CHAPTER 9

Approximation theory and stability

Polynomials of degree d have $d+1$ degrees of freedom, namely the $d+1$ coefficients relative to some polynomial basis. It turns out that each of these degrees of freedom can be utilised to gain approximation power so that the possible rate of approximation by polynomials of degree d is h^{d+1}, see Section 9.1. The meaning of this is that when a smooth function is approximated by a polynomial of degree d on an interval of length h, the error is bounded by Ch^{d+1}, where C is a constant that is independent of h. The exponent $d+1$ therefore controls how fast the error tends to zero with h.

When several polynomials are linked smoothly together to form a spline, each polynomial piece has $d+1$ coefficients, but some of these are tied up in satisfying the smoothness conditions. It therefore comes as a nice surprise that the approximation power of splines of degree d is the same as for polynomials, namely h^{d+1}, where h is now the largest distance between two adjacent knots. In passing from polynomials to splines we have therefore gained flexibility without sacrificing approximation power. We prove this in Section 9.2, by making use of some of the simple quasi-interpolants that we constructed in Chapter 8; it turns out that these produce spline approximations with the required accuracy.

The quasi-interpolants also allow us to establish two important properties of B-splines. The first is that B-splines form a stable basis for splines, see Section 9.3. This means that small perturbations of the B-spline coefficients can only lead to small perturbations in the spline, which is of fundamental importance for numerical computations. We have already seen that an important consequence of the stability of the B-spline basis is that the control polygon of a spline converges to the spline as the knot spacing tends to zero; this was proved in Section 4.1.

9.1 The distance to polynomials

We start by determining how well a given a real valued function f defined on an interval $[a, b]$ can be approximated by a polynomial of degree d. To measure the error in the approximation we will use the uniform norm which for a bounded function g defined on an interval $[a, b]$ is defined by

$$||g||_{\infty,[a,b]} = \sup_{a \le x \le b} \bigl|g(x)\bigr|.$$

Whenever we have an approximation p to f we can then measure the error by $||f - p||_{\infty,a,b}$. There are many possible approximations to f by polynomials of degree d, and the approximation that makes the error as small as possible is of course of special interest. This error is referred to as the *distance* from f to the space π_d of polynomials of degree $\leq d$ and is defined formally as

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_d) = \inf_{p \in \pi_d} ||f - p||_{\infty,[a,b]}.$$

In order to bound this approximation error, we have to place some restrictions on the functions that we approximate, and we will only consider functions with piecewise continuous derivatives. Such functions lie in a space that we denote $C^k_{\boldsymbol{\Delta}}[a, b]$ for some integer $k \geq 0$. A function f lies in this space if it has $k - 1$ continuous derivatives on the interval $[a, b]$, and the kth derivative $D^k f$ is continuous everywhere except for a finite number of points in the interior (a, b), given by $\boldsymbol{\Delta} = (z_j)$. At the points of discontinuity $\boldsymbol{\Delta}$ the limits from the left and right given by $D^k f(z_j+)$ and $D^k f(z_j-)$, should exist so all the jumps are finite. If there are no continuous derivatives we write $C_{\boldsymbol{\Delta}}[a, b] = C^0_{\boldsymbol{\Delta}}[a, b]$. Note that we will often refer to these spaces without stating explicitly what the singularities $\boldsymbol{\Delta}$ are.

An upper bound for the distance of f to polynomials of degree d is fairly simple to give by choosing a particular approximation, namely Taylor expansion.

Theorem 9.1. *Given a polynomial degree d and a function f in $C^{d+1}_{\boldsymbol{\Delta}}[a, b]$, then*

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_d) \leq K_d h^{d+1} ||D^{d+1} f||_{\infty,[a,b]},$$

where $h = b - a$ and

$$K_d = \frac{1}{2^{d+1}(d+1)!}$$

depends only on d.

Proof. Consider the truncated Taylor series of f at the midpoint $m = (a + b)/2$ of $[a, b]$.

$$T_d f(x) = \sum_{k=0}^{d} \frac{(x - m)^k}{k!} D^k f(m), \quad \text{for } x \in [a, b].$$

Since $T_d f$ is a polynomial of degree d we clearly have

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_d) \leq ||f - T_d f||_{\infty,[a,b]}. \tag{9.1}$$

To study the error we use the integral form of the remainder in the Taylor expansion,

$$f(x) - T_d f(x) = \frac{1}{d!} \int_m^x (x - y)^d D^{d+1} f(y) dy,$$

which is valid for any $x \in [a, b]$. If we restrict x to the interval $[m, b]$ we obtain

$$|f(x) - T_d f(x)| \leq ||D^{d+1} f||_{\infty,[a,b]} \frac{1}{d!} \int_m^x (x - y)^d dy.$$

The integral is given by

$$\frac{1}{d!} \int_m^x (x - y)^d dy = \frac{1}{(d+1)!} (x - m)^{d+1} \leq \frac{1}{(d+1)!} \left(\frac{h}{2}\right)^{d+1},$$

so for $x \geq m$ we have

$$|f(x) - T_d f(x)| \leq \frac{1}{2^{d+1}(d+1)!} h^{d+1} ||D^{d+1} f||_{\infty,[a,b]}.$$

By symmetry this estimate must also hold for $x \leq m$ and combining it with (9.1) completes the proof of the theorem. ∎

We remark that the best possible constant K_d can actually be computed. In fact, for each $f \in C^{d+1}[a,b]$ there is a point $\xi \in [a,b]$ such that

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_d) = \frac{2}{4^{d+1}(d+1)!} h^{d+1} |D^{d+1} f(\xi)|$$

Applying this formula to the function $f(x) = x^{d+1}$ we see that the exponent $d+1$ in h^{d+1} is best possible.

9.2 The distance to splines

Just as we defined the distance from a function f to the space of polynomials of degree d we can define the distance from f to a spline space. Our aim is to show that on one knot interval, the distance from f to a spline space of degree d is essentially the same as the distance from f to the space of polynomials of degree d on a slightly larger interval, see Theorem 9.2 and Corollary 9.11. Our strategy is to consider the cases $d = 0$, 1 and 2 separately and then generalise to degree d. The main ingredient in the proof is a family of simple approximation methods called quasi-interpolants. As well as leading to good estimates of the distance between f and a spline space, many of the quasi-interpolants are good, practical approximation methods.

