CHAPTER 10

Shape Preserving Properties of B-splines

In earlier chapters we have seen a number of examples of the close relationship between a spline function and its B-spline coefficients. This is especially evident in the properties of the Schoenberg operator, but the same phenomenon is apparent in the diagonal property of the blossom, the stability of the B-spline basis, the convergence of the control polygon to the spline it represents and so on. In the present chapter we are going to add to this list by relating the number of zeros of a spline to the number of sign changes in the sequence of its B-spline coefficients. From this property we shall obtain an accurate characterisation of when interpolation by splines is uniquely solvable. In the final section we show that the knot insertion matrix and the B-spline collocation matrix are totally positive, i.e., all their square submatrices have nonnegative determinants.

10.1 Bounding the number of zeros of a spline

In Section 4.5 of Chapter 4 we showed that the number of sign changes in a spline is bounded by the number of sign changes in its B-spline coefficients, a generalisation of Descartes' rule of signs for polynomials, Theorem 4.23. Theorem 4.25 is not a completely satisfactory generalisation of Theorem 4.23 since it does not allow multiple zeros. In this section we will prove a similar result that does allow multiple zeros, but we cannot allow the most general spline functions. we have to restrict ourselves to *connected splines*.

Definition 10.1. *A spline $f = \sum_{j=1}^{n} c_j B_{j,d}$ in $\mathbb{S}_{d,\boldsymbol{t}}$ is said to be* connected *if for each x in (t_1, t_{n+d+1}) there is some j such that $t_j < x < t_{j+d+1}$ and $c_j \neq 0$. A point x where this condition fails is called a* splitting point *for f.*

To develop some intuition about connected splines, let us see when a spline is not connected. A splitting point of f can be of two kinds:

(i) The splitting point x is not a knot. If $t_\mu < x < t_{\mu+1}$, then $t_j < x < t_{j+d+1}$ for $j = \mu - d, \ldots, \mu$ (assuming the knot vector is long enough) so we must have $c_{\mu-d} = \cdots = c_\mu = 0$. In other words f must be identically zero on $(t_\mu, t_{\mu+1})$. In this case f splits into two spline functions f_1 and f_2 with knot vectors $\boldsymbol{t}^1 = (t_j)_{j=1}^{\mu}$ and

$\boldsymbol{t}^2 = (t_j)_{j=\mu+1}^{n+d+1}$. We clearly have

$$f_1 = \sum_{j=1}^{\mu-d-1} c_j B_{j,d}, \qquad f_2 = \sum_{j=\mu+1}^{n} c_j B_{j,d}.$$

(ii) The splitting point x is a knot of multiplicity m, say

$$t_\mu < x = t_{\mu+1} = \cdots = t_{\mu+m} < t_{\mu+m+1}.$$

In this case we have $t_j < x < t_{j+1+d}$ for $j = \mu + m - d, \ldots, \mu$. We must therefore have $c_{\mu+m-d} = \cdots = c_\mu = 0$. (Note that if $m = d+1$, then no coefficients need to be zero). This means that all the B-splines that "cross" x do not contribute to $f(x)$. It therefore splits into two parts f_1 and f_2, but now the two pieces are not separated by an interval, but only by the single point x. The knot vector of f_1 is $\boldsymbol{t}^1 = (t_j)_{j=1}^{\mu+m}$ while the knot vector of f_2 is $\boldsymbol{t}^2 = (t_j)_{j=\mu+1}^{n+d+1}$. The two spline functions are given by

$$f_1 = \sum_{j=1}^{\mu+m-d-1} c_j B_{j,d}, \qquad f_2 = \sum_{j=\mu+1}^{n} c_j B_{j,d}.$$

Before getting on with our zero counts we need the following lemma.

Lemma 10.2. *Suppose that z is a knot that occurs m times in $\boldsymbol{t}$,*

$$t_i < z = t_{i+1} = \cdots = t_{i+m} < t_{i+m+1}$$

for some i. Let $f = \sum_j c_j B_{j,d}$ be a spline in $\mathbb{S}_{d,\boldsymbol{t}}$. Then

$$c_j = \frac{1}{d!} \sum_{k=0}^{d-m} (-1)^k D^{d-k} \rho_{j,d}(z) D^k f(z) \tag{10.1}$$

for all j such that $t_j < z < t_{j+d+1}$, where $\rho_{j,d}(y) = (y - t_{j+1}) \cdots (y - t_{j+d})$.

