CHAPTER 4

Knot insertion

In Chapter 1 we were led to B-splines, defined via the recurrence relation, as a convenient way to represent spline functions. In Chapters 2 and 3 we then established some of the basic properties of splines, with the recurrence relation as the major tool. We have seen that splines can be evaluated efficiently and stably, we have studied the smoothness of splines, we have shown that B-splines are linearly independent and that they form a basis for certain spaces of piecewise polynomials.

This chapter supplements the recurrence relation for B-splines with another very versatile tool, namely the idea of *knot insertion* or *knot refinement*. We have already seen that the control polygon of a spline provides a rough sketch of the spline itself. It turns out that the control polygon approaches the spline it represents as the distance between the knots of a spline is reduced, a fact that will be proved in Chapter 9. This indicates that it is of interest to see how the B-spline coefficients of a fixed spline depend on the knots.

Knot insertion amounts to what the name suggests, namely insertion of knots into an existing knot vector. The result is a new spline space with more B-splines and therefore more flexibility than the original spline space. This can be useful in many situations, for example in interactive design of spline curves. It turns out that the new spline space contains the original spline space as a subspace, so any spline in the original space can also be represented in terms of the B-splines in the refined space. As mentioned above, an important property of this new representation is that the control polygon will have moved closer to the spline itself. This provides us with a new and very powerful tool both for algorithmic manipulation and theoretical investigations of spline functions.

We start, in Section 9.4, by showing some simple examples of knot insertion. In Section 4.2 we then develop algorithms for expressing the B-spline coefficients relative to a refined knot vector in terms of the B-spline coefficients relative to the original knot vector. It turns out that the B-spline coefficients of a spline are completely characterised by three simple properties, and this is the topic of Section 4.3. This characterisation is often useful for developing the theory of splines, and in Section 4.4 this characterisation is used to obtain formulas for inserting one new knot into a spline function. Finally, in Section 4.5, we make use of knot insertion to prove that the number of sign changes in a spline is bounded by the number of sign changes in its control polygon; another instance of the close relationship between a spline and its control polygon.

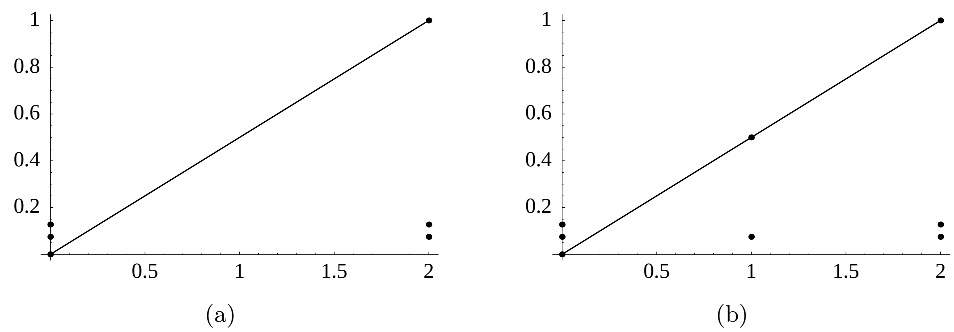


Figure 4.1. A line segment represented as a linear spline with no interior knots (a), and with one interior knot (b).

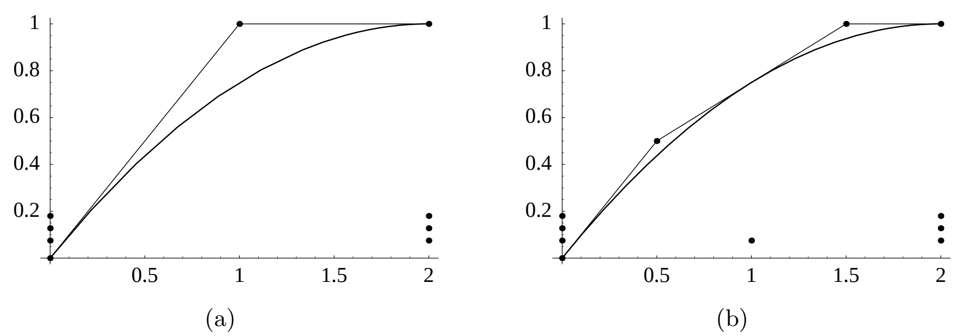


Figure 4.2. A piece of a parabola represented as a quadratic spline with no interior knots (a), and with one interior knot (b).

4.1 The control polygon relative to different knot vectors

In this introductory section we will consider some examples of knot insertion with the purpose of gaining an intuitive understanding of this important concept.

Figure 4.1 shows spline representations of a line segment. We all know that a straight line is uniquely determined by two points and in (a) the line segment is represented by its two end points. Although one may wonder what the point is, we can of course also represent the line segment by cutting it into smaller pieces and represent each of these pieces. This is what is shown in Figure 4.1 (b) where the line segment is represented by a linear spline with an interior knot at 1 which in effect means that we are using a redundant representation of three points to represent a line segment.

The redundancy in the representation is obvious and seems useless in the linear case. But let us increase the degree and consider a quadratic example. Figure 4.2 shows part of the parabola $y = (4x - x^2)/6$ represented as a spline without interior knots in (a) and with one interior knot in (b). In general, the representation in (b) requires a spline function and its first derivative to be continuous at $x = 1$, whereas a jump is allowed in the second derivative. The parabola in the figure is certainly continuous and has continuous first derivative at $x = 1$, but the jump in the second derivative happens to be 0. We knot at $x = 1$ is therefore redundant, but it has the nice effect of bringing the control polygon closer to the spline. We shall see later that there may be many other good reasons for inserting knots into a spline function.

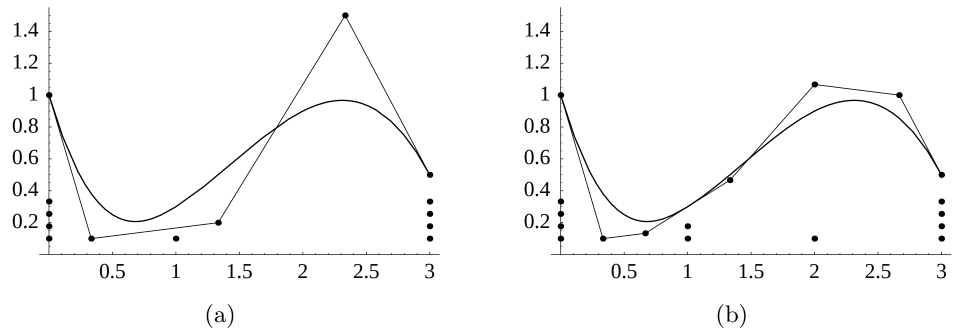


Figure 4.3. A cubic spline with one interior knot (a). In (b) the same spline is represented with two extra knots (the knot at $x = 1$ is now double).

An example with a cubic spline is shown in Figure 4.3. The situation is the same as before: The refined knot vector allows jumps in the second derivative at $x = 1$ and the third derivative at $x = 2$, but the jumps may be 0. For the specific spline in (a) these jumps are indeed 0, but one advantage of representing it in the refined spline space is that the control polygon comes closer to the spline.

The examples have hopefully shown that insertion of knots can be useful; at the very least it seems like it may be a useful tool for plotting splines. In the next sections we are going to develop algorithms for computing the B-spline coefficients on a refined knot vector and deduct various properties of the B-splines coefficients as functions of the knots. A proof of the fact that the control polygon converges to the spline it represents as the knot spacing goes to zero has to wait until Chapter 9.

4.2 Knot insertion

In this section we are going to develop two algorithms for computing the B-spline coefficients of a given spline relative to a refined knot vector. The two algorithms for knot insertion are closely related to Algorithms 2.20 and 2.21; in fact these two algorithms are special cases of the algorithms we develop here.

4.2.1 Basic idea

Knot insertion is exactly what the name suggests: extension of a given knot vector by adding new knots. Let us first define precisely what we mean by knot insertion, or knot refinement as it is also called.

Definition 4.1. *A knot vector $\boldsymbol{t}$ is said to be a refinement of a knot vector $\boldsymbol{\tau}$ if any real number occurs at least as many times in $\boldsymbol{t}$ as in $\boldsymbol{\tau}$.*

Note that if $\boldsymbol{t}$ is a refinement of $\boldsymbol{\tau}$ then $\boldsymbol{\tau}$ is a subsequence of $\boldsymbol{t}$, and this we will write $\boldsymbol{\tau} \subseteq \boldsymbol{t}$ even though knot vectors are sequences and not sets. The term knot insertion is used because in most situations the knot vector $\boldsymbol{\tau}$ is given and $\boldsymbol{t}$ is obtained by 'inserting' knots into $\boldsymbol{\tau}$. A simple example of a knot vector and a refinement is given by

$$\boldsymbol{\tau} = (0, 0, 0, 3, 4, 5, 5, 6, 6, 6) \qquad \text{and} \qquad \boldsymbol{t} = (0, 0, 0, 2, 2, 3, 3, 4, 5, 5, 5, 6, 6, 6).$$

Here two knots have been inserted at 2, one at 3 and one at 5.

With some polynomial degree d given, we can associate the spline spaces $\mathbb{S}_{d,\boldsymbol{\tau}}$ and $\mathbb{S}_{d,\boldsymbol{t}}$ with the two knot vectors $\boldsymbol{\tau}$ and $\boldsymbol{t}$. When $\boldsymbol{\tau}$ is a subsequence of $\boldsymbol{t}$, the two spline spaces are also related.

Lemma 4.2. *Let d be a positive integer and let $\boldsymbol{\tau}$ be a knot vector with at least $d+2$ knots. If $\boldsymbol{t}$ is a knot vector which contains $\boldsymbol{\tau}$ as a subsequence then $\mathbb{S}_{d,\boldsymbol{\tau}} \subseteq \mathbb{S}_{d,\boldsymbol{t}}$.*

Proof. Suppose first that both $\boldsymbol{\tau}$ and $\boldsymbol{t}$ are $d+1$-regular knot vectors with common knots at the ends. By the Curry-Schoenberg theorem (Theorem 3.25) we know that $\mathbb{S}_{d,\boldsymbol{t}}$ contains all splines with smoothness prescribed by the knot vector $\boldsymbol{t}$. Since all knots occur at least as many times in $\boldsymbol{t}$ as in $\boldsymbol{\tau}$, we see that at any knot, a spline f in $\mathbb{S}_{d,\boldsymbol{\tau}}$ is at least as smooth as required for a spline in $\mathbb{S}_{d,\boldsymbol{t}}$. But then $f \in \mathbb{S}_{d,\boldsymbol{\tau}}$ and $\mathbb{S}_{d,\boldsymbol{\tau}} \subseteq \mathbb{S}_{d,\boldsymbol{t}}$.

A proof in the general case where $\boldsymbol{\tau}$ and $\boldsymbol{t}$ are not $d+1$-regular with common knots at the ends, is outlined in exercise 5. ∎

Suppose that $f = \sum_{j=1}^{n} c_j B_{j,d,\boldsymbol{\tau}}$ is a spline in $\mathbb{S}_{d,\boldsymbol{\tau}}$ with B-spline coefficients $\boldsymbol{c} = (c_j)$. If $\boldsymbol{\tau}$ is a subsequence of $\boldsymbol{t}$, we know from Lemma 4.2 that $\mathbb{S}_{d,\boldsymbol{\tau}}$ is a subspace of $\mathbb{S}_{d,\boldsymbol{t}}$ so f must also lie in $\mathbb{S}_{d,\boldsymbol{t}}$. Hence there exist real numbers $\boldsymbol{b} = (b_i)$ with the property that $f = \sum_{i=1}^{m} b_i B_{i,d,\boldsymbol{t}}$, i.e., the vector $\boldsymbol{b}$ contains the B-spline coefficients of f in $\mathbb{S}_{d,\boldsymbol{t}}$. Knot insertion is therefore nothing but a change of basis from the B-spline basis in $\mathbb{S}_{d,\boldsymbol{\tau}}$ to the B-spline basis in $\mathbb{S}_{d,\boldsymbol{t}}$.