We consider a spline space $\mathbb{S}_{d,\boldsymbol{t}}$ where d is a nonnegative integer and $\boldsymbol{t} = (t_i)_{i=1}^{n+d+1}$ is a $d+1$ regular knot vector. We set

$$a = t_1, \quad b = t_{n+d+1}, \quad h_j = t_{j+1} - t_j, \quad h = \max_{1 \leq j \leq n} h_j.$$

Given a function f we consider the distance from f to $\mathbb{S}_{d,\boldsymbol{t}}$ defined by

$$\operatorname{dist}_{\infty,[a,b]}(f, \mathbb{S}_{d,\boldsymbol{t}}) = \inf_{g \in \mathbb{S}_{d,\boldsymbol{t}}} ||f - g||_{\infty,[a,b]}.$$

We want to show the following.

Theorem 9.2. *Let the polynomial degree d and the function f in $C_{\boldsymbol{\Delta}}^{d+1}[a,b]$ be given. Then for any spline space $\mathbb{S}_{d,\boldsymbol{t}}$*

$$\operatorname{dist}_{\infty,[a,b]}(f, \mathbb{S}_{d,\boldsymbol{t}}) \leq K_d h^{d+1} ||D^{d+1} f||_{\infty,[a,b]}, \tag{9.2}$$

where the constant K_d depends on d, but not on f, h or $\boldsymbol{t}$.

We will prove this theorem by constructing a spline $P_d f$ such that

$$|f(x) - P_d f(x)| \leq K_d h^{d+1} ||D^{d+1} f||_{\infty,[a,b]}, \quad x \in [a,b] \tag{9.3}$$

for a constant K_d depending only on d. The approximation $P_d f$ will be on the form

$$P_d f = \sum_{i=1}^{n} \lambda_i(f) B_{i,d}$$

where λ_i is a rule for computing the ith B-spline coefficient. We will restrict ourselves to rules λ_i like

$$\lambda_i(f) = \sum_{k=0}^{d} w_{i,k} f(x_{i,k})$$

where the points $(x_{i,k})_{k=0}^d$ all lie in one knot interval and $(w_{i,k})_{k=0}^d$ are suitable coefficients. These kinds of approximation methods are called *quasi-interpolants*.

9.2.1 The constant and linear cases

We first prove Theorem 9.2 in the low degree cases $d = 0$ and $d = 1$. For $d = 0$ the knots form a partition $a = t_1 < \cdots < t_{n+1} = b$ of $[a, b]$ and the B-spline $B_{i,0}$ is the characteristic function of the interval $[t_i, t_{i+1})$ for $i = 1, \ldots, n-1$, while $B_{n,0}$ is the characteristic function of the closed interval $[t_n, t_{n+1}]$. We consider the step function

$$g = P_0 f = \sum_{i=1}^{n} f(t_{i+1/2}) B_{i,0},$$

where $t_{i+1/2} = (t_i + t_{i+1})/2$. Fix $x \in [a, b]$ and let l be an integer such that $t_l \le x < t_{l+1}$. We then have

$$f(x) - P_0 f(x) = f(x) - f(t_{l+1/2}) = \int_{t_{l+1/2}}^{x} Df(y)dy$$

so

$$|f(x) - P_0 f(x)| \le |x - t_{l+1/2}| \; ||Df||_{\infty,[t_l,t_{l+1}]} \le \frac{h}{2} ||Df||_{\infty,[a,b]}.$$

In this way we obtain (9.2) with $K_0 = 1/2$.

In the linear case $d = 1$ we define $P_1 f$ to be the piecewise linear interpolant to f on $\boldsymbol{t}$

$$g = P_1 f = \sum_{i=1}^{n} f(t_{i+1}) B_{i,1}.$$

Proposition 5.2 gives an estimate of the error in linear interpolation and by applying this result on each interval we obtain

$$||f - P_1 f||_{\infty,[a,b]} \le \frac{h^2}{8} ||D^2 f||_{\infty,[a,b]}$$

which is (9.2) with $K_1 = 1/8$.

9.2.2 The quadratic case

Consider next the quadratic case $d = 2$. We shall approximate f by the quasi-interpolant $P_2 f$ that we constructed in Section 8.2.2. Its properties is summarised in the following lemma.

Lemma 9.3. *Suppose* $\boldsymbol{t} = (t_i)_{i=1}^{n+3}$ *is a knot vector with* $t_{i+3} > t_i$ *for* $i = 1, \ldots, n$. *The operator*

$$P_2 f = \sum_{i=1}^{n} \lambda_i(f) B_{i,2,\boldsymbol{t}}, \quad \text{with} \quad \lambda_i(f) = -\frac{1}{2} f(t_{i+1}) + 2f(t_{i+3/2}) - \frac{1}{2} f(t_{i+2}) \tag{9.4}$$

satisfies $P_2 p = p$ *for all* $p \in \pi_2$.

To show that (9.3) holds for $d = 2$ we now give a sequence of small lemmas.

Lemma 9.4. *Let* $P_2(f)$ *be as in* (9.4). *Then*

$$|\lambda_i(f)| \leq 3||f||_{\infty,[t_{i+1},t_{i+2}]}, \quad i = 1, \ldots, n. \tag{9.5}$$

Proof. Fix an integer i. Then

$$|\lambda_i(f)| = |-\frac{1}{2} f(t_{i+1}) + 2f(t_{i+3/2}) - \frac{1}{2} f(t_{i+2})| \leq (\frac{1}{2} + 2 + \frac{1}{2})||f||_{\infty,[t_{i+1},t_{i+2}]}$$

from which the result follows. ∎

Lemma 9.5. *For* $\ell = 3, \ldots, n$ *we can bound* $P_2 f$ *on a subinterval* $[t_\ell, t_{\ell+1}]$ *by*

$$||P_2 f||_{\infty,[t_\ell,t_{\ell+1}]} \leq 3||f||_{\infty,[t_{\ell-1},t_{\ell+2}]}. \tag{9.6}$$

Proof. Fix $x \in [t_\ell, t_{\ell+1}]$. Since the B-splines are nonnegative and form a partition of unity we have

$$\begin{aligned} |P_2 f(x)| &= \Big| \sum_{i=\ell-2}^{\ell} \lambda_i(f) B_{i,2,\boldsymbol{t}}(x) \Big| \leq \max_{\ell-2 \leq i \leq \ell} |\lambda_i(f)| \\ &\leq 3 \max_{\ell-2 \leq i \leq \ell} ||f||_{\infty,[t_{i+1},t_{i+2}]} = 3||f||_{\infty,[t_{\ell-1},t_{\ell+2}]}, \end{aligned}$$

where we used Lemma 9.4. This completes the proof. ∎

The following lemma shows that locally, the spline $P_2 f$ approximates f essentially as well as the best quadratic polynomial.

