CHAPTER 9

Approximation theory and stability

Polynomials of degree d have $d+1$ degrees of freedom, namely the $d+1$ coefficients relative to some polynomial basis. It turns out that each of these degrees of freedom can be utilised to gain approximation power so that the possible rate of approximation by polynomials of degree d is h^{d+1}, see Section 9.1. The meaning of this is that when a smooth function is approximated by a polynomial of degree d on an interval of length h, the error is bounded by Ch^{d+1}, where C is a constant that is independent of h. The exponent $d+1$ therefore controls how fast the error tends to zero with h.

When several polynomials are linked smoothly together to form a spline, each polynomial piece has $d+1$ coefficients, but some of these are tied up in satisfying the smoothness conditions. It therefore comes as a nice surprise that the approximation power of splines of degree d is the same as for polynomials, namely h^{d+1}, where h is now the largest distance between two adjacent knots. In passing from polynomials to splines we have therefore gained flexibility without sacrificing approximation power. We prove this in Section 9.2, by making use of some of the simple quasi-interpolants that we constructed in Chapter 8; it turns out that these produce spline approximations with the required accuracy.

The quasi-interpolants also allow us to establish two important properties of B-splines. The first is that B-splines form a stable basis for splines, see Section 9.3. This means that small perturbations of the B-spline coefficients can only lead to small perturbations in the spline, which is of fundamental importance for numerical computations. An important consequence of the stability of the B-spline basis is that the control polygon of a spline converges to the spline as the knot spacing tends to zero; this is proved in Section 9.4.

9.1 The distance to polynomials

We start by determining how well a given real valued function f defined on an interval $[a, b]$ can be approximated by a polynomial of degree d. We measure the error in the approximation with the uniform norm which for a bounded function g defined on an interval $[a, b]$ is defined by

$$\|g\|_{\infty,[a,b]} = \sup_{a \le x \le b} \big|g(x)\big|.$$

Whenever we have an approximation p to f we can use the norm and measure the error by $\|f-p\|_{\infty,[a,b]}$. There are many possible approximations to f by polynomials of degree d, and the approximation that makes the error as small as possible is of course of special interest. This approximation is referred to as the *best approximation* and the corresponding error is referred to as the *distance* from f to the space π_d of polynomials of degree $\leq d$. This is defined formally as

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_d) = \inf_{p \in \pi_d} \|f - p\|_{\infty,[a,b]}.$$

In order to bound this approximation error, we have to place some restrictions on the functions that we approximate, and we will only consider functions with piecewise continuous derivatives. Such functions lie in a space that we denote $\mathbb{C}^k_{\boldsymbol{\Delta}}[a,b]$ for some integer $k \geq 0$. A function f lies in this space if it has $k-1$ continuous derivatives on the interval $[a,b]$, and the kth derivative $D^k f$ is continuous everywhere except for a finite number of points in the interior (a,b), given by $\boldsymbol{\Delta} = (z_j)$. At the points of discontinuity $\boldsymbol{\Delta}$ the limits from the left and right, given by $D^k f(z_j+)$ and $D^k f(z_j-)$, should exist so all the jumps are finite. If there are no continuous derivatives we write $\mathbb{C}_{\boldsymbol{\Delta}}[a,b] = \mathbb{C}^0_{\boldsymbol{\Delta}}[a,b]$. Note that we will often refer to these spaces without stating explicitly what the singularities $\boldsymbol{\Delta}$ are.

It is quite simple to give an upper bound for the distance of f to polynomials of degree d by choosing a particular approximation, namely Taylor expansion.

Theorem 9.1. *Given a polynomial degree d and a function f in $\mathbb{C}^{d+1}_{\boldsymbol{\Delta}}[a,b]$, then*

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_d) \leq C_d h^{d+1} \|D^{d+1} f\|_{\infty,[a,b]},$$

where $h = b - a$ and the constant C_d only depends on d,

$$C_d = \frac{1}{2^{d+1}(d+1)!}.$$

Proof. Consider the truncated Taylor series of f at the midpoint $m = (a+b)/2$ of $[a,b]$,

$$T_d f(x) = \sum_{k=0}^{d} \frac{(x-m)^k}{k!} D^k f(m), \quad \text{for } x \in [a,b].$$

Since $T_d f$ is a polynomial of degree d we clearly have

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_d) \leq \|f - T_d f\|_{\infty,[a,b]}. \tag{9.1}$$

The error is given by the integral form of the remainder in the Taylor expansion,

$$f(x) - T_d f(x) = \frac{1}{d!} \int_m^x (x-y)^d D^{d+1} f(y) dy,$$

which is valid for any $x \in [a,b]$. If we restrict x to the interval $[m,b]$ we obtain

$$|f(x) - T_d f(x)| \leq \|D^{d+1} f\|_{\infty,[a,b]} \frac{1}{d!} \int_m^x (x-y)^d dy.$$