Proof. Recall from Theorem 8.5 that the B-spline coefficients of f can be written as

$$c_j = \lambda_j f = \frac{1}{d!} \sum_{k=0}^{d} (-1)^k D^{d-k} \rho_{j,d}(y) D^k f(y),$$

where y is a number such that $B_{j,d}(y) > 0$. In particular, we may choose $y = z$ for $j = i + m - d, \ldots, i$ so

$$c_j = \lambda_j f = \frac{1}{d!} \sum_{k=0}^{d} (-1)^k D^{d-k} \rho_{j,d}(z) D^k f(z), \tag{10.2}$$

for these values of j. But in this case $\rho_{j,d}(y)$ contains the factor $(y - t_{i+1}) \cdots (y - t_{i+m}) = (y - z)^m$ so $D^{d-k} \rho_{j,d}(z) = 0$ for $k > d - m$ and $j = i + m - d, \ldots, i$, i.e., for all values of j such that $t_j < z < t_{j+d+1}$. The formula (10.1) therefore follows from (10.2). ∎

In the situation of Lemma 10.2, we know from Lemma 2.6 that $D^k f$ is continuous at z for $k = 0, \ldots, d-m$, but $D^{d+1-m} f$ may be discontinuous. Equation (10.1) therefore shows that the B-spline coefficients of f can be computed solely from continuous derivatives of f at a point.

Lemma 10.3. *Let f be a spline that is connected. For each x in (t_1, t_{n+d+1}) there is then a nonnegative integer r such that $D^r f$ is continuous at x and $D^r f(x) \neq 0$.*

Proof. The claim is clearly true if x is not a knot, for otherwise f would be identically zero on an interval and therefore not connected. Suppose next that x is a knot of multiplicity m. Then the first discontinuous derivative at x is $D^{d-m+1} f$, so if the claim is not true, we must have $D^k f(x) = 0$ for $k = 0, \ldots, d-m$. But then we see from Lemma 10.2 that $c_l = \lambda_l f = 0$ for all l such that $t_l < x < t_{l+d+1}$. But this is impossible since f is connected. ∎

The lemma shows that we can count zeros of connected splines precisely as for smooth functions. If f is a connected spline then a zero must be of the form $f(z) = Df(z) = \cdots = D^{r-1} f(z) = 0$ with $D^r f(z) \neq 0$ for some integer r. Moreover $D^r f$ is continuous at z. The total number of zeros of f on (a, b), counting multiplicities, is denoted $Z(f) = Z_{(a,b)}(f)$. Recall from Definition 4.21 that $S^-(\boldsymbol{c})$ denotes the number of sign changes in the vector $\boldsymbol{c}$ (zeros are completely ignored).

Example 10.4. Below are some examples of zero counts of functions. For comparison we have also included counts of sign changes. All zero counts are over the whole real line.

$$\begin{aligned}
&Z(x) = 1, && S^-(x) = 1, && Z\big(x(1-x)^2\big) = 3, && S^-\big(x(1-x)^2\big) = 1,\\
&Z(x^2) = 2, && S^-(x^2) = 0, && Z\big(x^3(1-x)^2\big) = 5, && S^-\big(x^3(1-x)^2\big) = 1,\\
&Z(x^7) = 7, && S^-(x^7) = 1, && Z(-1 - x^2 + \cos x) = 2, && S^-(-1 - x^2 + \cos x) = 0.
\end{aligned}$$

We are now ready to prove a generalization of Theorem 4.23 that allows zeros to be counted with multiplicities.

Theorem 10.5. *Let $f = \sum_{j=1}^n c_j B_{j,d}$ be a spline in $\mathbb{S}_{d,\boldsymbol{t}}$ that is connected. Then*

$$Z_{(t_1, t_{n+d+1})}(f) \leq S^-(\boldsymbol{c}) \leq n - 1.$$

Proof. Let $z_1 < z_2 < \cdots < z_\ell$ be the zeros of f in the interval (t_1, t_{n+d+1}), each of multiplicity r_i; Lemma 10.2 shows that z_i occurs at most $d - r_i$ times in $\boldsymbol{t}$. For if z_i occured $m > d - r_i$ times in $\boldsymbol{t}$ then $d - m < r_i$, and hence all the derivatives of f involved in (10.1) would be zero for all j such that $t_j < z < t_{j+d+1}$. But this means that z is a splitting point for f which is impossible since f is connected.

Now we form a new knot vector $\hat{\boldsymbol{t}}$ where z_i occurs exactly $d - r_i$ times and the numbers $z_i - h$ and $z_i + h$ occur $d + 1$ times. Here h is a number that is small enough to ensure that there are no other zeros of f or knots from $\boldsymbol{t}$ other than z_i in $[z_i - h, z_i + h]$ for $1 \leq i \leq \ell$. Let $\hat{\boldsymbol{c}}$ be the B-spline coefficients of f relative to $\hat{\boldsymbol{t}}$. By Lemma 4.24 we then have $S^-(\hat{\boldsymbol{c}}) \leq S^-(\boldsymbol{c})$ so it suffices to prove that $Z_{(t_1, t_{n+d+1})}(f) \leq S^-(\hat{\boldsymbol{c}})$. But since