Lemma 9.6. *For* $\ell = 3, \ldots, n$ *the error* $f - P_2 f$ *on the interval* $[t_\ell, t_{\ell+1}]$ *is bounded by*

$$||f - P_2 f||_{\infty,[t_\ell,t_{\ell+1}]} \leq 4 \operatorname{dist}_{\infty,[t_{\ell-1},t_{\ell+2}]}(f, \pi_2). \tag{9.7}$$

Proof. Let $p \in \pi_2$ be any quadratic polynomial. Since $P_2 p = p$ and P_2 is a linear operator, application of (9.6) to $f - p$ yields

$$\begin{aligned} \big|f(x) - (P_2 f)(x)\big| &= \big|f(x) - p(x) - \big((P_2 f)(x) - p(x)\big)\big| \\ &\leq \big|f(x) - p(x)\big| + \big|P_2(f - p)(x)\big| \\ &\leq (1 + 3)||f - p||_{\infty,[t_{\ell-1},t_{\ell+2}]}. \end{aligned} \tag{9.8}$$

Since p is arbitrary we obtain (9.7). ∎

We can now prove (9.2) for $d = 2$. For any interval $[a, b]$ Theorem 9.1 with $d = 2$ gives

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_2) \leq K_2 h^3 ||D^3 f||_{\infty,[a,b]},$$

where $h = b - a$ and $K_2 = 1/(2^3 3!)$. Combining this estimate on $[a, b] = [t_{\ell-1}, t_{\ell+2}]$ with (9.7) we obtain (9.3) and hence (9.2).

9.2.3 The general case

The general case is analogous to the quadratic case, but the details are more complicated. Recall that for $d = 2$ we picked three points $x_{i,k} = t_{i+1} + k(t_{i+2} - t_{i+1})/2$ for $k = 0, 1, 2$ in each subinterval $[t_{i+1}, t_{i+2}]$ and then chose constants $w_{i,k}$ for $k = 0, 1, 2$ such that the operator

$$P_2 f = \sum_{i=1}^{n} \lambda_i(f) B_{i,2,\boldsymbol{t}}, \quad \text{with} \quad \lambda_i(f) = w_{i,0} f(x_{i,0}) + w_{i,1} f(x_{i,1}) + w_{i,2} f(x_{i,2}),$$

reproduced quadratic polynomials. We will follow the same strategy for general degree. The resulting quasi-interpolant is a special case of the one given in Theorem 8.7.

Suppose that $d \geq 2$ and fix an integer i such that $t_{i+d} > t_{i+1}$. We pick the largest subinterval $[a_i, b_i] = [t_l, t_{l+1}]$ of $[t_{i+1}, t_{i+d}]$ and define the uniformly spaced points

$$x_{i,k} = a_i + \frac{k}{d}(b_i - a_i), \quad \text{for } k = 0,\, 1,\, \ldots,\, d \tag{9.9}$$

in this interval. Given $f \in C_{\boldsymbol{\Delta}}[a, b]$ we define $P_d f \in \mathbb{S}_{d,\boldsymbol{t}}$ by

$$P_d f(x) = \sum_{i=1}^{n} \lambda_i(f) B_{i,d}(x), \quad \text{where} \quad \lambda_i(f) = \sum_{k=0}^{d} w_{i,k} f(x_{i,k}). \tag{9.10}$$

The following lemma shows how the coefficients $(w_{i,k})_{k=0}^{d}$ should be chosen so that $P_d p = p$ for all $p \in \pi_d$.

Lemma 9.7. *Suppose that in* (9.10) *the functionals* λ_i *are given by* $\lambda_i(f) = f(t_{i+1})$ *if* $t_{i+d} = t_{i+1}$, *while if* $t_{i+d} > t_{i+1}$ *we set*

$$w_{i,k} = \gamma_i(p_{i,k}), \quad k = 0, 1, \ldots, d, \tag{9.11}$$

where $\gamma_i(p_{i,k})$ *is the* i*th B-spline coefficient of the polynomial*

$$p_{i,k}(x) = \prod_{\substack{j=0 \\ j \neq k}}^{d} \frac{x - x_{i,j}}{x_{i,k} - x_{i,j}}. \tag{9.12}$$

Then the operator P_d *in* (9.10) *satisfies* $P_d p = p$ *for all* $p \in \pi_d$.

Proof. Suppose first that $t_{i+d} > t_{i+1}$. Any $p \in \pi_d$ can be written in the form

$$p(x) = \sum_{k=0}^{d} p(x_{i,k}) p_{i,k}(x). \tag{9.13}$$

For if we denote the function on the right by $q(x)$ then $q(x_{i,k}) = p(x_{i,k})$ for $k = 0,\, 1,\, \ldots,\, d$, and since $q \in \pi_d$ it follows by the uniqueness of the interpolating polynomial that $p = q$. Now, by linearity of γ_i we have

$$\begin{aligned} \lambda_i(p) &= \sum_{k=0}^{d} w_{i,k} p(x_{i,k}) = \sum_{k=0}^{d} \gamma_i(p_{i,k}) p(x_{i,k}) \\ &= \gamma_i\Big(\sum_{k=0}^{d} p_{i,k} p(x_{i,k})\Big) = \gamma_i(p). \end{aligned}$$

If $t_{i+1} = t_{i+d}$ we know that a spline of degree d with knots $\boldsymbol{t}$ agrees with its $i+1$st coefficient at t_{i+1}. In particular, for any polynomial p we have $\lambda_i(p) = f(t_{i+1}) = \gamma_i(p)$. Alltogether this means that

$$P_d(p) = \sum_{i=1}^{n} \lambda_i(p) B_{i,d}(x) = \sum_{i=1}^{n} \gamma_i(p) B_{i,d}(x) = p$$

which confirms the lemma. ∎

The B-spline coefficients of $p_{i,k}$ can be found from the following lemma.

Lemma 9.8. *Given a spline space $\mathbb{S}_{d,\boldsymbol{t}}$ and numbers $v_1, \ldots, v_d$. The ith B-spline coefficient of the polynomial $p(x) = (x - v_1) \ldots (x - v_d)$ can be written*

$$\gamma_i(p) = \frac{1}{d!} \sum_{(j_1,\ldots,j_d)\in\Pi_d} (t_{i+j_1} - v_1) \cdots (t_{i+j_d} - v_d), \tag{9.14}$$

where Π_d is the set of all permutations of the integers $1, 2, \ldots, d$.