Since $\mathbb{S}_{d,\boldsymbol{\tau}} \subseteq \mathbb{S}_{d,\boldsymbol{t}}$, all the B-splines in $\mathbb{S}_{d,\boldsymbol{\tau}}$ are also in $\mathbb{S}_{d,\boldsymbol{t}}$. We can therefore write

$$B_{j,d,\boldsymbol{\tau}} = \sum_{i=1}^{m} \alpha_{j,d}(i) B_{i,d,\boldsymbol{t}}, \quad j = 1, 2, \ldots, n, \tag{4.1}$$

for certain numbers $\alpha_{j,d}(i)$. In the matrix form we have used earlier this can be written

$$\boldsymbol{B}_{\boldsymbol{\tau}}^T = \boldsymbol{B}_{\boldsymbol{t}}^T \boldsymbol{A}, \tag{4.2}$$

where $\boldsymbol{B}_{\boldsymbol{\tau}}^T = (B_{1,d,\boldsymbol{\tau}}, \ldots, B_{n,d,\boldsymbol{\tau}})$ and $\boldsymbol{B}_{\boldsymbol{t}}^T = (B_{1,d,\boldsymbol{t}}, \ldots, B_{m,d,\boldsymbol{t}})$ are row vectors, and the $m \times n$-matrix $\boldsymbol{A} = \big(\alpha_{j,d}(i)\big)$ is the basis transformation matrix. Using this notation and remembering equation (4.2), we can write f in the form

$$f = \boldsymbol{B}_{\boldsymbol{t}}^T \boldsymbol{b} = \boldsymbol{B}_{\boldsymbol{\tau}}^T \boldsymbol{c} = \boldsymbol{B}_{\boldsymbol{\tau}}^T \boldsymbol{A} \boldsymbol{c}.$$

The linear independence of the B-splines in $\mathbb{S}_{d,\boldsymbol{\tau}}$ therefore means that $\boldsymbol{b}$ and $\boldsymbol{c}$ must be related by

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{c}, \qquad \text{or} \qquad b_i = \sum_{j=1}^{n} a_{i,j} c_j \quad \text{for } i = 1,\, 2,\, \ldots,\, m. \tag{4.3}$$

The basis transformation $\boldsymbol{A}$ is called *the knot insertion matrix of degree d from $\boldsymbol{\tau}$ to $\boldsymbol{t}$* and we will use the notation $\alpha_{j,d}(i) = \alpha_{j,d,\boldsymbol{\tau},\boldsymbol{t}}(i)$ for its entries. The discrete function $\alpha_{j,d}$ has many properties similar to those of $B_{j,d}$, and it is therefore called a *discrete B-spline on $\boldsymbol{t}$ with knots $\boldsymbol{\tau}$*.

To illustrate these ideas, let us consider a couple of simple examples of knot insertion for splines.

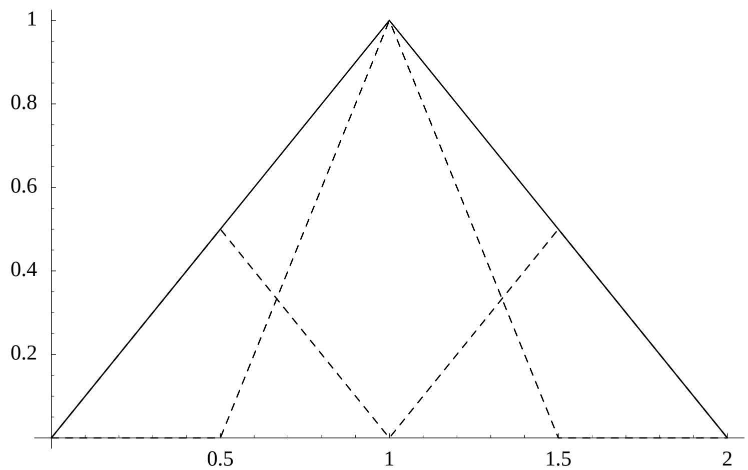


Figure 4.4. Refining a linear B-spline.

Example 4.3. Let us determine the transformation matrix $\boldsymbol{A}$ for splines with $d = 0$, when the coarse knot vector is given by $\boldsymbol{\tau} = (0, 1, 2)$, and the refined knot vector is $\boldsymbol{t} = (0, 1/2, 1, 3/2, 2) = (t_i)_{i=1}^5$. In this case

$$\mathbb{S}_{d,\boldsymbol{\tau}} = \text{span}\{B_{1,0,\boldsymbol{\tau}}, B_{2,0,\boldsymbol{\tau}}\} \quad \text{and} \quad \mathbb{S}_{d,\boldsymbol{t}} = \text{span}\{B_{1,0,\boldsymbol{t}}, B_{2,0,\boldsymbol{t}}, B_{3,0,\boldsymbol{t}}, B_{4,0,\boldsymbol{t}}\}.$$

We clearly have

$$B_{1,0,\boldsymbol{\tau}} = B_{1,0,\boldsymbol{t}} + B_{2,0,\boldsymbol{t}}, \quad B_{2,0,\boldsymbol{\tau}} = B_{3,0,\boldsymbol{t}} + B_{4,0,\boldsymbol{t}}.$$

This means that the knot insertion matrix in this case is given by

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Example 4.4. Let us also consider an example with linear splines. Let $d = 1$, and let $\boldsymbol{\tau}$ and $\boldsymbol{t}$ be as in the preceding example. In this case $\dim \mathbb{S}_{d,\boldsymbol{\tau}} = 1$ and we find that

$$B(x \mid 0, 1, 2) = \frac{1}{2} B(x \mid 0, 1/2, 1) + B(x \mid 1/2, 1, 3/2) + \frac{1}{2} B(x \mid 1, 3/2, 2).$$

The situation is shown in Figure 4.4. The linear B-spline on $\boldsymbol{\tau}$ is a weighted sum of the three B-splines (dashed) on $\boldsymbol{t}$. The knot insertion matrix $\boldsymbol{A}$ is therefore the 3×1-matrix, or row vector, given by

$$\boldsymbol{A} = \begin{pmatrix} 1/2 \\ 1 \\ 1/2 \end{pmatrix}.$$

4.2.2 Conversion between B-spline polynomials

We would obviously like to compute the B-spline coefficients on a refined knot vector by computer and therefore need a detailed and efficient algorithm. For this we need to study the matrix $\boldsymbol{A}$ in (4.2) and (4.3) in some more detail. We are going to use the strategy of considering what happens on individual knot intervals, which has proved successful in earlier chapters.

It will be helpful to specialise the linear algebra that led us to the two relations (4.2) and (4.3) to the space π_d of polynomials of degree d. Suppose we have two bases $\boldsymbol{p}^T =$

$(p_0, \ldots, p_d)$ and $\boldsymbol{q}^T = (q_0, \ldots, q_d)$ of π_d. We then know that there exists a nonsingular matrix $\boldsymbol{M}$ of dimension $d+1$ such that

$$\boldsymbol{p}^T = \boldsymbol{q}^T \boldsymbol{M}. \tag{4.4}$$

Let f be a polynomial of degree d with coefficients $\boldsymbol{b}$ relative to $\boldsymbol{p}$ and $\boldsymbol{c}$ relative to $\boldsymbol{q}$. Since $f = \boldsymbol{p}^T\boldsymbol{b} = \boldsymbol{q}^T\boldsymbol{M}\boldsymbol{b} = \boldsymbol{q}^T\boldsymbol{c}$ it follows that

$$\boldsymbol{c} = \boldsymbol{M}\boldsymbol{b}. \tag{4.5}$$

Conversely, it is not difficult to see that if the representations of any polynomial in two bases are related as in (4.5), then the bases must be related as in (4.4).

We are specially interested in polynomial bases obtained from B-splines. If $\boldsymbol{u} = (u_i)_{i=1}^{2d+2}$ is a knot vector with $u_{d+1} < u_{d+2}$, the theory in Chapter 3 shows that the corresponding B-splines form a basis for π_d on the interval $[u_{d+1}, u_{d+2}]$. On this interval the B-splines reduce to polynomials and therefore correspond to a polynomial basis $\boldsymbol{B}_{\boldsymbol{u}}$. And as all polynomials, these basis polynomials are defined on the whole real line (they can be computed for any x by always using $\mu = d+1$ in the spline evaluation algorithms in Chapter 2).

Suppose now that we have another knot vector $\boldsymbol{v} = (v_i)_{i=1}^{2d+2}$ with no relation to $\boldsymbol{u}$. This will give rise to a similar polynomial basis $\boldsymbol{B}_v$, and these two bases must be related by some matrix $\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}$,

$$\boldsymbol{B}_{\boldsymbol{v}}^T = \boldsymbol{B}_{\boldsymbol{u}}^T \boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}.$$

We want to find a formula for $\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}$ and to do this we consider the representation of the polynomial $f(x) = (y-x)^d$ where y is any real number. We know from Marsden's identity (Theorem 3.4) that the coefficients of f relative to the basis $\boldsymbol{B}_{\boldsymbol{u}}$ are the dual polynomials $\boldsymbol{\rho}_{\boldsymbol{u}} = \{\rho_{i,u}\}_{i=1}^{d+1}$ where $\rho_{i,\boldsymbol{u}}(y)$ is given by

$$\rho_{i,\boldsymbol{u}}(y) = (y - u_{i+1}) \cdots (y - u_{i+d}).$$

The B-spline coefficients of f relative to $\boldsymbol{B}_{\boldsymbol{v}}$ are given similarly by $\boldsymbol{\rho}_{\boldsymbol{v}}$, and the general discussion above shows that the two sets of coefficients must be related by the matrix $\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}$, as in (4.5),

$$\boldsymbol{\rho}_{\boldsymbol{v}}(y) = \boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}} \boldsymbol{\rho}_{\boldsymbol{u}}(y).$$

The ith component of this equation is

$$\rho_{i,\boldsymbol{v}}(y) = \big(\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}\big)_i \boldsymbol{\rho}_{\boldsymbol{u}}(y).$$

On the other hand we also know from Corollary 3.2 that

$$\rho_{i,\boldsymbol{v}}(y) = (y - v_{i+1}) \cdots (y - v_{i+d}) = \boldsymbol{R}_1(v_{i+1})\boldsymbol{R}_2(v_{i+2}) \cdots \boldsymbol{R}_d(v_{i+d})\boldsymbol{\rho}_{\boldsymbol{u}}(y),$$

where the matrices $\boldsymbol{R}_1, \ldots, \boldsymbol{R}_d$ are the bidiagonal B-spline matrices given in Theorem 2.18,

$$\boldsymbol{R}_k(x) = \boldsymbol{R}_{k,\boldsymbol{u}}^{d+1}(x) = \begin{pmatrix} \dfrac{u_{d+2} - x}{u_{d+2} - u_{d+2-k}} & \dfrac{x - u_{d+2-k}}{u_{d+2} - u_{d+2-k}} & & \\ & \ddots & \ddots & \\ & & \dfrac{u_{d+1+k} - x}{u_{d+1+k} - u_{d+1}} & \dfrac{x - u_{d+1}}{u_{d+1+k} - u_{d+1}} \end{pmatrix}.$$

Since the dual polynomials $\{\rho_{i,\boldsymbol{u}}\}_{i=1}^{d+1}$ are linearly indenpendent we therefore have

$$\big(\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}\big)_i = \boldsymbol{R}_1(v_{i+1})\boldsymbol{R}_2(v_{i+2})\cdots\boldsymbol{R}_d(v_{i+d}).$$

Let us sum up our findings so far.

Proposition 4.5. *Let $\boldsymbol{u} = (u_i)_{i=1}^{2d+2}$ and $\boldsymbol{v} = (v_i)_{i=1}^{2d+2}$ be two knot vectors with $u_{d+1} < u_{d+2}$ and $v_{d+1} < v_{d+2}$, and let $\boldsymbol{B}_{\boldsymbol{u}}$ and $\boldsymbol{B}_{\boldsymbol{v}}$ be the corresponding B-spline polynomials on the intervals $[u_{d+1}, u_{d+2}]$ and $[v_{d+1}, v_{d+2}]$ respectively. Then the two polynomial bases are related by*

$$\boldsymbol{B}_{\boldsymbol{v}}^T = \boldsymbol{B}_{\boldsymbol{u}}^T\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}} \tag{4.6}$$

where $\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}$ is a square matrix of dimension $d+1$ with rows given by

$$\big(\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}\big)_i = \boldsymbol{R}_1(v_{i+1})\boldsymbol{R}_2(v_{i+2})\cdots\boldsymbol{R}_d(v_{i+d}) \tag{4.7}$$

for $i = 1, \ldots, d+1$. Here $\boldsymbol{R}_k(x) = \boldsymbol{R}_{k,\boldsymbol{u}}^{d+1}$ for $k = 1, \ldots, d$ are the B-spline matrices of the interval $[\boldsymbol{\tau}_{d+1}, \boldsymbol{\tau}_{d+2}]$ defined in Theorem 2.18.

Although the expression (4.7) is slightly more complicated than what we encountered when developing algorithms for computing the value of splines and B-splines, those algorithms can easily be adapted to computing the matrix $\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}$ or converting from the representation in terms of $\boldsymbol{B}_{\boldsymbol{u}}$ to a representation in terms of $\boldsymbol{B}_{\boldsymbol{v}}$, see Algorithms 4.10 and 4.11 below. Note also that because of the symmetry in the construction, it is easy to find the inverse of the matrix $\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}$,

$$\boldsymbol{M}_{\boldsymbol{u},\boldsymbol{v}}^{-1} = \boldsymbol{M}_{\boldsymbol{v},\boldsymbol{u}},$$

i.e., just reverse the roles of $\boldsymbol{u}$ and $\boldsymbol{v}$.