$$Z_{(t_1, t_{n+d+1})}(f) = \sum_{i=1}^{\ell} Z_{(z_i - h, z_i + h)}(f),$$

it suffices to establish the theorem in the following situation: The knot vector is given by

$$\boldsymbol{t} = (\overbrace{z-h, \dots, z-h}^{d+1}, \overbrace{z, \dots, z}^{d-r}, \overbrace{z+h, \dots, z+h}^{d+1})$$

and z is a zero of f of multiplicity r. The key to proving the theorem in this more specialised situation is to show that

$$c_j = \frac{(d-r)!}{d!}(-1)^{d+1-j} h^r D^r f(z), \quad j = d+1-r, \dots, d+1, \tag{10.3}$$

as this means that the $r+1$ coefficients $(c_j)_{j=d+1-r}^{d+1}$ alternate in sign and $S^-(\boldsymbol{c}) \geq r = Z_{(z-h,z+h)}(f)$. Fix j in the range $d+1-r \leq j \leq d+1$. By equation (10.1) we have

$$c_j = \frac{1}{d!}\sum_{k=0}^{r}(-1)^k D^{d-k}\rho_{j,d}(z) D^k f(z) = \frac{(-1)^r}{d!} D^{d-r}\rho_{j,d}(z) D^r f(z),$$

since $D^j f(z) = 0$ for $j = 0 \dots, r-1$. With our special choice of knot vector we have

$$\rho_{j,d}(y) = (y-z+h)^{d+1-j}(y-z)^{d-r}(y-z-h)^{r-d-1+j}.$$

Taking $d-r$ derivatives we therefore obtain

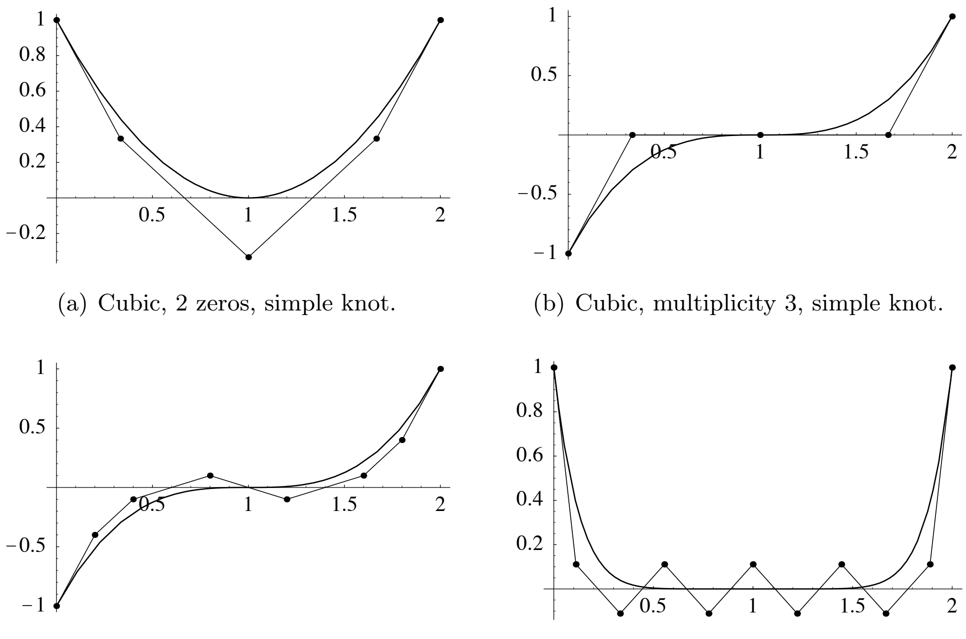
$$D^{d-r}\rho_{j,d}(z) = (d-r)! h^{d+1-j}(-h)^{r-d-1+j} = (d-r)!(-1)^{r-d-1+j} h^r$$

and (10.3) follows. ∎

Figures 10.1 (a)–(d) show some examples of splines with multiple zeros of the sort discussed in the proof of Theorem 10.5. All the knot vectors are $d+1$-regular on the interval $[0,2]$, with additional knots at $x=1$. In Figure 10.1 (a) there is one knot at $x=1$ and the spline is the polynomial $(x-1)^2$ which has a double zero at $x=1$. The control polygon models the spline in the normal way and has two sign changes. In Figure 10.1 (b) the knot vector is the same, but the spline is now the polynomial $(x-1)^3$. In this case the multiplicity of the zero is so high that the spline has a splitting point at $x=1$. The construction in the proof of Theorem 10.5 prescribes a knot vector with no knots at $x=1$ in this case. Figure 10.1 (c) shows the polynomial $(x-1)^3$ as a degree 5 spline on a 6-regular knot vector with a double knot at $x=1$. As promised by the theorem and its proof the coefficients change sign exactly three times. The spline in Figure 10.1 (d) is more extreme. It is the polynomial $(x-1)^8$ represented as a spline of degree 9 with one knot at $x=1$. The control polygon has the required 8 changes of sign.