Proof. By Theorem 4.16 we have

$$\gamma_i(p) = \mathcal{B}[p](t_{i+1}, \ldots, t_{i+d}),$$

where $\mathcal{B}[p]$ is the blossom of p. It therefore suffices to verify that the expression (9.14) for $\gamma_i(p)$ satisfies the three properties of the blossom, but this is immediate. ∎

As an example, for $d = 2$ the set of all permutations of $1, 2$ are $\Pi_2 = \{(1, 2), (2, 1)\}$ and therefore

$$\gamma_i\big((x - v_1)(x - v_2)\big) = \frac{1}{2}\Big((t_{i+1} - v_1)(t_{i+2} - v_2) + (t_{i+2} - v_1)(t_{i+1} - v_2)\Big).$$

We can now give a bound for $\lambda_i(f)$.

Theorem 9.9. *Let $P_d(f) = \sum_{i=1}^{n} \lambda_i(f) B_{i,d}$ be the operator in Lemma 9.7. Then*

$$|\lambda_i(f)| \le K_d ||f||_{\infty,[t_{i+1},t_{i+d}]}, \quad i = 1, \ldots, n, \tag{9.15}$$

where

$$K_d = \frac{2^d}{d!} [d(d-1)]^d \tag{9.16}$$

depends only on d.

Proof. Fix an integer i. From Lemma 9.8 we have

$$w_{i,k} = \sum_{(j_1,\ldots,j_d)\in\Pi_d} \prod_{r=1}^{d} \left(\frac{t_{i+j_r} - v_r}{x_{i,k} - v_r}\right) / d!, \tag{9.17}$$

where $(v_r)_{r=1}^d = (x_{i,0}, \ldots, x_{i,k-1}, x_{i,k+1}, \ldots, x_{i,d})$. and Π_d denotes the set of all permutations of the integers 1, 2, …, d. Since the numbers t_{i+j_r} and v_r belongs to the interval $[t_{i+1}, t_{i+d}]$ for all r we have the inequality

$$\prod_{r=1}^{d} (t_{i+j_r} - v_r) \le (t_{i+d} - t_{i+1})^d.$$

We also note that $x_{i,k} - v_r = (k-q)(b_i - a_i)/d$ for some q in the range $1 \le q \le d$ but with $q \ne k$. Taking the product over all r we therefore obtain

$$\prod_{r=1}^{d} |x_{i,k} - v_r| = \prod_{\substack{q=0\\q\ne k}}^{d} \frac{|k-q|}{d}(b_i - a_i) \ge k!(d-k)! \left(\frac{b_i - a_i}{d}\right)^d \ge k!(d-k)! \left(\frac{t_{i+d} - t_{i+1}}{d(d-1)}\right)^d$$

for all values of k and r since $[a_i, b_i]$ is the largest subinterval of $[t_{i+1}, t_{i+d}]$. Since the sum in (9.17) contains $d!$ terms, we find

$$\sum_{k=0}^{d} |w_{i,k}| \le \frac{[d(d-1)]^d}{d!} \sum_{k=0}^{d} \binom{d}{k} = \frac{2^d}{d!}[d(d-1)]^d = K_d$$

and hence

$$|\lambda_i(f)| \le ||f||_{\infty,[t_{i+1},t_{i+d}]} \sum_{k=0}^{d} |w_{i,k}| \le K_d ||f||_{\infty,[t_{i+1},t_{i+d}]} \tag{9.18}$$

which is the required inequality. ∎

From the bound for $\lambda_i(f)$ we easily obtain a bound for the norm of $P_d f$.

Theorem 9.10. *For $d+1 \le l \le n$ and $f \in C_{\boldsymbol{\Delta}}[a,b]$ we have the bound*

$$||P_d f||_{\infty,[t_l,t_{l+1}]} \le K_d ||f||_{\infty,[t_{l-d+1},t_{l+d}]}, \tag{9.19}$$

where K_d is the constant in Theorem 9.9.

Proof. Fix $x \in [t_l, t_{l+1}]$. Since the B-splines are nonnegative and form a partition of unity we have by Theorem 9.9

$$\begin{aligned} |P_d f(x)| = |\sum_{i=l-d}^{l} \lambda_i(f) B_{i,d,\boldsymbol{t}}(x)| &\le \max_{l-d\le i\le l} |\lambda_i(f)| \\ &\le K_d \max_{l-d\le i\le l} ||f||_{\infty,[t_{i+1},t_{i+d}]} = K_d ||f||_{\infty,[t_{l-d+1},t_{l+d}]} \end{aligned}$$

This completes the proof. ∎

The following corollary shows that $P_d f$ locally approximates f essentially as well as the best polynomial approximation of f of degree d.

Corollary 9.11. *For $l = d+1, \ldots, n$ the error $f - P_d f$ on the interval $[t_l, t_{l+1}]$ is bounded by*

$$||f - P_d f||_{\infty,[t_l,t_{l+1}]} \le (1 + K_d) \operatorname{dist}_{\infty,[t_{l-d+1},t_{l+d}]}(f, \pi_d), \tag{9.20}$$

where K_d is the constant in Theorem 9.9

Proof. We argue exactly as in the quadratic case. Let $p \in \pi_d$ be any polynomial in π_d. Since $P_d p = p$ and P_d is a linear operator we therefore have

$$\begin{aligned} \bigl|f(x) - (P_d f)(x)\bigr| &= \bigl|f(x) - p(x) - \bigl((P_d f)(x) - p(x)\bigr)\bigr| \\ &\le \bigl|f(x) - p(x)\bigr| + \bigl|P_d(f-p)(x)\bigr| \\ &\le (1 + K_d)||f - p||_{\infty,[t_{l-d+1},t_{l+d}]}. \end{aligned}$$

Since p is arbitrary we obtain (9.20). ∎

We can now prove (9.2) for general d. By Theorem 9.1 we have for any interval $[a, b]$

$$\text{dist}_{\infty,[a,b]}(f, \pi_d) \leq K_d h^{d+1} ||D^{d+1} f||_{\infty,[a,b]},$$

where $h = b - a$ and K_d only depends on d. Combining this estimate on $[a, b] = [t_{l-d+1}, t_{l+d}]$ with (9.20) we obtain (9.3) and hence (9.2).

9.3 Stability of the B-spline basis

In order to compute with polynomials or splines we need to choose a basis to represent the functions. If a basis is to be suitable for computer manipulations then it should be reasonably insensitive to round-off errors. In particular, functions with 'small' function

values should have 'small' coefficients and vice versa. A basis with this property is said to be *well conditioned* or *stable*. In this section we will study the relationship between a spline and its coefficients quantitatively by introducing the *condition number* of a basis.