4.2.3 Formulas and algorithms for knot insertion

We have seen how we can find formulas for conversion between two completely unrelated B-spline bases. Let us now apply this to the special situation of knot insertion.

Suppose as before that we have two knot vectors $\boldsymbol{\tau}$ and $\boldsymbol{t}$ with $\boldsymbol{\tau} \subseteq \boldsymbol{t}$ and a spline function $f = \sum_j c_j B_{j,d,\boldsymbol{\tau}} = \sum_i b_i B_{i,d,\boldsymbol{t}}$ which lies in $\mathbb{S}_{d,\boldsymbol{\tau}}$ and therefore also in $\mathbb{S}_{d,\boldsymbol{t}}$. Recall from (4.1) and (4.2) that the two spaces are related by the basis transformation matrix $\boldsymbol{A}$ whose (i,j)-entry we denote $\alpha_{j,d}(i)$. In other words we have

$$b_i = \sum_{j=1}^{n} \alpha_{j,d}(i)c_j \tag{4.8}$$

for $i = 1, \ldots, m$, and

$$B_{j,d,\boldsymbol{\tau}} = \sum_{i=1}^{m} \alpha_{j,d}(i)B_{i,d,\boldsymbol{t}} \tag{4.9}$$

for $j = 1, \ldots, n$. An important observation here is that a B-spline will usually consist of several polynomial pieces and according to (4.9), all the pieces of a B-spline in $\mathbb{S}_{d,\boldsymbol{\tau}}$ must be expressible as the same linear combination of the corresponding pieces of the B-splines in $\mathbb{S}_{d,\boldsymbol{t}}$. An example should help to clarify this.

Example 4.6. Suppose that $d = 3$ and that the knot vector $\boldsymbol{\tau} = (0,0,0,0,1,4,4,4,4)$ has been refined to $\boldsymbol{t} = (0,0,0,0,1,2,3,4,4,4,4)$. In $\mathbb{S}_{3,\boldsymbol{\tau}}$ we then have the five B-splines $\{B_{j,\boldsymbol{\tau}}\}_{j=1}^{5}$ and in $\mathbb{S}_{3,\boldsymbol{t}}$ we have seven B-splines $\{B_{i,\boldsymbol{t}}\}_{i=1}^{7}$ (we have dropped the degree from the notation as it will remain fixed in this example). Relation (4.9) therefore becomes

$$B_{j,\boldsymbol{\tau}} = \sum_{i=1}^{7} \alpha_j(i) B_{i,\boldsymbol{t}} \tag{4.10}$$

for $j = 1, \ldots, 5$. What does this really mean? It does of course mean that the B-splines in $\mathbb{S}_{3,\boldsymbol{\tau}}$ are linear combinations of the B-splines in $\mathbb{S}_{3,\boldsymbol{t}}$. But a consequence of this is that each polynomial piece of $B_{j,\boldsymbol{\tau}}$ can be written as a linear combination of the corresponding pieces of the B-splines in $\mathbb{S}_{3,\boldsymbol{t}}$.

Let us be more specific. The interval of interest is $[0,4]$ and a B-spline $B_{j,\boldsymbol{\tau}}$ in $\mathbb{S}_{3,\boldsymbol{\tau}}$ consists of two polynomial pieces within this interval, one piece on $[\tau_4,\tau_5] = [0,1]$ which we denote $B_{j,\boldsymbol{\tau}}^4$ and one piece on $[\tau_5,\tau_6] = [1,4]$ which we denote $B_{j,\boldsymbol{\tau}}^5$. Similarly, a B-spline $B_{i,\boldsymbol{t}}$ in $\mathbb{S}_{3,\boldsymbol{t}}$ consists of four polynomial pieces which we denote $B_{i,\boldsymbol{t}}^4$, $B_{i,\boldsymbol{t}}^5$, $B_{i,\boldsymbol{t}}^6$ and $B_{i,\boldsymbol{t}}^7$. With this notation, we can elaborate more on the meaning of relation (4.10).

If we restrict x to the interval $[0,1]$ we can write (4.10) as

$$B_{j,\boldsymbol{\tau}}^4 = \sum_{i=1}^{4} \alpha_j(i) B_{i,\boldsymbol{t}}^4$$

for $j = 1, \ldots, 5$, since the other B-splines in $\mathbb{S}_{3,\boldsymbol{t}}$ vanish on this interval. If we ignore $B_{5,\boldsymbol{\tau}}$, this is just a relation between two polynomial bases on B-spline form for the interval $[\tau_4,\tau_5]$, so we can use Proposition 4.5 to determine the coefficients $\big(\alpha_j(i)\big)_{i,j=1}^{4}$. We find that

$$\begin{pmatrix} \alpha_1(1) & \alpha_2(1) & \alpha_3(1) & \alpha_4(1) \\ \alpha_1(2) & \alpha_2(2) & \alpha_3(2) & \alpha_4(2) \\ \alpha_1(3) & \alpha_2(3) & \alpha_3(3) & \alpha_4(3) \\ \alpha_1(4) & \alpha_2(4) & \alpha_3(4) & \alpha_4(4) \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_1^4(t_2)\boldsymbol{R}_2^4(t_3)\boldsymbol{R}_3^4(t_4) \\ \boldsymbol{R}_1^4(t_3)\boldsymbol{R}_2^4(t_4)\boldsymbol{R}_3^4(t_5) \\ \boldsymbol{R}_1^4(t_4)\boldsymbol{R}_2^4(t_5)\boldsymbol{R}_3^4(t_6) \\ \boldsymbol{R}_1^4(t_5)\boldsymbol{R}_2^4(t_6)\boldsymbol{R}_3^4(t_7) \end{pmatrix}$$

where $\boldsymbol{R}_k^4 = \boldsymbol{R}_{k,\boldsymbol{\tau}}^4(x)$ for $k = 1, 2, 3$ are B-spline matrices for the interval $[\tau_4,\tau_5]$. We can also determine $\big(\alpha_5(i)\big)_{i=1}^{4}$ since $B_{5,\boldsymbol{\tau}}^4$ is identically zero. In fact the linear independence of the polynomials $\{B_{i,\boldsymbol{t}}^4\}_{i=1}^4$ on $[0,1]$ means that $\alpha_5(i) = 0$ for $i = 1, 2, 3, 4$.

If we move to the right, the next subinterval of $\boldsymbol{\tau}$ is $[\tau_5,\tau_6] = [1,4]$ while the next subinterval of $\boldsymbol{t}$ is $[t_5,t_6] = [1,2]$. On the smallest common subinterval $[1,2]$ relation (4.10) reduces to

$$B_{j,\boldsymbol{\tau}}^5 = \sum_{i=2}^{5} \alpha_j(i) B_{i,\boldsymbol{t}}^5$$

for $j = 1, \ldots, 5$. Similarly to the previous subinterval we can conclude that $\big(\alpha_1(i)\big)_{i=2}^{5}$ is zero since $B_{1,\boldsymbol{\tau}}^5$ is identically zero on this interval. The remaining $\alpha_j(i)$s involved in the sum can be determined from Proposition 4.5,

$$\begin{pmatrix} \alpha_2(2) & \alpha_3(2) & \alpha_4(2) & \alpha_5(2) \\ \alpha_2(3) & \alpha_3(3) & \alpha_4(3) & \alpha_5(3) \\ \alpha_2(4) & \alpha_3(4) & \alpha_4(4) & \alpha_5(4) \\ \alpha_2(5) & \alpha_3(5) & \alpha_4(5) & \alpha_5(5) \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_1^5(t_3)\boldsymbol{R}_2^5(t_4)\boldsymbol{R}_3^5(t_5) \\ \boldsymbol{R}_1^5(t_4)\boldsymbol{R}_2^5(t_5)\boldsymbol{R}_3^5(t_6) \\ \boldsymbol{R}_1^5(t_5)\boldsymbol{R}_2^5(t_6)\boldsymbol{R}_3^5(t_7) \\ \boldsymbol{R}_1^5(t_6)\boldsymbol{R}_2^5(t_7)\boldsymbol{R}_3^5(t_8) \end{pmatrix}.$$

If we move further to the right we come to the interval $[t_6,t_7] = [2,3]$ which is a subinterval of $[\tau_5,\tau_6] = [1,4]$. Relation (4.10) now becomes

$$B_{j,\boldsymbol{\tau}}^5 = \sum_{i=3}^{6} \alpha_j(i) B_{i,\boldsymbol{t}}^6$$

for $j = 1, \ldots, 5$. Again we can conclude that $\alpha_1(i) = 0$ for $i = 3, \ldots, 6$ while

$$\begin{pmatrix} \alpha_2(3) & \alpha_3(3) & \alpha_4(3) & \alpha_5(3) \\ \alpha_2(4) & \alpha_3(4) & \alpha_4(4) & \alpha_5(4) \\ \alpha_2(5) & \alpha_3(5) & \alpha_4(5) & \alpha_5(5) \\ \alpha_2(6) & \alpha_3(6) & \alpha_4(6) & \alpha_5(6) \end{pmatrix} = \begin{pmatrix} \boldsymbol{R}_1^5(t_4)\boldsymbol{R}_2^5(t_5)\boldsymbol{R}_3^5(t_6) \\ \boldsymbol{R}_1^5(t_5)\boldsymbol{R}_2^5(t_6)\boldsymbol{R}_3^5(t_7) \\ \boldsymbol{R}_1^5(t_6)\boldsymbol{R}_2^5(t_7)\boldsymbol{R}_3^5(t_8) \\ \boldsymbol{R}_1^5(t_7)\boldsymbol{R}_2^5(t_8)\boldsymbol{R}_3^5(t_9) \end{pmatrix}.$$

We can move one more interval to the left, to $[t_7, t_8] = [3, 4]$, which is also a subinterval of $[\tau_5, \tau_6] = [1, 4]$. On this interval we can conclude that $\alpha_1(i) = 0$ for $i = 4, \ldots, 7$ and determine the part of $\boldsymbol{A}$ given by $\big(\alpha_j(i)\big)_{i=4,j=2}^{7,5}$.

Note that many of the entries in the matrix $\boldsymbol{A}$ are determined several times in this example simply because a B-spline consists of several polynomial pieces. This is not really a problem as we will get the same value (up to round-off) each time.

Example 4.6 makes an important point clear: Since (4.9) is a relation between piecewise polynomials, the number $\alpha_j(i)$ must be the coefficient multiplying $B_{i,\boldsymbol{t}}$ in the representation of $B_{j,\boldsymbol{\tau}}$, irrespective of which polynomial piece we consider. Therefore, by considering relation (4.9) as a relation between polynomials on different intervals we get several possibilities for determining most entries in the matrix $\boldsymbol{A}$. This leaves us with the question of which polynomial pieces we should use to determine a certain entry in $\boldsymbol{A}$. Theorem 4.7 uses a standard choice, but it is worth remembering that other choices are possible.

For simplicity we will make the assumption that $\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+d+1}$ and $\boldsymbol{t} = (t_i)_{i=1}^{m+d+1}$ are both $d+1$-regular knot vectors with $d+1$ common knots at the two ends. Exercise 6 shows that this causes no loss of generality. The technique in Example 4.6 works in general and can be used to obtain an explicit formula for the knot insertion matrix $\boldsymbol{A}$.

Theorem 4.7. *Let the polynomial degree d be given, and let $\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+d+1}$ and $\boldsymbol{t} = (t_i)_{i=1}^{m+d+1}$ be two $d+1$-regular knot vectors with common knots at the ends and $\boldsymbol{\tau} \subseteq \boldsymbol{t}$. In row i of the knot insertion matrix $\boldsymbol{A}$ the entries are given by $\alpha_{j,d}(i) = 0$ for $j < \mu - d$ and $j > \mu$, where μ is determined by $\tau_\mu \le t_i < \tau_{\mu+1}$ and*

$$\boldsymbol{\alpha}_d(i)^T = \big(\alpha_{\mu-d,d}(i), \ldots, \alpha_{\mu,d}(i)\big) = \begin{cases} 1, & \text{if } d = 0, \\ \boldsymbol{R}_{1,\boldsymbol{\tau}}^{\mu}(t_{i+1}) \cdots \boldsymbol{R}_{d,\boldsymbol{\tau}}^{\mu}(t_{i+d}), & \text{if } d > 0, \end{cases} \tag{4.11}$$

and the matrix $\boldsymbol{R}_{k,\boldsymbol{\tau}}^{\mu}$ is defined in Theorem 2.18. If $f = \sum_j c_j B_{j,d,\boldsymbol{\tau}}$ is a spline in $\mathbb{S}_{d,\boldsymbol{\tau}}$, with B-spline coefficients $\boldsymbol{b}$ in $\mathbb{S}_{d,\boldsymbol{t}}$, then b_i is given by

$$b_i = \sum_{j=\mu-d}^{\mu} \alpha_{j,d}(i) c_j = \boldsymbol{R}_{1,\boldsymbol{\tau}}^{\mu}(t_{i+1}) \cdots \boldsymbol{R}_{d,\boldsymbol{\tau}}^{\mu}(t_{i+d}) \boldsymbol{c}_d, \tag{4.12}$$

where $\boldsymbol{c}_d = (c_{\mu-d}, \ldots, c_\mu)$.