10.2 Uniqueness of spline interpolation

Having established Theorem 10.5, we return to the problem of showing that the B-spline collocation matrix that occurs in spline interpolation, is nonsingular. We first consider Lagrange interpolation, and then turn to Hermite interpolation where we also allow interpolation derivatives.



(a) Cubic, 2 zeros, simple knot.

(b) Cubic, multiplicity 3, simple knot.

(c) Degree 5, multiplicity 3, double knot.

(d) Degree 9, multiplicity 8, simple knot.

Figure 10.1. Splines of varying degree with a varying number of zeros and knots at $x = 1$.

10.2.1 Lagrange Interpolation

In Chapter 8 we studied spline interpolation. With a spline space $\mathbb{S}_{d,\boldsymbol{t}}$ of dimension n and data $(y_i)_{i=1}^n$ given at n distinct points $x_1 < x_2 < \cdots < x_n$, the aim is to determine a spline $g = \sum_{i=1}^n c_i B_{i,d}$ in $\mathbb{S}_{d,\boldsymbol{t}}$ such that

$$g(x_i) = y_i, \qquad \text{for } i = 1, \ldots, n. \tag{10.4}$$

This leads to the linear system of equations

$$\boldsymbol{Ac} = \boldsymbol{y},$$

where

$$\boldsymbol{A} = \begin{pmatrix} B_{1,d}(x_1) & B_{2,d}(x_1) & \ldots & B_{n,d}(x_1) \\ B_{1,d}(x_2) & B_{2,d}(x_2) & \ldots & B_{n,d}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_{1,d}(x_n) & B_{2,d}(x_n) & \ldots & B_{n,d}(x_n) \end{pmatrix}, \qquad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

The matrix $\boldsymbol{A}$ is often referred to as the *B-spline collocation matrix.* Since $B_{i,d}(x)$ is nonzero only if $t_i < x < t_{i+d+1}$ (we may allow $t_i = x$ if $t_i = t_{i+d} < t_{i+d+1}$), the matrix $\boldsymbol{A}$ will in general be sparse. The following theorem tells us exactly when $\boldsymbol{A}$ is nonsingular.

Theorem 10.6. *Let $\mathbb{S}_{d,\boldsymbol{t}}$ be a given spline space, and let $x_1 < x_2 < \cdots < x_n$ be n distinct numbers. The collocation matrix $\boldsymbol{A}$ with entries $\big(B_{j,d}(x_i)\big)_{i,j=1}^n$ is nonsingular if and only if its diagonal is positive, i.e.,*

$$B_{i,d}(x_i) > 0 \qquad \text{for } i = 1, \ldots, n. \tag{10.5}$$

Proof. We start by showing that $\boldsymbol{A}$ is singular if a diagonal entry is zero. Suppose that $x_q \leq t_q$ (strict inequality if $t_q = t_{q+d} < t_{q+d+1}$) for some q so that $B_{q,d}(x_q) = 0$. By the support properties of B-splines we must have $a_{i,j} = 0$ for $i = 1, \ldots, q$ and $j = q, \ldots, n$. But this means that only the $n - q$ last entries of each of the last $n - q + 1$ columns of $\boldsymbol{A}$ can be nonzero; these columns must therefore be linearly dependent and $\boldsymbol{A}$ must be singular. A similar argument shows that $\boldsymbol{A}$ is also singular if $x_q \geq t_{q+d+1}$.

To show the converse, suppose that (10.5) holds but $\boldsymbol{A}$ is singular. Then there is a nonzero vector $\boldsymbol{c}$ such that $\boldsymbol{A}\boldsymbol{c} = 0$. Let $f = \sum_{i=1}^{n} c_i B_{i,d}$ denote the spline with B-spline coefficients $\boldsymbol{c}$. We clearly have $f(x_q) = 0$ for $q = 1, \ldots, n$. Let G denote the set

$$G = \cup_i \big\{(t_i, t_{i+d+1}) \mid c_i \neq 0\big\}.$$

Since each x in G must be in (t_i, t_{i+d+1}) for some i with $c_i \neq 0$, we note that G contains no splitting points of f. Note that if $x_i = t_i = t_{i+d} < t_{i+d+1}$ occurs at a knot of multiplicity $d + 1$, then $0 = f(x_i) = c_i$. To complete the proof, suppose first that G is an open interval. Since x_i is in G if $c_i \neq 0$, the number of zeros of f in G is greater than or equal to the number ℓ of nonzero coefficients in $\boldsymbol{c}$. Since we also have $S^-(\boldsymbol{c}) < \ell \leq Z_G(f)$, we have a contradiction to Theorem 10.5. In general G consists of several subintervals which means that f is not connected, but can be written as a sum of connected components, each defined on one of the subintervals. The above argument then leads to a contradiction on each subinterval, and hence we conclude that $\boldsymbol{A}$ is nonsingular. ∎

Theorem 10.6 makes it simple to ensure that the collocation matrix is nonsingular. We just place the knots and interpolation points in such a way that $t_i < x_i < t_{i+d+1}$ for $i = 1, \ldots, n$ (note again that if $t_i = t_{i+d} < t_{i+d+1}$, then $x_i = t_i$ is allowed).