We have already seen that the size of a spline is bounded by its B-spline coefficients. There is also a reverse inequality, i.e., a bound on the B-spline coefficients in terms of the size of f. There are several reasons why such inequalities are important. In Section 4.1 we made use of this fact to estimate how fast the control polygon converges to the spline as more and more knots are inserted. A more direct consequence is that small relative perturbations in the coefficients can only lead to small changes in the function values. Both properties reflect the fact that the B-spline basis is well conditioned.

9.3.1 A general definition of stability

The stability of a basis can be defined quite generally. Instead of considering polynomials, we can consider a general linear vector space where we can measure the size of the elements through a norm; this is called a *normed linear space.*

Definition 9.12. *Let $\mathbb{U}$ be a normed linear space. A basis (ϕ_j) for $\mathbb{U}$ is said to be stable with respect to a vector norm $||\cdot||$ if there are small positive constants C_1 and C_2 such that*

$$C_1^{-1}\big\|(c_j)\big\| \leq \Big\|\sum_j c_j \phi_j\Big\| \leq C_2 \big\|(c_j)\big\|, \tag{9.21}$$

for all sets of coefficients $\boldsymbol{c} = (c_j)$. Let C_1^* and C_2^* denote the smallest possible values of C_1 and C_2 such that (9.21) *holds. The condition number of the basis is then defined to be $\kappa = \kappa((\phi_i)_i) = C_1^* C_2^*$.*

At the risk of confusion, we have used the same symbol both for the norm in $\mathbb{U}$ and the vector norm of the coefficients. In our case $\mathbb{U}$ will of course be some spline space $\mathbb{S}_{d,\boldsymbol{t}}$ and the basis (ϕ_j) will be the B-spline basis. The norms we will consider are the p-norms which are defined by

$$||f||_p = ||f||_{p,[a,b]} = \left(\int_a^b |f(x)|^p dx\right)^{1/p}, \quad \text{and} \quad ||\boldsymbol{c}||_p = \left(\sum_j |c_j|^p\right)^{1/p}$$

where f is a function on the interval $[a, b]$ and $\boldsymbol{c} = (c_j)$ is a real vector, and p is a real number in the range $1 \leq p < \infty$ for any real number. For $p = \infty$ the norms are defined by

$$||f||_\infty = ||f||_{\infty,[a,b]} = \max_{a \leq x \leq b} |f(x)|, \quad \text{and} \quad ||\boldsymbol{c}||_\infty = \big\|(c_j)\big\|_\infty = \max_j |c_j|,$$

In practice, the most important norms are the 1-, 2- and ∞-norms.

In Definition 9.12 we require the constants C_1 and C_2 to be 'small', but how small is 'small'? There is no unique answer to this question, but it is typically required that C_1 and C_2 should be independent of the dimension n of $\mathbb{U}$, or at least grow very slowly with n. Note that we always have $\kappa \geq 1$, and $\kappa = 1$ if and only if we have equality in both inequalities in (9.21).

A stable basis is desirable for many reasons, and the constant $\kappa = C_1 C_2$ crops up in many different contexts. The condition number κ does in fact act as a sort of derivative of the basis and gives a measure of how much an error in the coefficients is magnified in a function value.

Proposition 9.13. *Suppose (ϕ_j) is a stable basis for $\mathbb{U}$. If $f = \sum_j c_j \phi_j$ and $g = \sum_j b_j \phi_j$ are two elements in $\mathbb{U}$ with $f \neq 0$, then*

$$\frac{||f - g||}{||f||} \leq \kappa \frac{||\boldsymbol{c} - \boldsymbol{b}||}{||\boldsymbol{c}||}, \tag{9.22}$$

where κ is the condition number of the basis as in Definition 9.12.

Proof. From (9.21), we have the two inequalities $||f - g|| \leq C_2 ||(c_j - b_j)||$ and $1/||f|| \leq C_1/||(c_j)||$. Multiplying these together gives the result. ∎

If we think of g as an approximation to f, then (9.22) says that the relative error in $f - g$ is bounded by at most κ times the relative error in the coefficients. If κ is small, then a small relative error in the coefficients gives a small relative error in the function values. This is important in floating point calculations on a computer. A function is usually represented by its coefficients relative to some basis. Normally, the coefficients are real numbers that must be represented inexactly as floating point numbers in a computer. This round-off error means that the computed

spline, here g, will differ from the exact f. Proposition 9.13 shows that this is not so serious if the perturbed coefficients of g are close to those of f and the basis is stable.

Proposition 9.13 also provides some information as to what are acceptable values of C_1^* and C_2^*. If for example $\kappa = C_1^* C_2^* = 100$ we risk losing 2 decimal places in evaluation of a function; exactly how much accuracy one can afford to lose will of course vary.

One may wonder whether there are any unstable polynomial bases. It turns out that the power basis 1, x, x^2, ..., on the interval $[0,1]$ is unstable even for quite low degrees. Already for degree 10, one risks losing as much as 4 or 5 decimal digits in the process of computing the value of a polynomial on the interval $[0,1]$ relative to this basis, and other operations such as numerical root finding is even more sensitive.

9.3.2 The condition number of the B-spline basis. Infinity norm

Since splines and B-splines are defined via the knot vector, it is quite conceivable that the condition number of the B-spline basis could become arbitrarily large for certain knot configurations, for example in the limit when two knots merge into one. We will now prove that the condition number of the B-spline basis can be bounded independently of the knot vector so it cannot grow beyond all bounds when the knots vary.

The best constant C_2^* in Definition 9.12 can be found quite easily for the B-spline basis.

Lemma 9.14. *In all spline spaces $\mathbb{S}_{d,\boldsymbol{t}}$ the bound*

$$\Big\|\sum_{i=1}^{m} b_i B_{i,d}\Big\|_{\infty,[t_1,t_{m+1+d}]} \leq ||\boldsymbol{b}||_\infty$$

holds. Equality holds if $b_i = 1$ for all i and the knot vector $\boldsymbol{t} = (t_i)_{i=0}^{n+d}$ is $d+1$-extended; in this case $C_2^* = 1$.

Proof. This follows since the B-splines are nonnegative and sum to one. ∎

To find a bound for the constant C_1 we shall use the operator P_d given by (9.3). We recall that P_d reproduces polynomials of degree d, i.e., $P_d p = p$ for all $p \in \pi_d$. We now show that more is true; we have in fact that P_d reproduces all splines in $\mathbb{S}_{d,\boldsymbol{t}}$.