Proof. We note that (4.12) follows from the general discussion earlier in this chapter so we focus on the proof of (4.11). For degree $d = 0$ this is easy so we focus on the general case. We fix the integer i and are going to show how row no. i of $\boldsymbol{A}$ can be determined. Row i consists of the numbers $\big(\alpha_j(i)\big)_{j=1}^{n}$ where $\alpha_j(i)$ gives the coefficient of $B_{i,\boldsymbol{t}}$ in the linear combination of the B-splines in $\mathbb{S}_{d,\boldsymbol{t}}$ that make up $B_{j,\boldsymbol{\tau}}$, see (4.9). We will deduce (4.11) by considering different polynomial pieces of the B-splines that are involved. Let μ

be as stated in the theorem, and let ν be the largest integer such that $t_\nu = t_i$. We then have the two bases of B-spline polynomials,

$$\boldsymbol{B}^\mu_{\boldsymbol{\tau}} = (B^\mu_{\mu-d,\boldsymbol{\tau}}, \dots, B^\mu_{\mu,d})^T,$$
$$\boldsymbol{B}^\nu_{\boldsymbol{t}} = (B^\nu_{\nu-d,\boldsymbol{t}}, \dots, B^\nu_{\nu,\boldsymbol{t}})^T.$$

The first basis consists of the polynomial pieces of the nonzero B-splines in $\mathbb{S}_{d,\boldsymbol{\tau}}$ on the interval $[\tau_\mu, \tau_{\mu+1}]$ and the other consists of the polynomial pieces of the nonzero B-splines in $\mathbb{S}_{d,\boldsymbol{t}}$ on the interval $[t_\nu, t_{\nu+1}]$. Note that the definition of ν means that $B^\nu_{i,\boldsymbol{t}}$ is one of the B-spline polynomials in $\boldsymbol{B}^\nu_{\boldsymbol{t}}$. From Proposition 4.5 we know that these two bases are related by a $(d+1)\times(d+1)$-matrix $\boldsymbol{M}_{\boldsymbol{\tau},\boldsymbol{t}}$. Each row of this matrix is associated with one of the B-spline polynomials in the basis $\boldsymbol{B}^\nu_{\boldsymbol{t}}$ and the row associated with $B^\nu_{i,\boldsymbol{t}}$ is given by

$$\boldsymbol{R}^\mu_{1,\boldsymbol{\tau}}(t_{i+1}) \cdots \boldsymbol{R}^\mu_{d,\boldsymbol{\tau}}(t_{i+d}).$$

On other hand, we also know that the matrix $\boldsymbol{M}_{\boldsymbol{\tau},\boldsymbol{t}}$ is a submatrix of the knot insertion matrix $\boldsymbol{A}$,

$$\boldsymbol{M}_{\boldsymbol{\tau},\boldsymbol{t}} = \big(\alpha_j(\ell)\big)^{\mu,\nu}_{j=\mu-d,\,\ell=\nu-d},$$

since the two bases $\boldsymbol{B}^\mu_{\boldsymbol{\tau}}$ and $\boldsymbol{B}^\nu_{\boldsymbol{t}}$ are part of the two B-spline bases for $\mathbb{S}_{d,\boldsymbol{\tau}}$ and $\mathbb{S}_{d,\boldsymbol{t}}$. In particular we have

$$\big(\alpha_{\mu-d}(i), \dots, \alpha_\mu(i)\big) = \boldsymbol{R}^\mu_{1,\boldsymbol{\tau}}(t_{i+1}) \cdots \boldsymbol{R}^\mu_{d,\boldsymbol{\tau}}(t_{i+d}).$$

What remains is to prove that the other entries in row i of $\boldsymbol{A}$ are zero. Suppose that $j < \mu-d$. By the support properties of B-splines we must then have $B_{j,\boldsymbol{\tau}}(x) = B^\mu_{j,\boldsymbol{\tau}}(x) = 0$ for $x \in [t_\nu, t_{\nu+1}]$. When x varies in this interval we have

$$0 = B^\mu_{j,\boldsymbol{\tau}}(x) = \sum_{\ell=\nu-d}^{\nu} \alpha_j(\ell) B^\nu_{\ell,\boldsymbol{t}}(x).$$

From the linear independence of the B-spline polynomials $\{B_{\ell,\boldsymbol{t}}\}^\nu_{\ell=\nu-d}$ we can then conclude that $\alpha_j(\ell) = 0$ for $\ell = \nu-d, \dots, \nu$. In particular we have $\alpha_j(i) = 0$. The case $j > \mu$ is similar. ∎

Theorem 4.7 shows that the knot insertion matrix is banded: In any row, there are first some zeros, then some nonzero entries, and then more zeros. As we have already noted there are several possibilities when it comes to computing the nonzero entries since a B-spline consists of different polynomial pieces which are all transformed in the same way. In Theorem 4.7 we compute the nonzero entries in row i by considering the knot interval in $\boldsymbol{t}$ which has t_i as its left end and the knot interval in $\boldsymbol{\tau}$ whose left end is closest to t_i. In general, there are many other possibilities. With i given, we could for example choose μ by requiring that $\tau_{\mu+d} \le t_{i+d+1} < \tau_{\mu+d+1}$.

It should be noted that, in general, not all the $d+1$ entries of row i of $\boldsymbol{A}$ given by (4.11) will be nonzero. It is in fact quite easy to see that $\alpha_j(i)$ will only be nonzero if the whole support of $B_{i,\boldsymbol{t}}$ is a subset of the support of $B_{j,\boldsymbol{\tau}}$. More specifically, it can be shown that if there are r new knots among $t_{i+1}, \dots, t_{i+d}$ then there will be $r+1$ nonzero entries in row i of $\boldsymbol{A}$.

Note that if no new knots are inserted ($\boldsymbol{\tau} = \boldsymbol{t}$) then the two sets of B-spline coefficients $\boldsymbol{c}$ and $\boldsymbol{b}$ are obviously the same. Equation (4.12) then shows that

$$c_i = \boldsymbol{R}^{\mu}_{1,\boldsymbol{\tau}}(\tau_{i+1}) \cdots \boldsymbol{R}^{\mu}_{d,\boldsymbol{\tau}}(\tau_{i+d})\boldsymbol{c}_d. \tag{4.13}$$

This simple observation will be useful later.

A couple of examples will illustrate the use of Theorem 4.7.

Example 4.8. We consider quadratic splines ($d = 2$) on the knot vector $\boldsymbol{\tau} = (-1,-1,-1,0,1,1,1)$, and insert two new knots, at $-1/2$ and $1/2$ so $\boldsymbol{t} = (-1,-1,-1,-1/2,0,1/2,1,1,1)$. We note that $\tau_3 \le t_i < \tau_4$ for $1 \le i \le 4$ so the first three entries of the first four rows of the 6×4 knot insertion matrix $\boldsymbol{A}$ are given by

$$\boldsymbol{\alpha}_2(i) = \boldsymbol{R}^3_{1,\boldsymbol{\tau}}(t_{i+1})\boldsymbol{R}^3_{2,\boldsymbol{\tau}}(t_{i+2})$$

for $i = 1, \ldots, 4$. Since

$$\boldsymbol{R}^3_{1,\boldsymbol{\tau}}(x) = \begin{pmatrix} -x & 1+x \end{pmatrix}, \qquad \boldsymbol{R}^3_{2,\boldsymbol{\tau}}(x) = \begin{pmatrix} -x & 1+x & 0 \\ 0 & (1-x)/2 & (1+x)/2 \end{pmatrix},$$

we have from (4.11)

$$\boldsymbol{\alpha}_2(i) = \frac{1}{2}\begin{pmatrix} 2t_{i+1}t_{i+2}, & 1 - t_{i+1} - t_{i+2} - 3t_{i+1}t_{i+2}, & (1+t_{i+1})(1+t_{i+2}) \end{pmatrix}.$$

Inserting the correct values for t_{i+1} and t_{i+2} and adding one zero at the end of each row, we find that the first four rows of $\boldsymbol{A}$ are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 1/4 & 3/4 & 0 \end{pmatrix}.$$

To determine the remaining two rows of $\boldsymbol{A}$ we have to move to the interval $[\tau_4, \tau_5) = [0, 1)$. Here we have

$$\boldsymbol{R}^4_{1,\boldsymbol{\tau}}(x) = \begin{pmatrix} 1-x & x \end{pmatrix} \qquad \boldsymbol{R}^4_{2,\boldsymbol{\tau}}(x) = \begin{pmatrix} (1-x)/2 & (1+x)/2 & 0 \\ 0 & 1-x & x \end{pmatrix},$$

so

$$\boldsymbol{a}_2(i) = \boldsymbol{R}^4_{1,\boldsymbol{\tau}}(t_{i+1})\boldsymbol{R}^4_{2,\boldsymbol{\tau}}(t_{i+2}) = \frac{1}{2}\begin{pmatrix} (1-t_{i+1})(1-t_{i+2}), & 1 + t_{i+1} + t_{i+2} - 3t_{i+1}t_{i+2}, & 2t_{i+1}t_{i+2} \end{pmatrix}.$$

Evaluating this for $i = 5, 6$ and inserting one zero as the first entry, we obtain the last two rows as

$$\begin{pmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To see visually the effect of knot insertion, let $f = B_{1,2,\boldsymbol{\tau}} - 2B_{2,2,\boldsymbol{\tau}} + 2B_{3,2,\boldsymbol{\tau}} - B_{4,2,\boldsymbol{\tau}}$ be a spline in $\mathbb{S}_{d,\boldsymbol{\tau}}$ with B-spline coefficients $\boldsymbol{c} = (1,-2,2,-1)^T$. Its coefficients $\boldsymbol{b} = (b_i)_{i=1}^6$ are then given by

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 3/4 & 1/4 & 0 \\ 0 & 1/4 & 3/4 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ -1 \\ 1 \\ 1/2 \\ -1 \end{pmatrix}.$$

Figure 4.5 (a) shows a plot of f together with its control polygons relative to $\boldsymbol{\tau}$ and $\boldsymbol{t}$. We note that the control polygon relative to $\boldsymbol{t}$ is much closer to f and that both control polygons give a rough estimate of f.

The knot insertion process can be continued. If we insert one new knot halfway between each old knot in $\boldsymbol{t}$, we obtain the new knot vector

$$\boldsymbol{t}^1 = (-1,-1,-1,-3/4,-1/2,-1/4,0,1/4,1/2,3/4,1,1,1).$$

A plot of f and its control polygon relative to this knot vector is shown in Figure 4.5 (b).

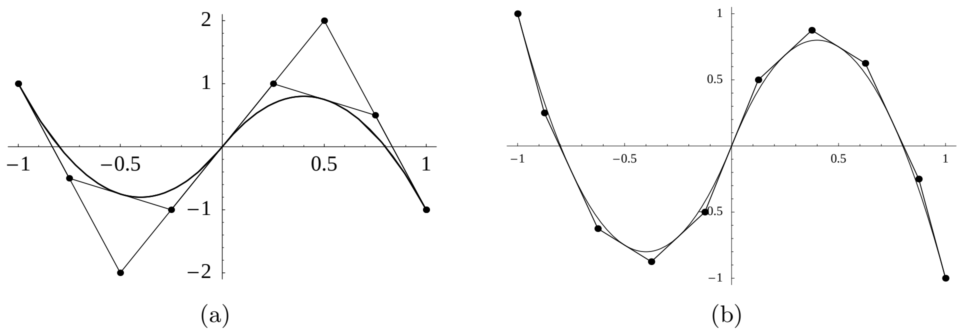


Figure 4.5. A quadratic spline together with its control polygon relative to a coarse and a finer knot vector (a), and the same spline as in (a) with its control polygon relative to an even more refined knot vector (b).