10.2.2 Hermite Interpolation

In earlier chapters, particularly in Chapter 8, we made use of polynomial interpolation with Hermite data—data based on derivatives as well as function values. This is also of interest for splines, and as for polynomials this is conveniently indicated by allowing the interpolation point to coalesce. If for example $x_1 = x_2 = x_3 = x$, we take x_1 to signify interpolation of function value at x, the second occurrence of x signifies interpolation of first derivative, and the third tells us to interpolate second derivative at x. If we introduce the notation

$$\lambda_{\boldsymbol{x}}(i) = \max_j \{j \mid x_{i-j} = x_i\}$$

and assume that the interpolation points are given in nondecreasing order as $x_1 \leq x_2 \leq \cdots \leq x_n$, then the interpolation conditions are

$$D^{\lambda_{\boldsymbol{x}}(i)} g(x_i) = D^{\lambda_{\boldsymbol{x}}(i)} f(x_i) \tag{10.6}$$

where f is a given function and g is the spline to be determined. Since we are dealing with splines of degree d we cannot interpolate derivatives of higher order than d; we therefore assume that $x_i < x_{i+d+1}$ for $i = 1, \ldots, n - d - 1$. At a point of discontinuity (10.6) is to be interpreted according to our usual convention of taking limits from the right. The (i, j)-entry of the collocation matrix $\boldsymbol{A}$ is now given by

$$a_{i,j} = D^{\lambda_{\boldsymbol{x}}(i)} B_{j,d}(x_i),$$

and as before the interpolation problem is generally solvable if and only if the collocation matrix is nonsingular. Also as before, it turns out that the collocation matrix is nonsingular if and only if $t_i \leq x_i < t_{i+d+1}$, where equality is allowed in the first inequality only if $D^{\lambda_{\boldsymbol{x}}(i)} B_{i,d}(x_i) \neq 0$. This result will follow as a special case of our next theorem where we consider an even more general situation.

At times it is of interest to know exactly when a submatrix of the collocation matrix is nonsingular. The submatrices we consider are obtained by removing the same number of rows and columns from $\boldsymbol{A}$. Any columns may be removed, or equivalently, we consider a subset $\{B_{j_1,d}, \ldots, B_{j_\ell,d}\}$ of the B-splines. When removing rows we have to be a bit more careful. The convention is that if a row with derivatives of order r at z is included, then we also include all the lower order derivatives at z. This is most easily formulated by letting the sequence of interpolation points only contain ℓ points as in the following theorem.

Theorem 10.7. *Let $\mathbb{S}_{d,\boldsymbol{t}}$ be a spline space and let $\{B_{j_1,d}, \ldots, B_{j_\ell,d}\}$ be a subsequence of its B-splines. Let $x_1 \leq \cdots \leq x_\ell$ be a sequence of interpolation points with $x_i \leq x_{i+d+1}$ for $i = 1, \ldots, \ell - d - 1$. Then the $\ell \times \ell$ matrix $\boldsymbol{A}(\boldsymbol{j})$ with entries given by*

$$a_{i,q} = D^{\lambda_{\boldsymbol{x}}(i)} B_{j_q,d}(x_i)$$

for $i = 1, \ldots, \ell$ and $q = 1, \ldots, \ell$ is nonsingular if and only if

$$t_{j_i} \leq x_i < t_{j_i+d+1}, \qquad \text{for } i = 1, \ldots, \ell, \tag{10.7}$$

where equality is allowed in the first inequality if $D^{\lambda_{\boldsymbol{x}}(i)} B_{j_i,d}(x_i) \neq 0$.

Proof. The proof follows along the same lines as the proof of Theorem 10.6. The most challenging part is the proof that condition (10.7) is necessary so we focus on this. Suppose that (10.7) holds, but $\boldsymbol{A}(\boldsymbol{j})$ is singular. Then we can find a nonzero vector $\boldsymbol{c}$ such that $\boldsymbol{A}(\boldsymbol{j})\boldsymbol{c} = \boldsymbol{0}$. Let $f = \sum_{i=1}^{\ell} c_i B_{j_i,d}$ denote the spline with $\boldsymbol{c}$ as its B-spline coefficients, and let G denote the set

$$G = \cup_{i=1}^{\ell} \{(t_{j_i}, t_{j_i+d+1}) \mid c_i \neq 0\}.$$

To carry through the argument of Theorem 10.6 we need to verify that in the exceptional case where $x_i = t_{j_i}$ then $c_i = 0$.