Theorem 9.15. *The operator*

$$P_d f = \sum_{i=1}^{n} \lambda_i(f) B_{i,d}$$

given by (9.3) *reproduces all splines in $\mathbb{S}_{d,\boldsymbol{t}}$, $P_d f = f$ for all $f \in \mathbb{S}_{d,\boldsymbol{t}}$.*

Proof. We first show that

$$\lambda_j(B_{k,d}) = \delta_{j,k}, \qquad \text{for } j,k = 1,\,\ldots,\,n. \tag{9.23}$$

Fix i and let

$$I_i = [a_i, b_j] = [t_{l_i}, t_{l_i+1}]$$

be the interval used to define $\lambda_i(f)$. We consider the polynomials

$$\phi_k = B_{k,d}|_{I_i} \qquad \text{for } l_i - d \leq k \leq l_i$$

obtained by restricting the B-splines $\{B_{k,d}\}_{k=l_i-d}^{l_i}$ to the interval I_i. Since P_d reproduces π_d we have

$$\phi_k(x) = (P_d\phi_k)(x) = \sum_{j=l_i-d}^{l_i} \lambda_j(\phi_k)\phi_j(x)$$

for x in the interval I_i. By the linear independence of the the polynomials (ϕ_k) we therefore obtain

$$\lambda_j(B_{k,d}) = \lambda_j(\phi_k) = \delta_{j,k}, \qquad \text{for } j,k = l_i - d,\,\ldots,\,l_i.$$

In particular we have $\lambda_i B_{i,d} = 1$ since $l_i - d \leq i \leq l_i$. For $k < l_i - d$ or $k > l_i$ the support of $B_{k,d}$ has empty intersection with I_i so $\lambda_i(B_{k,d}) = 0$ for these values of k. Thus (9.23) holds for all k.

To complete the proof suppose $f = \sum_{k=1}^{n} c_k B_{k,d}$ is a spline in $\mathbb{S}_{d,\boldsymbol{t}}$. From (9.23) we then have

$$Qf = \sum_{j=1}^{n}\Big(\sum_{k=1}^{n} c_k \lambda_j(B_{k,d})\Big)B_{j,d} = \sum_{j=1}^{n} c_j B_{j,d} = f. \quad ∎$$

To obtain an upper bound for C_1^* we note that the leftmost inequality in (9.21) is equivalent to

$$|b_i| \leq C_1||f||, \quad i = 1, \ldots, m.$$

Lemma 9.16. *There is a constant K_d, depending only on the polynomial degree d, such that for all splines $f = \sum_{i=1}^m b_i B_{i,d}$ in some given spline space $\mathbb{S}_{d,\boldsymbol{t}}$ the inequality*

$$|b_i| \leq K_d||f||_{[t_{i+1}, t_{i+d}]} \tag{9.24}$$

holds for all i.

Proof. Consider the operator P_d given in Lemma 9.7. Since $P_d f = f$ we have $b_i = \lambda_i(f)$. The result now follows from (9.15) ∎

Note that if $[a, b] \subseteq [c, d]$, then $||f||_{\infty,[a,b]} \leq ||f||_{\infty,[c,d]}$. From (9.24), we therefore conclude that $|b_i| \leq K_d\|f\|_{\infty,[t_1,t_{m+1+d}]}$ for all i or briefly $||\boldsymbol{b}|| \leq K_d\|f\|$. The constant K_d can therefore be used as C_1 in Definition 9.12 in the case where the norm is the ∞-norm. Combining the two lemmas we obtain the following theorem.

Theorem 9.17. *There is a constant K_1, depending only on the polynomial degree d, such that for all spline spaces $\mathbb{S}_{d,\boldsymbol{t}}$ and all splines $f = \sum_{i=1}^m b_i B_{i,d} \in \mathbb{S}_{d,\boldsymbol{t}}$ with B-spline coefficients $\boldsymbol{b} = (b_i)_{i=1}^m$ the inequalities*

$$K_1^{-1}||\boldsymbol{b}||_\infty \leq ||f||_{\infty,[t_1,t_{m+d}]} \leq ||\boldsymbol{b}||_\infty \tag{9.25}$$

hold.

The condition number of the B-spline basis on the knot vector $\boldsymbol{t}$ with respect to the ∞-norm is usually denoted $\kappa_{d,\infty,\boldsymbol{t}}$. By taking the supremum over all knot vectors we obtain the knot independent condition number $\kappa_{d,\infty}$,

$$\kappa_{d,\infty} = \sup_{\boldsymbol{t}} \kappa_{d,\infty,\boldsymbol{t}}.$$

Theorem 9.17 shows that $\kappa_{d,\infty}$ is bounded above by K_1.

The estimate K_d for C_1^* given by (9.16) is a number which grows quite rapidly with d and does not indicate that the B-spline basis is stable . However, it is possible to find better estimates for the condition number, and it is known that the B-spline basis is very stable , at least for moderate values of d. To determine the condition number is relatively simple for $d \leq 2$; we have $\kappa_{0,\infty} = \kappa_{1,\infty} = 1$ and $\kappa_{2,\infty} = 3$. For $d \geq 3$ it has recently been shown that $\kappa_{d,\infty} = O(2^d)$. The first few values are known numerically to be $\kappa_{3,\infty} \approx 5.5680$ and $\kappa_{4,\infty} \approx 12.088$.

9.3.3 The condition number of the B-spline basis. p-norm

With $1 \leq p \leq \infty$ and q such that $1/p + 1/q = 1$ we recall the Hölder inequality for functions

$$\int_a^b |f(x)g(x)|dx \leq ||f||_p||g||_q,$$

and the Hölder inequality for sums

$$\sum_{i=1}^m |b_i c_i| \leq ||(b_i)_{i=1}^m||_p||(c_i)_{i=1}^m||_q.$$

We also note that for any polynomial $g \in \pi_d$ and any interval $[a,b]$ we have

$$|g(x)| \le \frac{C}{b-a}\int_a^b |g(x)|\,dx, \quad x \in [a,b], \tag{9.26}$$

where the constant C only depends on the degree d. This follows on $[a,b]=[0,1]$ since all norms on a finite dimensional vector space are equivalent, and then on an arbitrary interval $[a,b]$ by a change of variable.

In order to generalise the stability result (9.25) to arbitrary p-norms we need to scale the B-splines differently. We define the p-norm B-splines to be identically zero if $t_{i+d+1}=t_i$ and

$$B^p_{i,d,\boldsymbol{t}} = \left(\frac{d+1}{t_{i+d+1}-t_i}\right)^{1/p} B_{i,d,\boldsymbol{t}}, \tag{9.27}$$

otherwise.