Example 4.9. Let us again consider quadratic splines on a uniform knot vector with multiple knots at the ends,

$$\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+3} = (3, 3, 3, 4, 5, 6, \ldots, n, n+1, n+1, n+1),$$

and form $\boldsymbol{t}$ by inserting one knot half way between each pair of old knots,

$$\boldsymbol{t} = (t_i)_{i=1}^{2n+1} = (3, 3, 3, 7/2, 4, 9/2, 5, \ldots, n, (2n+1)/2, n+1, n+1, n+1).$$

Since $\dim \mathbb{S}_{d,\boldsymbol{\tau}} = n$ and $\dim \mathbb{S}_{d,\boldsymbol{t}} = 2n-2$, the knot insertion matrix $\boldsymbol{A}$ is now a $(2n-2) \times n$ matrix. As in Example 4.8 we find that the first three columns of the first four rows of $\boldsymbol{A}$ are

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 3/4 & 1/4 \\ 0 & 1/4 & 3/4 \end{pmatrix}.$$

To determine rows $2\mu-3$ and $2\mu-2$ with $4 \le \mu \le n-1$, we need the matrices $\boldsymbol{R}^{\mu}_{1,\boldsymbol{\tau}}$ and $\boldsymbol{R}^{\mu}_{2,\boldsymbol{\tau}}$ which are given by

$$\boldsymbol{R}^{\mu}_{1,\boldsymbol{\tau}}(x) = \begin{pmatrix} \mu+1-x & x-\mu \end{pmatrix}, \qquad \boldsymbol{R}^{\mu}_{2,\boldsymbol{\tau}}(x) = \begin{pmatrix} (\mu+1-x)/2 & (x+1-\mu)/2 & 0 \\ 0 & (\mu+2-x)/2 & (x-\mu)/2 \end{pmatrix}.$$

Observe that $\tau_i = i$ for $i = 3, \ldots, n+1$ and $t_i = (i+3)/2$ for $i = 3, \ldots, 2n-1$. Entries $\mu-2$, $\mu-1$ and μ of row $2\mu-3$ are therefore given by

$$\boldsymbol{R}^{\mu}_{1,\boldsymbol{\tau}}(t_{2\mu-2})\boldsymbol{R}^{\mu}_{2,\boldsymbol{\tau}}(t_{2\mu-1}) = \boldsymbol{R}^{\mu}_{1,\boldsymbol{\tau}}(\mu+1/2)\boldsymbol{R}^{\mu}_{2,\boldsymbol{\tau}}(\mu+1) = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 0 & 3/4 & 1/4 \end{pmatrix}.$$

Similarly, entries $\mu-3$, $\mu-2$ and μ of row $2\mu-2$ are given by

$$\boldsymbol{R}^{\mu}_{1,\boldsymbol{\tau}}(t_{2\mu-1})\boldsymbol{R}^{\mu}_{2,\boldsymbol{\tau}}(t_{2\mu}) = \boldsymbol{R}^{\mu}_{1,\boldsymbol{\tau}}(\mu+1)\boldsymbol{R}^{\mu}_{2,\boldsymbol{\tau}}(\mu+3/2) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} -1/4 & 5/4 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} = \begin{pmatrix} 0 & 1/4 & 3/4 \end{pmatrix}.$$

Finally, we find as in Example 4.8 that the last three entries of the last two rows are

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The complete knot insertion matrix is therefore

$$
\boldsymbol{A} = \begin{pmatrix}
1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 3/4 & 1/4 & 0 & \dots & 0 & 0 & 0 \\
0 & 1/4 & 3/4 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 3/4 & 1/4 & \dots & 0 & 0 & 0 \\
0 & 0 & 1/4 & 3/4 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \dots & 3/4 & 1/4 & 0 \\
0 & 0 & 0 & 0 & \dots & 1/4 & 3/4 & 0 \\
0 & 0 & 0 & 0 & \dots & 0 & 1/2 & 1/2 \\
0 & 0 & 0 & 0 & \dots & 0 & 0 & 1
\end{pmatrix}.
$$

The formula for $\boldsymbol{\alpha}_d(i)$ shows very clearly the close relationship between B-splines and discrete B-splines, and it will come as no surprise that $\alpha_{j,d}(i)$ satisfies a recurrence relation similar to that of B-splines, see Definition 2.1. The recurrence for $\alpha_{j,d}(i)$ is obtained by setting $x = t_{i+d}$ in the recurrence (2.1) for $B_{j,d}(x)$,

$$
\alpha_{j,d}(i) = \frac{t_{i+d} - \tau_j}{\tau_{j+d} - \tau_j}\alpha_{j,d-1}(i) + \frac{\tau_{j+1+d} - t_{i+d}}{\tau_{j+1+d} - \tau_{j+1}}\alpha_{j+1,d-1}(i), \tag{4.14}
$$

starting with $\alpha_{j,0}(i) = B_{j,0}(t_i)$.

The two evaluation algorithms for splines, Algorithms 3.17 and 3.18, can be adapted to knot insertion quite easily. For historical reasons these algorithms are usually referred to as *the Oslo algorithms*.

Algorithm 4.10 (Oslo-Algorithm 1)**.** *Let the polynomial degree d, and the two $d+1$-regular knot vectors $\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+d+1}$ and $\boldsymbol{t} = (t_i)_{i=1}^{m+d+1}$ with common knots at the ends be given. To compute the $m \times n$ knot insertion matrix $\boldsymbol{A} = \big(\alpha_{j,d}(i)\big)_{i,j=1}^{m,n}$ from $\boldsymbol{\tau}$ to $\boldsymbol{t}$ perform the following steps:*

1. *For $i = 1, \dots, m$.*

1.1 *Determine μ such that $\tau_\mu \le t_i < \tau_{\mu+1}$.*

1.2 *Compute entries $\mu - d, \dots, \mu$ of row i by evaluating*

$$
\boldsymbol{\alpha}_d(i)^T = \big(\alpha_{\mu-d,d}(i), \dots, \alpha_{\mu,d}(i)\big)^T = \begin{cases} 1, & \text{if } d = 0. \\ \boldsymbol{R}_1(t_{i+1}) \cdots \boldsymbol{R}_d(t_{i+d}), & \text{if } d > 0. \end{cases}
$$

All other entries in row i are zero.

An algorithm for converting a spline from a B-spline representation in $\mathbb{S}_{d,\boldsymbol{\tau}}$ to $\mathbb{S}_{d,\boldsymbol{t}}$ is as follows.

Algorithm 4.11 (Oslo-Algorithm 2)**.** *Let the polynomial degree d, and the two $d+1$-regular knot vectors $\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+d+1}$ and $\boldsymbol{t} = (t_i)_{i=1}^{m+d+1}$ with common knots at the ends be given together with the spline f in $\mathbb{S}_{d,\boldsymbol{\tau}}$ with B-spline coefficients $\boldsymbol{c} = (c_j)_{j=1}^{n}$. To compute the B-spline coefficients $\boldsymbol{b} = (b_i)_{i=1}^{m}$ of f in $\mathbb{S}_{d,\boldsymbol{t}}$ perform the following steps:*

1. *For $i = 1, \dots, m$.*

1.1 *Determine μ such that $\tau_\mu \le t_i < \tau_{\mu+1}$.*

1.2 *Set* $\boldsymbol{c}_d = (c_j)_{j=\mu-d}^{\mu}$ *and compute* b_i *by evaluating*

$$b_i = \begin{cases} c_\mu, & \text{if } d = 0. \\ \boldsymbol{R}_1(t_{i+1}) \cdots \boldsymbol{R}_d(t_{i+d})\boldsymbol{c}_d, & \text{if } d > 0. \end{cases}$$

4.3 B-spline coefficients as functions of the knots

Knot insertion allows us to represent the same spline function on different knot vectors. In fact, any spline function can be given any real numbers as knots, as long as we also include the original knots. It therefore makes sense to consider the B-spline coefficients as functions of the knots, and we shall see that this point of view allows us to characterise the B-spline coefficients completely by three simple properties.

Initially, we assume that the spline $f = \sum_{j=1}^{n} c_j B_{j,d,\boldsymbol{\tau}}$ is a polynomial represented on a $d+1$-extended knot vector $\boldsymbol{\tau}$. On the knot interval $[\tau_\mu, \tau_{\mu+1})$ we know that f can be written as

$$f(x) = \boldsymbol{R}_1(x) \cdots \boldsymbol{R}_d(x)\boldsymbol{c}_d, \tag{4.15}$$

where $\boldsymbol{c}_d = (c_{\mu-d}, \ldots, c_\mu)^T$, see Section 2.3. Since f is assumed to be a polynomial this representation is valid for all real numbers x, although when x is outside $[\tau_\mu, \tau_{\mu+1})$ it is no longer a true B-spline representation.

Consider the function

$$F(x_1, \ldots, x_d) = \boldsymbol{R}_1(x_1) \cdots \boldsymbol{R}_d(x_d)\boldsymbol{c}_d. \tag{4.16}$$

We recognise the right-hand side of this expression from equation (4.12) in Theorem 4.7: If we have a knot vector that includes the knots $(x_0, x_1, \ldots, x_d, x_{d+1})$, then $F(x_1, \ldots, x_d)$ gives the B-spline coefficient that multiplies the B-spline $B(x \mid x_0, \ldots, x_{d+1})$ in the representation of the polynomial f on the knot vector $\boldsymbol{x}$. When f is a polynomial, it turns out that the function F is completely independent of the knot vector $\boldsymbol{\tau}$ that underlie the definition of the $\boldsymbol{R}$-matrices in (4.16). The function F is referred to as the *blossom* of f, and the whole theory of splines can be built from properties of this function.

4.3.1 The blossom

In this subsection we develop some of the properties of the blossom. We will do this in an abstract fashion, by starting with a formal definition of the blossom. In the next subsection we will then show that the function F in (4.16) satisfies this definition.

Definition 4.12. *A function on the form $f(x) = ax$, where a is a real number, is called a linear function. A function on the form $f(x) = ax + b$ with a and b real constants is called an affine function. A function of d variables $f(x_1, \ldots, x_d)$ is said to be affine if it is affine viewed as a function of each x_i for $i = 1, \ldots, d$, with the other variables fixed. A symmetric affine function is an affine function that is not altered when the order of the variables is changed.*

It is common to say that a polynomial $p(x) = a+bx$ of degree one is a linear polynomial, even when a is nonzero. According to Definition 4.12 such a polynomial is an affine polynomial, and this (algebraic) terminology will be used in the present section. Outside this section however, we will use the term linear polynomial.

For a linear function of one variable we have

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y), \quad x, y \in \mathbb{R} \tag{4.17}$$

for all real numbers α and β, while for an affine function f with $b \neq 0$ equation (4.17) only holds if $\alpha + \beta = 1$. This is in fact a complete characterisation of affine functions: If (4.17) holds with $\alpha + \beta = 1$, then f is affine, see exercise 9.

A general affine function of 2 variables is given by

$$\begin{aligned} f(x_1, x_2) &= a x_2 + b = (a_2 x_1 + b_2) x_2 + a_1 x_1 + b_1 \\ &= c_0 + c_1 x_1 + c_2 x_2 + c_{1,2} x_1 x_2. \end{aligned} \tag{4.18}$$

Similarly, an affine function of three variables is a function on the form

$$f(x_1, x_2, x_3) = c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_{1,2} x_1 x_2 + c_{1,3} x_1 x_3 + c_{2,3} x_2 x_3 + c_{1,2,3} x_1 x_2 x_3.$$

In general, an affine function can be written as a linear combination of 2^d terms. This follows by induction as in (4.18) where we passed from one argument to two.

A symmetric and affine function satisfies the equation

$$f(x_1, x_2, \ldots, x_d) = f(x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_d}),$$

for any permutation $(\pi_1, \pi_2, \ldots, \pi_d)$ of the numbers 1, 2, …, d. We leave it as an exercise to show that symmetric, affine functions of two and three variables can be written in the form

$$\begin{aligned} f(x_1, x_2) &= a_0 + a_1(x_1 + x_2) + a_2 x_1 x_2, \\ f(x_1, x_2, x_3) &= a_0 + a_1(x_1 + x_2 + x_3) + a_2(x_1 x_2 + x_1 x_3 + x_2 x_3) + a_3 x_1 x_2 x_3. \end{aligned}$$

We are now ready to give the definition of the blossom of a polynomial.

Definition 4.13. *Let p be a polynomial of degree at most d. The blossom $\mathcal{B}[p](x_1, \ldots, x_d)$ of p is a function of d variables with the properties:*

1. *Symmetry. The blossom is a symmetric function of its arguments,*

   $$\mathcal{B}[p](x_1, \ldots, x_d) = \mathcal{B}[p](x_{\pi_1}, \ldots, x_{\pi_d})$$

   for any permutation $\pi_1, \ldots, \pi_d$ of the integers 1, …, d.

2. *Affine. The blossom is affine in each of its variables,*

   $$\mathcal{B}[p](\ldots, \alpha x + \beta y, \ldots) = \alpha \mathcal{B}[p](\ldots, x, \ldots) + \beta \mathcal{B}[p](\ldots, y, \ldots)$$

   whenever $\alpha + \beta = 1$.

3. *Diagonal property. The blossom agrees with p on the diagonal,*

   $$\mathcal{B}[p](x, \ldots, x) = p(x)$$

   for all real numbers x.

The blossom of a polynomial exists and is unique.