Set $r = \lambda_{\boldsymbol{x}}(i)$ and suppose that the knot t_{j_i} occurs m times in $\boldsymbol{t}$ and that $t_{j_i} = x_i$ so $D^r B_{j_i,d}(x_i) \neq 0$. In other words

$$t_\mu < x_i = t_{\mu+1} = \cdots = t_{\mu+m} < t_{\mu+m+1}$$

for some integer μ, and in addition $j_i = \mu + k$ for some integer k with $1 \leq k \leq m$. Note that f satisfies

$$f(x_i) = Df(x_i) = \cdots = D^r f(x_i) = 0.$$

(Remember that if a derivative is discontinuous at x_i we take limits from the right.) Recall from Lemma 2.6 that all B-splines have continuous derivatives up to order $d - m$ at x_i. Since $D^r B_{j_i}$ clearly is discontinuous at x_i, it must be true that $r > d - m$. We therefore have $f(x_i) = Df(x_i) = \cdots = D^{d-m} f(x_i) = 0$ and hence $c_{\mu+m-d} = \cdots = c_\mu = 0$ by Lemma 10.2. The remaining interpolation conditions at x_i are $D^{d-m+1} f(x_i) = D^{d-m+2} f(x_i) = \cdots =$

$D^r f(x_i) = 0$. Let us consider each of these in turn. By the continuity properties of B-splines we have $D^{d-m+1}B_{\mu+1}(x_i) \neq 0$ and $D^{d-m+1}B_{\mu+\nu} = 0$ for $\nu > 1$. This means that

$$0 = D^{d-m+1}f(x_i) = c_{\mu+1}D^{d-m+1}B_{\mu+1}(x_i)$$

and $c_{\mu+1} = 0$. Similarly, we also have

$$0 = D^{d-m+2}f(x_i) = c_{\mu+2}D^{d-m+2}B_{\mu+2}(x_i),$$

and hence $c_{\mu+2} = 0$ since $D^{d-m+2}B_{\mu+2}(x_i) \neq 0$. Continuing this process we find

$$0 = D^r f(x_i) = c_{\mu+r+m-d}D^r B_{\mu+r+m-d}(x_i),$$

so $c_{\mu+r+m-d} = 0$ since $D^r B_{\mu+r+m-d}(x_i) \neq 0$. This argument also shows that j_i cannot be chosen independently of r; we must have $j_i = \mu + r + m - d$.

For the rest of the proof it is sufficient to consider the case where G is an open interval, just as in the proof of Theorem 10.6. Having established that $c_i = 0$ if $x_i = t_{j_i}$, we know that if $c_i \neq 0$ then $x_i \in G$. The number of zeros of f in G (counting multiplicities) is therefore greater than or equal to the number of nonzero coefficients. But this is impossible according to Theorem 10.5. ∎

10.3 Total positivity

In this section we are going to deduce another interesting property of the knot insertion matrix and the B-spline collocation matrix, namely that they are totally positive. We follow the same strategy as before and establish this first for the knot insertion matrix and then obtain the total positivity of the collocation matrix by recognising it as a submatrix of a knot insertion matrix.

Definition 10.8. *A matrix $\boldsymbol{A}$ in $\mathbb{R}^{m,n}$ is said to be totally positive if all its square submatrices have nonnegative determinant. More formally, let $\boldsymbol{i} = (i_1, i_2, \ldots, i_\ell)$ and $\boldsymbol{j} = (j_1, j_2, \ldots, j_\ell)$ be two integer sequences such that*

$$1 \leq i_1 < i_2 < \cdots < i_\ell \leq m, \tag{10.8}$$

$$1 \leq i_1 < i_2 < \cdots < i_\ell \leq n, \tag{10.9}$$

and let $\boldsymbol{A}(\boldsymbol{i},\boldsymbol{j})$ denote the submatrix of $\boldsymbol{A}$ with entries $(a_{i_p,j_q})_{p,q=1}^{\ell}$. Then $\boldsymbol{A}$ is totally positive if $\det \boldsymbol{A}(\boldsymbol{i},\boldsymbol{j}) \geq 0$ *for all sequences $\boldsymbol{i}$ and $\boldsymbol{j}$ on the form* (10.8) *and* (10.9), *for all ℓ with $1 \leq \ell \leq \min\{m, n\}$.*

We first show that knot insertion matrices are totally positive.