Theorem 9.18. *There is a constant K, depending only on the polynomial degree d, such that for all $1 \le p \le \infty$, all spline spaces $\mathbb{S}_{d,\boldsymbol{t}}$ and all splines $f=\sum_{i=1}^m b_i B^p_{i,d} \in \mathbb{S}_{d,\boldsymbol{t}}$ with p-norm B-spline coefficients $\boldsymbol{b}=(b_i)_{i=1}^m$ the inequalities*

$$K^{-1}||\boldsymbol{b}||_p \le ||f||_{p,[t_1,t_{m+d}]} \le ||\boldsymbol{b}||_p \tag{9.28}$$

hold.

Proof. We first prove the upper inequality. Let $\gamma_i=(d+1)/(t_{i+d+1}-t_i)$ for $i=1,\ldots,m$ and set $[a,b]=[t_1,t_{m+d+1}]$. Using the Hölder inequality for sums we have

$$\sum_i |b_i B^p_{i,d}| = \sum_i |b_i \gamma_i^{1/p} B^{1/p}_{i,d}| B^{1/q}_{i,d} \le \Big(\sum_i |b_i|^p \gamma_i B_{i,d}\Big)^{1/p} \Big(\sum_i B_{i,d}\Big)^{1/q}.$$

Raising this to the pth power and using the partition of unity property we obtain the inequality

$$\big|\sum_i b_i B^p_{i,d}(x)\big|^p \le \sum_i |b_i|^p \gamma_i B_{i,d}(x), \quad x \in \mathbb{R}.$$

Therefore, recalling that $\int B_{i,d}(x)dx = 1/\gamma_i$ we find

$$||f||^p_{p,[a,b]} = \int_a^b \big|\sum_i b_i B^p_{i,d}(x)\big|^p\,dx \le \sum_i |b_i|^p \gamma_i \int_a^b B_{i,d}(x)\,dx = \sum_i |b_i|^p.$$

Taking pth roots proves the upper inequality.

Consider now the lower inequality. Recall from (9.24) that we can bound the B-spline coefficients in terms of the infinity norm of the function. In terms of the coefficients b_i of the p-norm B-splines we obtain from (9.24) for all i

$$\left(\frac{d+1}{t_{i+d+1}-t_i}\right)^{1/p} |b_i| \le K_1 \max_{t_{i+1}\le x \le t_{i+d}} |f(x)|,$$

where the constant K_1 only depends on d. Taking max over a larger subinterval, using (9.26), and then Hölder for integrals we find

$$\begin{aligned}|b_i| &\le K_1(d+1)^{-1/p}\big(t_{i+d+1}-t_i\big)^{1/p}|\max_{t_i\le x\le t_{i+d+1}}|f(x)| \\ &\le CK_1(d+1)^{-1/p}\big(t_{i+d+1}-t_i\big)^{-1+1/p}\int_{t_i}^{t_{i+d+1}}|f(y)|\,dy \\ &\le CK_1(d+1)^{-1/p}\Big(\int_{t_i}^{t_{i+d+1}}|f(y)|^p\,dy\Big)^{1/p}\end{aligned}$$

Raising both sides to the pth power and summing over i we obtain

$$\sum_i |b_i|^p \le C^pK_1^p(d+1)^{-1}\sum_i\int_{t_i}^{t_{i+d+1}}|f(y)|^p\,dy \le C^pK_1^p||f||_{p,[a,b]}^p.$$

Taking pth roots we obtain the lower inequality in (9.28) with $K = CK_1$. ∎

Exercises for Chapter 9

9.1 In this exercise we will study the order of approximation by the Schoenberg Variation Diminishing Spline Approximation of degree $d \ge 2$. This approximation is given by

$$V_d f = \sum_{i=1}^n f(t_i^*)B_{i,d}, \quad \text{with} \quad t_i^* = \frac{t_{i+1}+\cdots t_{i+d}}{d}.$$

Here $B_{i,d}$ is the ith B-spline of degree d on a $d+1$-regular knot vector $\boldsymbol{t} = (t_i)_{i=1}^{n+d+1}$. We assume that $t_{i+d} > t_i$ for $i = 2,\ldots,n$. Moreover we define the quantities

$$a = t_1, \quad b = t_{n+d+1}, \quad h = \max_{1\le i\le n} t_{i+1}-t_i.$$

We want to show that $V_d f$ is an $O(h^2)$ approximation to a sufficiently smooth f.

We first consider the more general spline approximation

$$\tilde{V}_d f = \sum_{i=1}^n \lambda_i(f)B_{i,d}, \quad \text{with} \quad \lambda_i(f) = w_{i,0}f(x_{i,0}) + w_{i,1}f(x_{i,1}).$$

Here $x_{i,0}$ and $x_{i,1}$ are two distinct points in $[t_i, t_{i+d}]$ and $w_{i,0}, w_{i,1}$ are constants, $i = 1,\ldots,n$.

Before attempting to solve this exercise the reader might find it helpful to review Section 9.2.2

a) Suppose for $i = 1,\ldots,n$ that $w_{i,0}$ and $w_{i,1}$ are such that

$$\begin{aligned} w_{i,0} + w_{i,1} &= 1 \\ x_{i,0}w_{i,0} + x_{i,1}w_{i,1} &= t_i^* \end{aligned}$$

Show that then $\tilde{V}_d p = p$ for all $p \in \pi_1$. (Hint: Consider the polynomials $p(x) = 1$ and $p(x) = x$.)

b) Show that if we set $x_{i,0} = t_i^*$ for all i then $\tilde{V}_d f = V_d f$ for all f, regardless of how we choose the value of $x_{i,1}$.

In the rest of this exercise we set $\lambda_i(f) = f(t_i^*)$ for $i = 1, \ldots, n$, i.e. we consider $V_d f$. We define the usual uniform norm on an interval $[c, d]$ by

$$||f||_{[c,d]} = \sup_{c \le x \le d} |f(x)|, \quad f \in C_{\boldsymbol{\Delta}}[c, d].$$

c) Show that for $d + 1 \le l \le n$

$$||V_d f||_{[t_l, t_{l+1}]} \le ||f||_{[t_{l-d}^*, t_l^*]}, \quad f \in C_{\boldsymbol{\Delta}}[a, b].$$

d) Show that for $f \in C_{\boldsymbol{\Delta}}[t_{l-d}^*, t_l^*]$ and $d + 1 \le l \le n$

$$||f - V_d f||_{[t_l, t_{l+1}]} \le 2 \operatorname{dist}_{[t_{l-d}^*, t_l^*]}(f, \pi_1).$$

e) Explain why the following holds for $d + 1 \le l \le n$

$$\operatorname{dist}_{[t_{l-d}^*, t_l^*]}(f, \pi_1) \le \frac{(t_l^* - t_{l-d}^*)^2}{8} ||D^2 f||_{[t_{l-d}^*, t_l^*]}.$$

f) Show that the following $O(h^2)$ estimate holds

$$||f - V_d f||_{[a,b]} \le \frac{d^2}{4} h^2 ||D^2 f||_{[a,b]}.$$

(Hint: Verify that $t_l^* - t_{l-d}^* \le hd$.)