Theorem 4.14. *Each polynomial p of degree d has a unique blossom $\mathcal{B}[p](x_1,\ldots,x_d)$. The blossom acts linearly on p, i.e., if p_1 and p_2 are two polynomials and c_1 and c_2 are two real constants then*

$$\mathcal{B}[c_1p_1+c_2p_2](x_1,\ldots,x_d)=c_1\mathcal{B}[p_1](x_1,\ldots,x_d)+c_2\mathcal{B}[p_2](x_1,\ldots,x_d). \tag{4.19}$$

Proof. The proof of uniqueness follows along the lines sketched at the beginning of this section for small d. Start with a general affine function F of d variables

$$F(x_1,\ldots,x_d)=c_0+\sum_{j=1}^{d}\sum_{1\le i_1<\cdots<i_j\le d}c_{i_1,\ldots,i_j}x_{i_1}\cdots x_{i_j}.$$

Symmetry forces all the coefficients multiplying terms of the same degree to be identical. To see this we note first that

$$F(1,0,\ldots,0)=c_0+c_1=F(0,\ldots,1,\ldots,0)=c_0+c_i$$

for all i with $1\le i\le d$. Hence we have $c_1=\cdots=c_d$. To prove that the terms of degree j all have the same coefficients we use induction and set j of the variables to 1 and the rest to 0. By the induction hypothesis we know that all the terms of degree less than j are symmetric; denote the contribution from these terms by p_{j-1}. Symmetry then gives

$$p_{j-1}+c_{1,2,\ldots,j}=p_{j-1}+c_{1,2,\ldots,j-1,j+1}=\cdots=p_{j-1}+c_{d-j+1,\ldots,d}.$$

From this we conclude that all the coefficients multiplying terms of degree j must be equal. We can therefore write F as

$$F(x_1,\ldots,x_d)=a_0+\sum_{j=1}^{d}a_j\sum_{1\le i_1<\cdots<i_j\le d}x_{i_1}\cdots x_{i_j}, \tag{4.20}$$

for suitable constants $(a_j)_{j=0}^d$. From the diagonal property $F(x,\ldots,x)=f(x)$ the coefficients $(a_j)_{j=0}^d$ are all uniquely determined (since 1, x, …, x^d is basis for π_d).

The linearity of the blossom with regards to p follows from its uniqueness: The right-hand side of (4.19) is affine in each of the x_i, it is symmetric, and it reduces to $c_1p_1(x)+c_2p_2(x)$ on the diagonal $x_1=\cdots=x_d=x$. ∎

Recall that the elementary symmetric polynomials

$$s_j(x_1,\ldots,x_d)=\Bigl(\sum_{1\le i_1<\cdots<i_j\le d}x_{i_1}x_{i_2}\cdots x_{i_j}\Bigr)\Big/\binom{d}{j}$$

that appear in (4.20) (apart from the binomial coefficient) agree with the B-spline coefficients of the polynomial powers,

$$\sigma_{k,d}^{j}=s_j(\tau_{k+1},\ldots,\tau_{k+d}),$$

see Corollary 3.5. In fact, the elementary symmetric polynomials are the blossoms of the powers,

$$\mathcal{B}[x^j](x_1, \ldots, x_d) = s_j(x_1, \ldots, x_d) \qquad \text{for } j = 0, \ldots, d.$$

They can also be defined by the relation

$$(x - x_1) \cdots (x - x_d) = \sum_{k=0}^{d} (-1)^{d-k} \binom{d}{k} s_{d-k}(x_1, \ldots, x_d) x^k.$$

Note that the blossom depends on the degree of the polynomial in a nontrivial way. If we consider the polynomial $p(x) = x$ to be of degree one, then $\mathcal{B}[p](x_1) = x_1$. But we can also think of p as a polynomial of degree three (the cubic and quadratic terms are zero); then we obviously have $\mathcal{B}[p](x_1, x_2, x_3) = (x_1 + x_2 + x_3)/3$.

4.3.2 B-spline coefficients as blossoms

Earlier in this chapter we have come across a function that is both affine and symmetric. Suppose we have a knot vector $\boldsymbol{\tau}$ for B-splines of degree d. On the interval $[\tau_\mu, \tau_{\mu+1})$ the only nonzero B-splines are $\boldsymbol{B}_d = (B_{\mu-d,d}, \ldots, B_{\mu,d})^T$ which can be expressed in terms of matrices as

$$\boldsymbol{B}_d(x)^T = \boldsymbol{R}_1(x) \cdots \boldsymbol{R}_d(x).$$

If we consider the polynomial piece $f = \boldsymbol{B}_d^T \boldsymbol{c}_d$ with coefficients $\boldsymbol{c}_d = (c_{\mu-d}, \ldots, c_\mu)^T$ we can define a function F of d variables by

$$F(x_1, \ldots, x_d) = \boldsymbol{R}_1(x_1) \cdots \boldsymbol{R}_d(x_d) \boldsymbol{c}_d. \tag{4.21}$$

From equation(4.12) we recognise $F(x_1, \ldots, x_d)$ as the coefficient multiplying a B-spline with knots $x_0, x_1, \ldots, x_{d+1}$ in the representation of the polynomial f.

Equation (3.7) in Lemma 3.3 shows that F is a symmetric function. It is also affine in each of its variables. To verify this, we note that because of the symmetry it is sufficient to check that it is affine with respect to the first variable. Recall from Theorem 2.18 that $\boldsymbol{R}_1 = \boldsymbol{R}_{1,\boldsymbol{\tau}}$ is given by

$$\boldsymbol{R}_1(x) = \left(\frac{\tau_{\mu+1} - x}{\tau_{\mu+1} - \tau_\mu}, \quad \frac{x - \tau_\mu}{\tau_{\mu+1} - \tau_\mu} \right)$$

which is obviously an affine function of x.

The function F is also related to the polynomial f in that $F(x, \ldots, x) = f(x)$. We have proved the following lemma.

Lemma 4.15. *Let $f = \sum_{j=\mu-d}^{\mu} c_j B_{j,d}$ be a polynomial represented in terms of the B-splines in $\mathbb{S}_{d,\boldsymbol{\tau}}$ on the interval $[\tau_\mu, \tau_{\mu+1})$, with coefficients $\boldsymbol{c}_d = (c_{\mu-d}, \ldots, c_\mu)^T$. Then the function*

$$F(x_1, \ldots, x_d) = \boldsymbol{R}_1(x_1) \cdots \boldsymbol{R}_d(x_d) \boldsymbol{c}_d$$

is symmetric and affine, and agrees with f on the diagonal,

$$F(x, \ldots, x) = f(x).$$

Lemma 4.15 and Theorem 4.14 show that the blossom of f is given by

$$\mathcal{B}[f](x_1, \ldots, x_d) = \boldsymbol{R}_1(x_1) \cdots \boldsymbol{R}_1(x_d) \boldsymbol{c}_d.$$

Blossoming can be used to give explicit formulas for the B-spline coefficients of a spline.

Theorem 4.16. *Let* $f = \sum_{j=1}^{n} c_j B_{j,d,\boldsymbol{\tau}}$ *be a spline on a* $d+1$*-regular knot vector* $\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+d+1}$. *It's B-spline coefficients are then given by*

$$c_j = \mathcal{B}[f_k](\tau_{j+1}, \dots, \tau_{j+d}), \quad \text{for } k = j,\, j+1,\, \dots,\, j+d, \tag{4.22}$$

provided $\tau_k < \tau_{k+1}$. *Here* $f_k = f|_{(\tau_k, \tau_{k+1})}$ *is the restriction of* f *to the interval* (τ_k, τ_{k+1}).

Proof. Let us first restrict x to the interval $[\tau_\mu, \tau_{\mu+1})$ and only consider one polynomial piece f_μ of f. From Lemma 4.15 we know that $\mathcal{B}[f_\mu](x_1, \dots, x_d) = \boldsymbol{R}_1(x_1) \cdots \boldsymbol{R}_d(x_d)\boldsymbol{c}_d$, where $\boldsymbol{c}_d = (c_j)_{j=\mu-d}^{\mu}$ are the B-spline coefficients of f active on the interval $[\tau_\mu, \tau_{\mu+1})$. From (4.13) we then obtain

$$c_j = \mathcal{B}[f_\mu](\tau_{j+1}, \dots, \tau_{j+d}) \tag{4.23}$$

which is (4.22) in this special situation.

To prove (4.22) in general, fix j and choose the integer k in the range $j \le k \le j+d$. We then have

$$f_k(x) = \sum_{i=k-d}^{k} c_i B_{i,d}(x), \tag{4.24}$$

By the choice of k we see that the sum in (4.24) includes the term $c_j B_{j,d}$. Equation (4.22) therefore follows by applying (4.23) to f_k. ∎

The affine property allows us to perform one important operation with the blossom; we can change the arguments.

Lemma 4.17. *The blossom of* p *satisfies the relation*

$$\mathcal{B}[p](\dots, x, \dots) = \frac{b-x}{b-a}\mathcal{B}[p](\dots, a \dots) + \frac{x-a}{b-a}\mathcal{B}[p](\dots, b, \dots) \tag{4.25}$$

for all real numbers a, b *and* x *with* $a \ne b$.

Proof. Observe that x can be written as an affine combination of a and b,

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b.$$

Equation (4.25) then follows from the affine property of the blossom. ∎

The next result will be useful later.

Lemma 4.18. *Let* $\mathcal{B}_x\big[p(x,y)\big]$ *denote the blossom of* p *with respect to the variable* x. *Then*

$$\mathcal{B}_x\big[(y-x)^k\big](x_1, \dots, x_d) = \frac{k!}{d!} D^{d-k}\big((y-x_1)\cdots(y-x_d)\big), \tag{4.26}$$

for $k = 0,\, 1,\, \dots,\, d$, *and*

$$\mathcal{B}_x\big[(y_1-x)\cdots(y_\ell-x)\big](x_1, \dots, x_d) = \frac{(d-\ell)!}{d!} \sum_{1 \le i_1, \dots, i_\ell \le d} (y_1 - x_{i_1}) \cdots (y_\ell - x_{i_\ell}), \tag{4.27}$$

where the sum is over all distinct choices $i_1,\, \dots,\, i_\ell$ *of* ℓ *integers from the* d *integers* $1,\, \dots,\, d$.

Proof. For $k = d$ equation (4.26) follows since the right-hand side is symmetric and affine in each of the variables x_i and it agrees with $(y-x)^d$ on the diagonal $x_1 = \cdots = x_d = x$. The general result is then obtained by differentiating both sides k times.

Equation (4.27) follows since the right-hand side is affine, symmetric and reduces to $(y_1 - x)\cdots(y_\ell - x)$ when $x = x_1 = \cdots = x_d$, i.e., it must be the blossom of $(y-x)^d$. ∎

4.4 Inserting one knot at a time

With blossoming we have a simple but powerful tool for determining the B-spline coefficients of splines. Here we will apply blossoming to develop an alternative knot insertion strategy. Instead of inserting all new knots simultaneously we can insert them sequentially. We insert one knot at a time and update the B-spline coefficients between each insertion. This leads to simple, explicit formulas.

Lemma 4.19 (Böhm's method)**.** *Let $\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+d+1}$ be a given knot vector and let $\boldsymbol{t} = (t_i)_{i=1}^{n+d+2}$ be the knot vector obtained by inserting a knot z in $\boldsymbol{\tau}$ in the interval $[\tau_\mu, \tau_{\mu+1})$. If*

$$f = \sum_{j=1}^{n} c_j B_{j,d,\boldsymbol{\tau}} = \sum_{i=1}^{n+1} b_i B_{i,d,\boldsymbol{t}},$$

then $(b_i)_{i=1}^{n+1}$ can be expressed in terms of $(c_j)_{j=1}^{n}$ through the formulas

$$b_i = \begin{cases} c_i, & \text{if } 1 \le i \le \mu - d; \\ \dfrac{z - \tau_i}{\tau_{i+d} - \tau_i} c_i + \dfrac{\tau_{i+d} - z}{\tau_{i+d} - \tau_i} c_{i-1}, & \text{if } \mu - d + 1 \le i \le \mu; \\ c_{i-1}, & \text{if } \mu + 1 \le i \le n+1. \end{cases} \tag{4.28}$$

Proof. Observe that for $j \le \mu$ we have $\tau_j = t_j$. For $i \le \mu - d$ and with k an integer such that $i \le k \le i + d$ it therefore follows from (4.22) that

$$b_i = \mathcal{B}[f^k](t_{i+1}, \ldots, t_{i+d}) = \mathcal{B}[f^k](\tau_{i+1}, \ldots, \tau_{i+d}) = c_i.$$

Similarly, we have $t_i = \tau_{i-1}$ for $i \ge \mu + 1$ so

$$b_i = \mathcal{B}[f^k](t_{i+1}, \ldots, t_{i+d}) = \mathcal{B}[f^k](\tau_i, \ldots, \tau_{i+d-1}) = c_{i-1}$$

for such values of i.