The integral is given by

$$\frac{1}{d!} \int_m^x (x-y)^d dy = \frac{1}{(d+1)!} (x-m)^{d+1} \leq \frac{1}{(d+1)!} \left(\frac{h}{2}\right)^{d+1},$$

so for $x \geq m$ we have

$$\left|f(x) - T_d f(x)\right| \leq \frac{1}{2^{d+1}(d+1)!} h^{d+1} \|D^{d+1} f\|_{\infty,[a,b]}.$$

By symmetry this estimate must also hold for $x \leq m$. Combining the estimate with (9.1) completes the proof. ∎

It is in fact possible to compute the best possible constant C_d. It turns out that for each $f \in \mathbb{C}^{d+1}[a,b]$ there is a point $\xi \in [a,b]$ such that

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_d) = \frac{2}{4^{d+1}(d+1)!} h^{d+1} |D^{d+1} f(\xi)|$$

Applying this formula to the function $f(x) = x^{d+1}$ we see that the exponent $d+1$ in h^{d+1} is best possible.

9.2 The distance to splines

Just as we defined the distance from a function f to the space of polynomials of degree d we can define the distance from f to a spline space. Our aim is to show that on one knot interval, the distance from f to a spline space of degree d is essentially the same as the distance from f to the space of polynomials of degree d on a slightly larger interval, see Theorem 9.2 and Corollary 9.12. Our strategy is to consider the cases $d = 0$, 1 and 2 separately and then generalise to degree d. The main ingredient in the proof is to construct a simple but good approximation method that we can use in the same way that Taylor expansion was used in the polynomial case above. Some of the quasi-interpolants that we constructed in Chapter 8 will do this job very nicely.

We consider a spline space $\mathbb{S}_{d,\boldsymbol{\tau}}$ where d is a nonnegative integer and $\boldsymbol{\tau} = (\tau_i)_{i=1}^{n+d+1}$ is a $d+1$ regular knot vector and set

$$a = \tau_1, \quad b = \tau_{n+d+1}, \quad h_j = \tau_{j+1} - \tau_j, \quad h = \max_{1 \leq j \leq n} h_j.$$

Given a function f we consider the distance from f to $\mathbb{S}_{d,\boldsymbol{\tau}}$ defined by

$$\operatorname{dist}_{\infty,[a,b]}(f, \mathbb{S}_{d,\boldsymbol{\tau}}) = \inf_{g \in \mathbb{S}_{d,\boldsymbol{\tau}}} \|f - g\|_{\infty,[a,b]}.$$

We want to show the following.

Theorem 9.2. *Let the polynomial degree d and the function f in $\mathbb{C}^{d+1}_{\boldsymbol{\Delta}}[a,b]$ be given. The distance between f and the spline space $\mathbb{S}_{d,\boldsymbol{\tau}}$ is bounded by*

$$\operatorname{dist}_{\infty,[a,b]}(f, \mathbb{S}_{d,\boldsymbol{\tau}}) \leq D_d h^{d+1} \|D^{d+1} f\|_{\infty,[a,b]}, \tag{9.2}$$

where the constant D_d depends on d, but not on f or $\boldsymbol{\tau}$.

We will prove this theorem by constructing a spline $P_d f$ such that

$$|f(x) - P_d f(x)| \leq D_d h^{d+1} \|D^{d+1} f\|_{\infty,[a,b]}, \quad x \in [a,b] \tag{9.3}$$

for a constant D_d that depends only on d. The approximation $P_d f$ will be a quasi-interpolant on the form

$$P_d f = \sum_{i=1}^{n} \lambda_i(f) B_{i,d}$$

where λ_i is a rule for computing the ith B-spline coefficient. We will restrict ourselves to rules λ_i like

$$\lambda_i(f) = \sum_{k=0}^{d} w_{i,k} f(x_{i,k})$$

where the points $(x_{i,k})_{k=0}^{d}$ all lie in one knot interval and $(w_{i,k})_{k=0}^{d}$ are suitable coefficients.