9.2 In this exercise we want to perform a numerical simulation experiment to determine the order of approximation by the quadratic spline approximations

$$V_2 f = \sum_{i=1}^{n} f(t_i^*) B_{i,2}, \quad \text{with} \quad t_i^* = \frac{t_{i+1} + t_{i+2}}{2},$$
$$P_2 f = \sum_{i=1}^{n} \Big(-\frac{1}{2} f(t_{i+1}) + 2 f(t_i^*) - \frac{1}{2} f(t_{i+2}) \Big) B_{i,2}.$$

We want to test the hypotheses $f - V_2 f = O(h^2)$ and $f - P_2 f = O(h^3)$ where $h = \max_i t_{i+1} - t_i$. We test these on the function $f(x) = \sin x$ on $[0, \pi]$ for various values of h. Consider for $m \ge 0$ and $n_m = 2 + 2^m$ the 3-regular knot vector $\boldsymbol{t}^m = (t_i^m)_{i=1}^{n_m+3}$ on the interval $[0, \pi]$ with uniform spacing $h_m = \pi 2^{-m}$. We define

$$V_2^m f = \sum_{i=1}^{n} f(t_{i+3/2}^m) B_{i,2}^m, \quad \text{with} \quad t_i^m = \frac{t_{i+1}^m + t_{i+2}^m}{2},$$
$$P_2^m f = \sum_{i=1}^{n} \Big(-\frac{1}{2} f(t_{i+1}^m) + 2 f(t_{i+3/2}^m) - \frac{1}{2} f(t_{i+2}^m) \Big) B_{i,2}^m,$$

and $B_{i,2}^m$ is the ith quadratic B-spline on $\boldsymbol{t}^m$. As approximations to the norms $||f - V_2^m f||_{[0,\pi]}$ and $||f - P_2^m f||_{[0,\pi]}$ we use

$$E_V^m = \max_{0\le j\le 100} |f(j\pi/100) - V_2^m f(j\pi/100)|,$$
$$E_P^m = \max_{0\le j\le 100} |f(j\pi/100) - P_2^m f(j\pi/100)|.$$

Write a computer program to compute numerically the values of E_V^m and E_P^m for $m = 0, 1, 2, 3, 4, 5$, and the ratios E_V^m/E_V^{m-1} and E_P^m/E_P^{m-1} for $1 \le m \le 5$. What can you deduce about the approximation order of the two methods?

Make plots of $V_2^m f$, $P_2^m f$, $f - V_2^m f$, and $f - P_2^m f$ for some values of m.

9.3 Suppose we have $m \ge 3$ data points $\big(x_i, f(x_i)\big)_{i=1}^m$ sampled from a function f, where the abscissas $\boldsymbol{x} = (x_i)_{i=1}^m$ satisfy $x_1 < \cdots < x_m$. In this exercise we want to derive a local quasi-interpolation scheme which only uses the data values at the x_i's and which has $O(h^3)$ order of accuracy if the y-values are sampled from a smooth function f. The method requires m to be odd.

From $\boldsymbol{x}$ we form a 3-regular knot vector by using every second data point as a knot

$$\boldsymbol{t} = (t_j)_{j=1}^{n+3} = (x_1, x_1, x_1, x_3, x_5, \ldots, x_{m-2}, x_m, x_m, x_m), \tag{9.29}$$

where $n = (m + 3)/2$. In the quadratic spline space $\mathbb{S}_{2,\boldsymbol{t}}$ we can then construct the spline

$$Q_2 f = \sum_{j=1}^{n} \lambda_j(f) B_{j,2}, \tag{9.30}$$

where the B-spline coefficients $\lambda_j(f)_{j=1}^n$ are defined by the rule

$$\lambda_j(f) = \frac{1}{2}\Big(- \theta_j^{-1} f(x_{2j-3}) + \theta_j^{-1}(1 + \theta_j)^2 f(x_{2j-2}) - \theta_j f(x_{2j-1})\Big), \tag{9.31}$$

for $j = 1, \ldots, n$. Here $\theta_1 = \theta_n = 1$ and

$$\theta_j = \frac{x_{2j-2} - x_{2j-3}}{x_{2j-1} - x_{2j-2}}$$

for $j = 2, \ldots, n - 1$.

a) Show that Q_2 simplifies to P_2 given by (9.4) when the data abscissas are uniformly spaced.

b) Show that $Q_2 p = p$ for all $p \in \pi_2$ and that because of the multiple abscissas at the ends we have $\lambda_1(f) = f(x_1)$, $\lambda_n(f) = f(x_m)$, so only the original data are used to define $Q_2 f$. (Hint: Use the formula in Exercise 1.

c) Show that for $j = 1, \ldots, n$ and $f \in C_{\boldsymbol{\Delta}}[x_1, x_m]$

$$|\lambda_j(f)| \le (2\theta + 1)||f||_{\infty,[t_{j+1},t_{j+2}]},$$

where

$$\theta = \max_{1\le j\le n} \{\theta_j^{-1}, \theta_j\}.$$

d) Show that for $l = 3, \ldots, n$, $f \in C_{\mathbf{\Delta}}[x_1, x_m]$, and $x \in [t_l, t_{l+1}]$

$$|Q_2(f)(x)| \leq (2\theta + 1)||f||_{\infty,[t_{l-1},t_{l+2}]}.$$

e) Show that for $l = 3, \ldots, n$ and $f \in C_{\mathbf{\Delta}}[x_1, x_m]$

$$||f - Q_2 f||_{\infty,[t_l,t_{l+1}]} \leq (2\theta + 2)\,\mathrm{dist}_{[t_{l-1},t_{l+2}]}(f, \pi_2).$$

f) Show that for $f \in C^3_{\mathbf{\Delta}}[x_1, x_m]$ we have the $O(h^3)$ estimate

$$||f - Q_2 f||_{\infty,[x_1,x_m]} \leq K(\theta)|\Delta x|^3||D^3 f||_{\infty,[x_1,x_m]},$$

where

$$|\Delta x| = \max_j |x_{j+1} - x_j|$$

and the constant $K(\theta)$ only depends on θ.