When i satisfies $\mu - d + 1 \le i \le \mu$ we note that z will appear in the sequence $(t_{i+1}, \ldots, t_{i+d})$. From (4.22) we therefore obtain

$$b_i = \mathcal{B}[f^\mu](t_{i+1}, \ldots, z, \ldots, t_{i+d}) = \mathcal{B}[f^\mu](\tau_{i+1}, \ldots, z, \ldots, \tau_{i+d-1})$$

since we now may choose $k = \mu$. Applying Lemma 4.17 with $x = z$, $a = \tau_i$ and $b = \tau_{i+d}$ yields

$$b_i = \frac{\tau_{i+d} - z}{\tau_{i+d} - \tau_i} \mathcal{B}[f^\mu](\tau_{i+1}, \ldots, \tau_i, \ldots, \tau_{i+d}) + \frac{z - \tau_i}{\tau_{i+d} - \tau_i} \mathcal{B}[f^\mu](\tau_i, \ldots, \tau_{i+d}, \ldots, \tau_{i+d-1}).$$

Exploiting the symmetry of the blossom and again applying (4.22) leads to the middle formula in (4.28). ∎

It is sometimes required to insert the same knot several times; this can of course be accomplished by applying the formulas in (4.28) several times. Since blossoms have the property $\mathcal{B}[f](z,\ldots,z) = f(z)$, we see that inserting a knot d times in a spline of degree d gives as a by-product the function value of f at z. This can be conveniently illustrated by listing old and new coefficients in a triangular scheme. Consider the following triangle ($d = 3$),

$$\begin{array}{ccccccccccccc}
\cdots & c^0_{\mu-4} & & c^0_{\mu-3} & & c^0_{\mu-2} & & c^0_{\mu-1} & & c^0_{\mu} & & c^0_{\mu+1} & \cdots \\
& & & & c^1_{\mu-2} & & c^1_{\mu-1} & & c^1_{\mu} & & & & \\
& & & & & c^2_{\mu-1} & & c^2_{\mu} & & & & & \\
& & & & & & c^3_{\mu} & & & & & &
\end{array}$$

In the first row we have the coefficients of f on the original knot vector $\boldsymbol{\tau}$. After inserting z in $(\tau_\mu, \tau_{\mu+1})$ once, the coefficients relative to the knot vector $\boldsymbol{\tau}^1 = \boldsymbol{\tau} \cup \{z\}$ are

$$(\ldots, c^0_{\mu-4}, c^0_{\mu-3}, c^1_{\mu-2}, c^1_{\mu-1}, c^1_{\mu}, c^0_{\mu}, c^0_{\mu+1}, \ldots),$$

i.e., we move down one row in the triangle. Suppose that z is inserted once more. The new B-spline coefficients on $\boldsymbol{\tau}^2 = \boldsymbol{\tau}^1 \cup \{z\}$ are now found by moving down to the second row, across this row, and up the right hand side,

$$(\ldots, c^0_{\mu-4}, c^0_{\mu-3}, c^1_{\mu-2}, c^2_{\mu-1}, c^2_{\mu}, c^1_{\mu}, c^0_{\mu}, c^0_{\mu+1}, \ldots).$$

Similarly, if z is inserted 3 times, we move around the whole triangle. We can also insert z a full $d = 4$ times. We then simply repeat c^3_μ two times in the last row.

Lemma 4.19 shows that Oslo Algorithm 2 (Algorithm 4.11) is not always efficient. To compute a new coefficient in the case where only one new knot is inserted requires at most one convex combination according to Lemma 4.19 while Algorithm 4.11 requires the computation of a full triangle (two nested loops). More efficient versions of the Oslo algorithms can be developed, but this will not be considered here.

The simplicity of the formulas (4.28) indicates that the knot insertion matrix $\boldsymbol{A}$ must have a simple structure when only one knot is inserted. Setting $\boldsymbol{c} = (c_i)_{i=1}^n$ and $\boldsymbol{b} = (b_i)_{i=1}^{n+1}$ and remembering that $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{c}$, we see that $\boldsymbol{A}$ is given by the $(n+1) \times n$ matrix

$$\boldsymbol{A} = \begin{pmatrix}
1 & 0 & & & & & & & \\
& \ddots & \ddots & & & & & & \\
& & & 1 & 0 & & & & \\
& & & 1-\lambda_{\mu-d+1} & \lambda_{\mu-d+1} & & & & \\
& & & & \ddots & \ddots & & & \\
& & & & & 1-\lambda_\mu & \lambda_\mu & & \\
& & & & & 0 & 1 & & \\
& & & & & & \ddots & \ddots & \\
& & & & & & & 0 & 1
\end{pmatrix}, \tag{4.29}$$

where $\lambda_i = (z - \tau_i)/(\tau_{i+d} - \tau_i)$ for $\mu - d + 1 \le i \le \mu$. All the entries off the two diagonals are zero and such matrices are said to be bi-diagonal. Since z lies in the interval $[\tau_\mu, \tau_{\mu+1})$ all the entries in $\boldsymbol{A}$ are nonnegative. This property generalises to arbitrary knot insertion matrices.

Lemma 4.20. *Let* $\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+d+1}$ *and* $\boldsymbol{t} = (t_i)_{i=1}^{m+d+1}$ *be two knot vectors for splines of degree* d *with* $\boldsymbol{\tau} \subseteq \boldsymbol{t}$*. All the entries of the knot insertion matrix* $\boldsymbol{A}$ *from* $\mathbb{S}_{d,\boldsymbol{\tau}}$ *to* $\mathbb{S}_{d,\boldsymbol{t}}$ *are nonnegative and* $\boldsymbol{A}$ *can be factored as*

$$\boldsymbol{A} = \boldsymbol{A}_{m-n}\boldsymbol{A}_{m-n-1}\cdots\boldsymbol{A}_1, \tag{4.30}$$

where $\boldsymbol{A}_i$ *is a bi-diagonal* $(n+i)\times(n+i-1)$*-matrix with nonnegative entries.*

Proof. Let us denote the $m-n$ knots that are in $\boldsymbol{t}$ but not in $\boldsymbol{\tau}$ by $(z_i)_{i=1}^{m-n}$. Set $\boldsymbol{t}^0 = \boldsymbol{\tau}$ and $\boldsymbol{t}^i = \boldsymbol{t}^{i-1} \cup (z_i)$ for $i = 1, \ldots, m-n$. Denote by $\boldsymbol{A}_i$ the knot insertion matrix from $\boldsymbol{t}^{i-1}$ to $\boldsymbol{t}^i$. By applying Böhm's method $m-n$ times we obtain (4.30). Since all the entries in each of the matrices $\boldsymbol{A}_i$ are nonnegative the same must be true of $\boldsymbol{A}$. ∎

4.5 Bounding the number of sign changes in a spline

In this section we will make use of Böhm's method for knot insertion to prove that the number of spline changes in a spline function is bounded by the number of sign changes in its B-spline coefficient vector. This provides a generalisation of an interesting property of polynomials known as Descartes' rule of signs. Bearing the name of Descartes, this result is of course classical, but it is seldom mentioned in elementary mathematics textbooks. Before stating Descartes' rule of signs let us record what we mean by sign changes in a definition.

Definition 4.21. *Let* $\boldsymbol{c} = (c_i)_{i=1}^n$ *be a vector of real numbers. The number of sign changes in* $\boldsymbol{c}$ *(zeros are ignored) is denoted* $S^-(\boldsymbol{c})$*. The number of sign changes in a function* f *in an interval* (a,b) *is denoted* $S^-_{(a,b)}(f) = S^-(f)$*, provided this number is finite. It is given by the largest possible integer* r *such that an increasing sequence of* $r+1$ *real numbers* $x_1 < \cdots < x_{r+1}$ *in* (a,b) *can be found with the property that* $S^-\big(f(x_1),\ldots,f(x_{r+1})\big) = r$*.*

Example 4.22. Let us consider some simple examples of counting sign changes. It is easily checked that

$$\begin{aligned} S^-(1,-2) &= 1, & S^-(1,0,-1,3) &= 2,\\ S^-(1,0,2) &= 0, & S^-(2,0,0,0,-1) &= 1,\\ S^-(1,-1,2) &= 2, & S^-(2,0,0,0,1) &= 0. \end{aligned}$$

As stated in the definition, we simply count sign changes by counting the number of jumps from positive to negative values and from negative to positive, ignoring all components that are zero.

Descartes' rule of signs bounds the number of zeros in a polynomial by the number of sign changes in its coefficients. Recall that z is a zero of f of multiplicity $r \geq 1$ if $f(z) = Df(z) = \cdots = D^{r-1}f(z) = 0$ but $D^r f(z) \neq 0$.

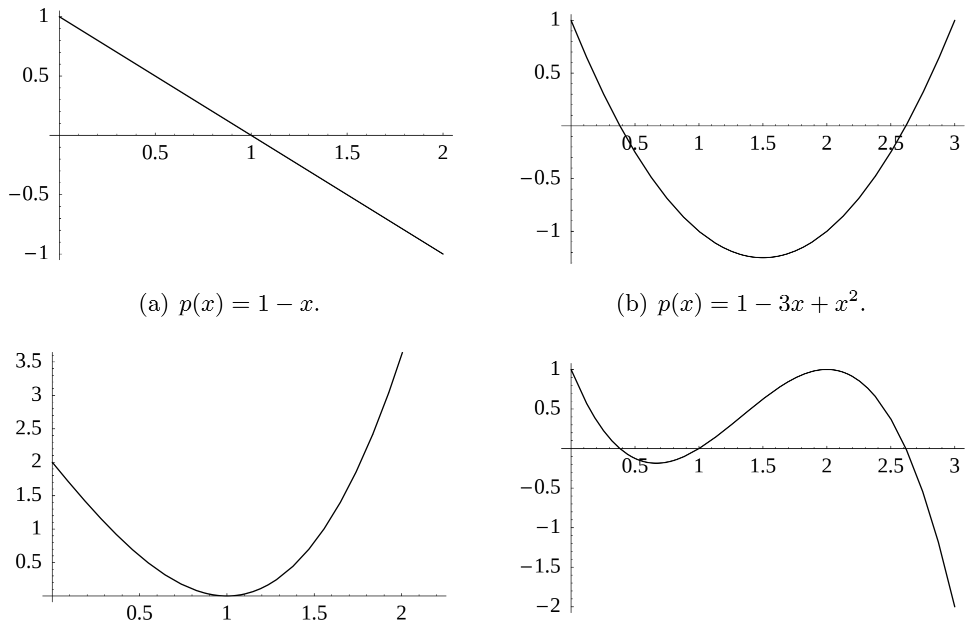
Theorem 4.23 (Descartes' rule of signs)**.** *Let* $p = \sum_{i=0}^d c_i x^i$ *be a polynomial of degree* d *with coefficients* $\boldsymbol{c} = (c_0,\ldots,c_d)^T$*, and let* $Z(p)$ *denote the total number of zeros of* p *in the interval* $(0,\infty)$*, counted with multiplicities. Then*

$$Z(p) \leq S^-(\boldsymbol{c}),$$

i.e., the number of zeros of p *is bounded by the number of sign changes in its coefficients.*

Figures 4.6 (a)–(d) show some polynomials and their zeros in $(0,\infty)$.

Our aim is to generalise this result to spline functions, written in terms of B-splines. This is not so simple because it is difficult to count zeros for splines. In contrast to



(a) $p(x) = 1 - x$.

(b) $p(x) = 1 - 3x + x^2$.

(c) $p(x) = 2 - 3x + x^2$.

(d) $p(x) = 1 - 4x + 4x^2 - x^3$.

Figure 4.6. Illustrations of Descartes' rule of signs: the number of zeros in $(0, \infty)$ is no greater than the number of strong sign changes in the coefficients.

polynomials, a spline may for instance be zero on an interval without being identically zero. In this section we will therefore only consider zeros that are also sign changes. In the next section we will then generalise and allow multiple zeros.