Theorem 10.9. *Let $\boldsymbol{\tau}$ and $\boldsymbol{t}$ be two knot vectors with $\boldsymbol{\tau} \subseteq \boldsymbol{t}$. Then the knot insertion matrix from $\mathbb{S}_{d,\boldsymbol{\tau}}$ to $\mathbb{S}_{d,\boldsymbol{t}}$ is totally positive.*

Proof. Suppose that there are k more knots in $\boldsymbol{t}$ than in $\boldsymbol{\tau}$; our proof is by induction on k. We first note that if $k = 0$, then $\boldsymbol{A} = I$, the identity matrix, while if $k = 1$, then $\boldsymbol{A}$ is a bi-diagonal matrix with one more rows than columns. Let us denote the entries of $\boldsymbol{A}$ by $\big(\alpha_j(i)\big)_{i,j=1}^{n+1,n}$ (if $k = 0$ the range of i is $1, \ldots, n$). In either case all the entries are nonnegative and $\alpha_j(i) = 0$ for $j < i - 1$ and $j > i$. Consider now the determinant of $\boldsymbol{A}(\boldsymbol{i},\boldsymbol{j})$. If $j_\ell \geq i_\ell$ then $j_\ell > i_q$ for $q = 1, \ldots, \ell-1$ so $\alpha_{j_\ell}(i_q) = 0$ for $q < \ell$. This means that only the last entry of the last column of $\boldsymbol{A}(\boldsymbol{i},\boldsymbol{j})$ is nonzero. The other possibility is that $j_\ell \leq i_\ell - 1$ so that $j_q < i_\ell - 1$ for $q < \ell$. Then $\alpha_{j_q}(i_\ell) = 0$ for $q < \ell$ so only the last entry of the last row of $\boldsymbol{A}(\boldsymbol{i},\boldsymbol{j})$ is nonzero. Expanding the determinant either by the last column or last row we therefore have $\det \boldsymbol{A}(\boldsymbol{i},\boldsymbol{j}) = \alpha_{j_\ell}(i_\ell) \det \boldsymbol{A}(\boldsymbol{i}',\boldsymbol{j}')$ where $\boldsymbol{i}' = (i_1, \ldots, i_{\ell-1})$ and $\boldsymbol{j}' = (j_1, \ldots, j_{\ell-1})$. Continuing this process we find that

$$\det \boldsymbol{A}(\boldsymbol{i},\boldsymbol{j}) = \alpha_{j_1}(i_1)\alpha_{j_2}(i_2)\cdots\alpha_{j_\ell}(i_\ell)$$

which clearly is nonnegative.

For $k \geq 2$, we make use of the factorization

$$\boldsymbol{A} = \boldsymbol{A}_k \cdots \boldsymbol{A}_1 = \boldsymbol{A}_k \boldsymbol{B}, \tag{10.10}$$

where each $\boldsymbol{A}_r$ corresponds to insertion of one knot and $\boldsymbol{B} = \boldsymbol{A}_{k-1} \cdots \boldsymbol{A}_1$ is the knot insertion matrix for inserting $k-1$ of the knots. By the induction hypothesis we know that both $\boldsymbol{A}_k$ and $\boldsymbol{B}$ are totally positive; we must show that $\boldsymbol{A}$ is totally positive. Let $(\boldsymbol{a}_i)$ and $(\boldsymbol{b}_i)$ denote the rows of $\boldsymbol{A}$ and $\boldsymbol{B}$, and let $\big(\alpha_j(i)\big)_{i,j=1}^{m,m-1}$ denote the entries of $\boldsymbol{A}_k$. From (10.10) we have

$$\boldsymbol{a}_i = \alpha_{i-1}(i)\boldsymbol{b}_{i-1} + \alpha_i(i)\boldsymbol{b}_i \qquad \text{for } i = 1, \ldots, m,$$

where $\alpha_0(1) = \alpha_m(m) = 0$. Let $\boldsymbol{a}_i(\boldsymbol{j})$ and $\boldsymbol{b}_i(\boldsymbol{j})$ denote the vectors obtained by keeping only entries $(j_q)_{q=1}^{\ell}$ of $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ respectively. Row q of $\boldsymbol{A}(\boldsymbol{i},\boldsymbol{j})$ of $\boldsymbol{A}$ is then given by

$$\boldsymbol{a}_{i_q}(\boldsymbol{j}) = \alpha_{i_q-1}(i_q)\boldsymbol{b}_{i_q-1}(\boldsymbol{j}) + \alpha_{i_q}(i_q)\boldsymbol{b}_{i_q}(\boldsymbol{j}).$$

