis for $\mathbb{S}_d^{-1}(\boldsymbol{\Delta})$ is given by the functions $\{\theta_{i,k}\}_{i=1,k=0}^{N-1,d}$ defined by

$$\theta_{i,k}(x) = (x-\xi_i)_+^k,$$

where

$$a_+^k = \begin{cases} a^k, & \text{if } a > 0; \\ 0, & \text{otherwise}; \end{cases}$$

except that we use the convention $0^0 = 1$.

c) Let J denote the jump-operator defined in Definition 3.10. Show that

$$J_{\xi_i}(D^\ell \theta_{j,k}) = k!\delta_{\ell,k}\delta_{i,j}$$

where $\delta_{m,n} = 1$ if $m = n$ and zero otherwise.

d) Let $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ be as in Definition 3.23. Show that $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ is a subspace of $\mathbb{S}_d^{-1}(\boldsymbol{\Delta})$. Show also that if $f = \sum_{i=1}^{N-1}\sum_{k=0}^{d} c_{i,k}\eta_{i,k}$ is in $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$ then $c_{i,k} = 0$ for $k = 0, 1, \ldots, r_i$ and $i = 2, 3, \ldots, N-1$. Hint: Make use of (c). Conclude that $\{\theta_{i,k}\}_{i=1,k=r_i}^{N-1,d}$, where $r_1 = 0$, is a basis for $\mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta})$, and that

$$\dim \mathbb{S}_d^{\boldsymbol{r}}(\boldsymbol{\Delta}) = (N-1)d + 1 - \sum_{i=2}^{N-1} r_i.$$