To bound the number of sign changes of a spline we will investigate how knot insertion influences the number of sign changes in the B-spline coefficients. Let $\mathbb{S}_{d,\boldsymbol{\tau}}$ and $\mathbb{S}_{d,\boldsymbol{t}}$ be two spline spaces of degree d, with $\mathbb{S}_{d,\boldsymbol{\tau}} \subseteq \mathbb{S}_{d,\boldsymbol{t}}$. Recall from Section 4.4 that to get from the knot vector $\boldsymbol{\tau}$ to the refined knot vector $\boldsymbol{t}$, we can insert one knot at a time. If there are ℓ more knots in $\boldsymbol{\tau}$ than in $\boldsymbol{t}$, this leads to a factorisation of the knot insertion matrix $\boldsymbol{A}$ as

$$\boldsymbol{A} = \boldsymbol{A}_\ell \boldsymbol{A}_{\ell-1} \cdots \boldsymbol{A}_1, \tag{4.31}$$

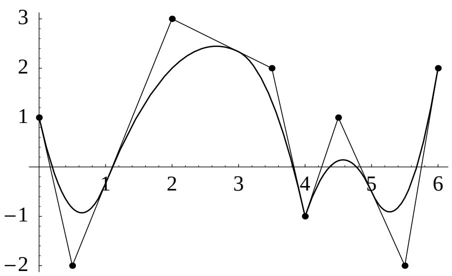
where $\boldsymbol{A}_k$ is a $(n+k) \times (n+k-1)$ matrix for $k = 1, \ldots, \ell$, if $\dim \mathbb{S}_{d,\boldsymbol{\tau}} = n$. Each of the matrices $\boldsymbol{A}_k$ corresponds to insertion of only one knot, and all the nonzero entries of the bi-diagonal matrix $\boldsymbol{A}_k$ are found in positions (i, i) and $(i+1, i)$ for $i = 1, \ldots, n+k-1$, and these entries are all nonnegative (in general many of them will be zero).

We start by showing that the number of sign changes in the B-spline coefficients is reduced when the knot vector is refined.

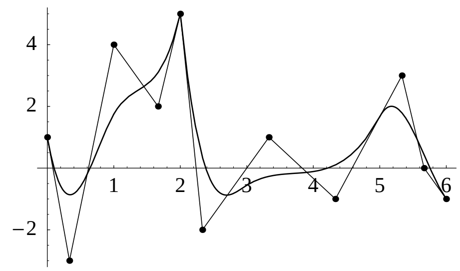
Lemma 4.24. *Let $\mathbb{S}_{d,\boldsymbol{\tau}}$ and $\mathbb{S}_{d,\boldsymbol{t}}$ be two spline spaces such that $\boldsymbol{t}$ is a refinement of $\boldsymbol{\tau}$. Let $f = \sum_{j=1}^{n} c_j B_{j,d,\boldsymbol{\tau}} = \sum_{i=1}^{m} b_i B_{i,d,\boldsymbol{t}}$ be a spline in $\mathbb{S}_{d,\boldsymbol{\tau}}$ with B-spline coefficients $\boldsymbol{c}$ in $\mathbb{S}_{d,\boldsymbol{\tau}}$ and $\boldsymbol{b}$ in $\mathbb{S}_{d,\boldsymbol{t}}$. Then $\boldsymbol{b}$ has no more sign changes than $\boldsymbol{c}$, i.e.,*

$$S^-(\boldsymbol{A}\boldsymbol{c}) = S^-(\boldsymbol{b}) \le S^-(\boldsymbol{c}), \tag{4.32}$$

where $\boldsymbol{A}$ is the knot insertion matrix from $\boldsymbol{\tau}$ to $\boldsymbol{t}$.



(a)



(b)

Figure 4.7. A quadratic spline (a) and a cubic spline (b) with their control polygons.

Proof. Since we can insert the knots one at a time, it clearly suffices to show that (4.32) holds in the case where there is only one more knot in $\boldsymbol{t}$ than in $\boldsymbol{\tau}$. In this case we know from Lemma 4.19 that $\boldsymbol{A}$ is bidiagonal so

$$b_i = \alpha_{i-1}(i)c_{i-1} + \alpha_i(i)c_i, \qquad \text{for } i = 1, \ldots n+1,$$

where $\big(\alpha_j(i)\big)_{i,j=1}^{n+1,n}$ are the entries of $\boldsymbol{A}$ (for convenience of notation we have introduced two extra entries that are zero, $\alpha_0(1) = \alpha_{n+1}(n+1) = 0$). Since $\alpha_{i-1}(i)$ and $\alpha_i(i)$ both are nonnegative, the sign of b_i must be the same as either c_{i-1} or c_i (or be zero). Since the number of sign changes in a vector is not altered by inserting zeros or a number with the same sign as one of its neighbours we have

$$S^-(\boldsymbol{c}) = S^-(b_1, c_1, b_2, c_2, \ldots, b_{n-1}, c_{n-1}, b_n, c_n, b_{n+1}) \geq S^-(\boldsymbol{b}).$$

The last inequality follows since the number of sign changes in a vector is always reduced when entries are removed. ∎

From Lemma 4.24 we can quite easily bound the number of sign changes in a spline in terms of the number of sign changes in its B-spline coefficients.

Theorem 4.25. *Let* $f = \sum_{j=1}^{n} c_j B_{j,d}$ *be a spline in* $\mathbb{S}_{d,\boldsymbol{\tau}}$. *Then*

$$S^-(f) \leq S^-(\boldsymbol{c}) \leq n-1. \tag{4.33}$$

Proof. Suppose that $S^-(f) = \ell$, and let $(x_i)_{i=1}^{\ell+1}$ be $\ell+1$ points chosen so that $S^-(f) = S^-\big(f(x_1), \ldots, f(x_{\ell+1})\big)$. We form a new knot vector $\boldsymbol{t}$ that includes $\boldsymbol{\tau}$ as a subsequence, but in addition each of the x_i occurs exactly $d+1$ times in $\boldsymbol{t}$. From our study of knot insertion we know that f may be written $f = \sum_j b_j B_{j,d,\boldsymbol{t}}$ for suitable coefficients (b_j), and from Lemma 2.6 we know that each of the function values $f(x_i)$ will appear as a B-spline coefficient in $\boldsymbol{b}$. We therefore have

$$S^-(f) \leq S^-(\boldsymbol{b}) \leq S^-(\boldsymbol{c}),$$

the last inequality following from Lemma 4.24. The last inequality in (4.33) follows since an n-vector can only have $n-1$ sign changes. ∎

The validity of Theorem 4.25 can be checked with the two plots in Figure 4.7 as well as all other figures which include both a spline function and its control polygon.

Exercises for Chapter 4

4.1 In this exercise we are going to study a change of polynomial basis from the Bernstein basis to the Monomial basis. Recall that the Bernstein basis of degree d is defined by

$$B_j^d(x) = \binom{d}{j} x^j (1-x)^{d-j}, \qquad \text{for } j = 0,\, 1,\, \dots,\, d. \tag{4.34}$$

A polynomial p of degree d is said to be written in Monomial form if $p(x) = \sum_{j=0}^{d} b_j x^j$ and in Bernstein form if $p(x) = \sum_{j=0}^{d} c_j B_j^d(x)$. In this exercise the binomial formula

$$(a+b)^d = \sum_{k=0}^{d} \binom{d}{k} a^k b^{d-k} \tag{4.35}$$

will be useful.

a) By applying (4.35), show that

$$B_j^d(x) = \sum_{i=j}^{d} (-1)^{i-j} \binom{d}{j} \binom{d-j}{i-j} x^i, \qquad \text{for } j = 0,\, 1,\, \dots,\, d.$$

Also show that $\binom{d}{j}\binom{d-j}{i-j} = \binom{d}{i}\binom{i}{j}$ for $i = j,\, \dots,\, d$ and $j = 0,\, \dots,\, d$.

b) The two basis vectors $\boldsymbol{B}_d = \big(B_0^d(x), \dots, B_d^d(x)\big)^T$ and $\boldsymbol{P}_d = (1, x, \dots, x^d)^T$ are related by $\boldsymbol{B}_d^T = \boldsymbol{P}_d^T \boldsymbol{A}_d$ where $\boldsymbol{A}_d$ is a $(d+1) \times (d+1)$-matrix $\boldsymbol{A}_d$. Show that the entries of $\boldsymbol{A}_d = (a_{i,j})_{i,j=0}^{d}$ are given by

$$a_{i,j} = \begin{cases} 0, & \text{if } i < j, \\ (-1)^{i-j} \binom{d}{i}\binom{i}{j}, & \text{otherwise.} \end{cases}$$

c) Show that the entries of $\boldsymbol{A}_d$ satisfy the recurrence relation

$$a_{i,j} = \beta_i \left(a_{i-1,j-1} - a_{i-1,j} \right), \qquad \text{where } \beta_i = (d-i+1)/i.$$

Give a detailed algorithm for computing $\boldsymbol{A}_d$ based on this formula.

d) Explain how we can find the coefficients of a polynomial relative to the Monomial basis if $\boldsymbol{A}_d$ is known and the coefficients relative to the Bernstein basis are known.

4.2 In this exercise we are going to study the opposite conversion of that in Exercise 1, namely from the Monomial basis to the Bernstein basis.

a) With the aid of (4.35), show that for all x and t in $\mathbb{R}$ we have

$$\big(tx + (1-x)\big)^d = \sum_{k=0}^{d} B_k^d(x) t^k. \tag{4.36}$$

The function $G(t) = \big(tx + (1-x)\big)^d$ is called a generating function for the Bernstein polynomials.

b) Show that $\sum_{k=0}^{d} B_k^d(x) = 1$ for all x by choosing a suitable value for t in (4.36).

c) Find two different expressions for $G^{(j)}(1)/j!$ and show that this leads to the formulas

$$\binom{d}{j} x^j = \sum_{i=j}^{d} \binom{i}{j} B_k^d(x), \qquad \text{for } j = 0, \ldots, d. \tag{4.37}$$

d) Show that the entries of the matrix $\boldsymbol{B}_d = (b_{i,j})_{i,j=0}^{d}$ such that $\boldsymbol{P}_d^T = \boldsymbol{B}_d^T \boldsymbol{B}_d$ are given by

$$b_{i,j} = \begin{cases} 0, & \text{if } i < j, \\ \binom{i}{j} / \binom{d}{j}, & \text{otherwise.} \end{cases}$$

4.3 Let $\boldsymbol{P}$ denote the cubic Bernstein basis on the interval $[0,1]$ and let $\boldsymbol{Q}$ denote the cubic Bernstein basis on the interval $[2,3]$. Determine the matrix $\boldsymbol{A}_3$ such that $\boldsymbol{P}(x)^T = \boldsymbol{Q}(x)^T \boldsymbol{A}_3$ for all real numbers x.

4.4 Let $\boldsymbol{A}$ denote the knot insertion matrix for the linear ($d = 1$) B-splines on $\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+2}$ to the linear B-splines in $\boldsymbol{t} = (t_i)_{i=1}^{m+2}$. We assume that $\boldsymbol{\tau}$ and $\boldsymbol{t}$ are 2-extended with $\tau_1 = t_1$ and $\tau_{n+2} = t_{m+2}$ and $\boldsymbol{\tau} \subseteq \boldsymbol{t}$.

a) Determine $\boldsymbol{A}$ when $\boldsymbol{\tau} = (0, 0, 1/2, 1, 1)$ and $\boldsymbol{t} = (0, 0, 1/4, 1/2, 3/4, 1, 1)$.

b) Device a detailed algorithm that computes $\boldsymbol{A}$ for general $\boldsymbol{\tau}$ and $\boldsymbol{t}$ and requires $O(m)$ operations.

c) Show that the matrix $\boldsymbol{A}^T \boldsymbol{A}$ is tridiagonal.

4.5 Prove Lemma 4.2 in the general case where $\boldsymbol{\tau}$ and $\boldsymbol{t}$ are not $d + 1$-regular. Hint: Augment both $\boldsymbol{\tau}$ and $\boldsymbol{t}$ by inserting $d + 1$ identical knots at the beginning and end.

4.6 Prove Theorem 4.7 in the general case where the knot vectors are not $d + 1$-regular with common knots at the ends. Hint: Use the standard trick of augmenting $\boldsymbol{\tau}$ and $\boldsymbol{t}$ with $d + 1$ identical knots at both ends to obtain new knot vectors $\hat{\boldsymbol{\tau}}$ and $\hat{\boldsymbol{t}}$. The knot insertion matrix from $\boldsymbol{\tau}$ to $\boldsymbol{t}$ can then be identified as a sub-matrix of the knot insertion matrix from $\hat{\boldsymbol{\tau}}$ to $\hat{\boldsymbol{t}}$.

4.7 Show that if $\boldsymbol{\tau}$ and $\boldsymbol{t}$ are $d + 1$-regular knot vectors with $\boldsymbol{\tau} \subseteq \boldsymbol{t}$ whose knots agree at the ends then $\sum_j \alpha_{j,d}(i) = 1$.

4.8 Implement Algorithm 4.11 and test it on two examples. Verify graphically that the control polygon converges to the spline as more and more knots are inserted.

4.9 Let f be a function that satisfies the identity

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \tag{4.38}$$

for all real numbers x and y and all real numbers α and β such that $\alpha + \beta = 1$. Show that then f must be an affine function. Hint: Use the alternative form of equation (4.38) found in Lemma 4.17.

4.10 Find the cubic blossom $\mathcal{B}[p](x_1, x_2, x_3)$ when p is given by:

a) $p(x) = x^3$.

b) $p(x) = 1$.

c) $p(x) = 2x + x^2 - 4x^3$.

d) $p(x) = 0$.

e) $p(x) = (x - a)^2$ where a is some real number.