Using the linearity of the determinant in row q we therefore have

$$\begin{aligned}
\det \begin{pmatrix} \boldsymbol{a}_{i_1}(\boldsymbol{j}) \\ \vdots \\ \boldsymbol{a}_{i_q}(\boldsymbol{j}) \\ \vdots \\ \boldsymbol{a}_{i_\ell}(\boldsymbol{j}) \end{pmatrix} &= \det \begin{pmatrix} \boldsymbol{a}_{i_1}(\boldsymbol{j}) \\ \vdots \\ \alpha_{i_q-1}(i_q)\boldsymbol{b}_{i_q-1}(\boldsymbol{j}) + \alpha_{i_q}(i_q)\boldsymbol{b}_{i_q}(\boldsymbol{j}) \\ \vdots \\ \boldsymbol{a}_{i_\ell}(\boldsymbol{j}) \end{pmatrix} \\
&= \alpha_{i_q-1}(i_q) \det \begin{pmatrix} \boldsymbol{a}_{i_1}(\boldsymbol{j}) \\ \vdots \\ \boldsymbol{b}_{i_q-1}(\boldsymbol{j}) \\ \vdots \\ \boldsymbol{a}_{i_\ell}(\boldsymbol{j}) \end{pmatrix} + \alpha_{i_q}(i_q) \det \begin{pmatrix} \boldsymbol{a}_{i_1}(\boldsymbol{j}) \\ \vdots \\ \boldsymbol{b}_{i_q}(\boldsymbol{j}) \\ \vdots \\ \boldsymbol{a}_{i_\ell}(\boldsymbol{j}) \end{pmatrix}.
\end{aligned}$$

By expanding the other rows similarly we find that $\det \boldsymbol{A}(\boldsymbol{i},\boldsymbol{j})$ can be written as a sum of determinants of submatrices of $\boldsymbol{B}$, multiplied by products of $\alpha_j(i)$'s. By the induction hypothesis all these quantities are nonnegative, so the determinant of $\boldsymbol{A}(\boldsymbol{i},\boldsymbol{j})$ must also be nonnegative. Hence $\boldsymbol{A}$ is totally positive. ∎

Knowing that the knot insertion matrix is totally positive, we can prove a similar property of the B-spline collocation matrix, even in the case where multiple collocation points are allowed.

Theorem 10.10. *Let $\mathbb{S}_{d,\boldsymbol{\tau}}$ be a spline space and let $\{B_{j_1,d}, \ldots, B_{j_\ell,d}\}$ be a subsequence of its B-splines. Let $x_1 \leq \cdots \leq x_\ell$ be a sequence of interpolation points with $x_i \leq x_{i+d+1}$ for $i = 1, \ldots, \ell - d - 1$, and denote by $\boldsymbol{A}(\boldsymbol{j})$ the $\ell \times \ell$ matrix with entries given by*

$$a_{i,q} = D^{\lambda_{\boldsymbol{x}}(i)} B_{j_q,d}(x_i)$$

for $i = 1, \ldots, \ell$ and $q = 1, \ldots, \ell$. Then

$$\det \boldsymbol{A}(\boldsymbol{j}) \geq 0.$$

Proof. We first prove the claim in the case $x_1 < x_2 < \cdots < x_\ell$. By inserting knots of multiplicity $d+1$ at each of $(x_i)_{i=1}^{\ell}$ we obtain a knot vector $\boldsymbol{t}$ that contains $\boldsymbol{\tau}$ as a subsequence. If $t_{i-1} < t_i = t_{i+d} < t_{i+d+1}$ we know from Lemma 2.6 that $B_{j,d,\boldsymbol{\tau}}(t_i) = \alpha_{j,d}(i)$. This means that the matrix $\boldsymbol{A}(\boldsymbol{j})$ appears as a submatrix of the knot insertion matrix from $\boldsymbol{\tau}$ to $\boldsymbol{t}$. It therefore follows from Theorem 10.9 that $\det \boldsymbol{A}(\boldsymbol{j}) \geq 0$ in this case.

To prove the theorem in the general case we consider a set of distinct collocation points $y_1 < \cdots < y_\ell$ and let $\boldsymbol{A}(\boldsymbol{j}, \boldsymbol{y})$ denote the corresponding collocation matrix. Set $\lambda^i = \lambda_{\boldsymbol{x}}(i)$ and let ρ_i denote the linear functional given by

$$\rho_i f = \lambda^i!\,[y_{i-\lambda^i}, \ldots, y_i]f \tag{10.11}$$

for $i = 1, \ldots, \ell$. Here $[\cdot, \ldots, \cdot]f$ is the divided difference of f. By standard properties of divided differences we have

$$\rho_i B_{j,d} = \sum_{s=i-\lambda^i}^{i} \gamma_{i,s} B_{j,d}(y_s)$$

and $\gamma_{i,i} > 0$. Denoting by $\boldsymbol{D}$ the matrix with (i,j)-entry $\rho_i B_{j,d}$, we find by properties of determinants and (10.11) that

$$\det \boldsymbol{D} = \gamma_{1,1} \cdots \gamma_{\ell,\ell} \det \boldsymbol{A}(\boldsymbol{j}, \boldsymbol{y}).$$

If we now let $\boldsymbol{y}$ tend to $\boldsymbol{x}$ we know from properties of the divided difference functional that $\rho_i B_j$ tends to $D^{\lambda^i} B_j$ in the limit. Hence $\boldsymbol{D}$ tends to $\boldsymbol{A}(\boldsymbol{j})$ so $\det \boldsymbol{A}(\boldsymbol{j}) \geq 0$. ∎