9.2.1 The constant and linear cases

We first prove Theorem 9.2 in the simplest cases $d = 0$ and $d = 1$. For $d = 0$ the knots form a partition $a = \tau_1 < \cdots < \tau_{n+1} = b$ of $[a, b]$ and the B-spline $B_{i,0}$ is the characteristic function of the interval $[\tau_i, \tau_{i+1})$ for $i = 1, \ldots, n-1$, while $B_{n,0}$ is the characteristic function of the closed interval $[\tau_n, \tau_{n+1}]$. We consider the step function

$$g(x) = P_0 f(x) = \sum_{i=1}^{n} f(\tau_{i+1/2}) B_{i,0}(x), \tag{9.4}$$

where $\tau_{i+1/2} = (\tau_i + \tau_{i+1})/2$. Fix $x \in [a, b]$ and let μ be an integer such that $\tau_\mu \le x < \tau_{\mu+1}$. We then have

$$f(x) - P_0 f(x) = f(x) - f(\tau_{\mu+1/2}) = \int_{\tau_{\mu+1/2}}^{x} Df(y)dy$$

so

$$\bigl|f(x) - P_0 f(x)\bigr| \le |x - \tau_{\mu+1/2}|\, \|Df\|_{\infty,[\tau_\mu,\tau_{\mu+1}]} \le \frac{h}{2}\|Df\|_{\infty,[a,b]}.$$

In this way we obtain (9.2) with $D_0 = 1/2$.

In the linear case $d = 1$ we define $P_1 f$ to be the piecewise linear interpolant to f on $\boldsymbol{\tau}$ defined by

$$g = P_1 f = \sum_{i=1}^{n} f(\tau_{i+1}) B_{i,1}. \tag{9.5}$$

Proposition 5.2 gives an estimate of the error in linear interpolation and by applying this result on each interval we obtain

$$\|f - P_1 f\|_{\infty,[a,b]} \le \frac{h^2}{8}\|D^2 f\|_{\infty,[a,b]}$$

which is (9.2) with $D_1 = 1/8$.

9.2.2 The quadratic case

The quadratic case $d = 2$ is more involved. We shall approximate f by the quasi-interpolant $P_2 f$ that we constructed in Section 8.2.2 and then estimate the error. The relevant properties of P_2 are summarised in the following lemma.

Lemma 9.3. *Suppose $\boldsymbol{\tau} = (\tau_i)_{i=1}^{n+3}$ is a knot vector with $\tau_{i+3} > \tau_i$ for $i = 1, \ldots, n$ and set $t_{i+3/2} = (t_{i+1} + t_{i+2})/2$. The operator*

$$P_2 f = \sum_{i=1}^{n} \lambda_i(f) B_{i,2,\boldsymbol{\tau}} \quad \textit{with} \quad \lambda_i(f) = -\frac{1}{2} f(\tau_{i+1}) + 2f(\tau_{i+3/2}) - \frac{1}{2} f(\tau_{i+2}) \tag{9.6}$$

is linear and satisfies $P_2 f = f$ for all $f \in \mathbb{S}_{2,\boldsymbol{\tau}}$.

Note that since the knot vector is 3-regular we have $\lambda_1(f) = f(t_2)$ and $\lambda_n(f) = f(t_{n+1})$. We also note that since P_2 reproduces all splines in $\mathbb{S}_{d,\boldsymbol{\tau}}$ it certainly reproduces all quadratic polynomial. This fact that will be useful in the proof of Lemma 9.6.

Our aim is to show that (9.3) holds for $d = 2$ and we are going to do this by establishing a sequence of lemmas. The first lemma shows that $\lambda_i(f)$ can become at most 3 times as large as f, irrespective of what the knot vector is.

Lemma 9.4. *Let $P_2(f)$ be as in (9.6). Then*

$$\left|\lambda_i(f)\right| \leq 3\|f\|_{\infty,[\tau_{i+1},\tau_{i+2}]}, \quad \textit{for } i = 1, \ldots, n. \tag{9.7}$$

Proof. Fix an integer i. Then

$$\left|\lambda_i(f)\right| = \left|-\frac{1}{2} f(\tau_{i+1}) + 2f(\tau_{i+3/2}) - \frac{1}{2} f(\tau_{i+2})\right| \leq \left(\frac{1}{2} + 2 + \frac{1}{2}\right)\|f\|_{\infty,[\tau_{i+1},\tau_{i+2}]}$$

from which the result follows. ∎

Since the B-spline coefficients of $P_2 f$ are bounded it is easy to see that the spline $P_2 f$ is also bounded by the same constant.

Lemma 9.5. *Select some interval $[\tau_\mu, \tau_{\mu+1})$ of $[\tau_3, \tau_{n+1})$. On this interval the spline $P_2 f$ is bounded by*

$$\|P_2 f\|_{\infty,[\tau_\mu,\tau_{\mu+1}]} \leq 3\|f\|_{\infty,[\tau_{\mu-1},\tau_{\mu+2}]}. \tag{9.8}$$

Proof. Fix $x \in [\tau_\mu, \tau_{\mu+1}]$. Since the B-splines are nonnegative and form a partition of unity we have

$$\begin{aligned} |P_2 f(x)| &= \Big|\sum_{i=\mu-2}^{\mu} \lambda_i(f) B_{i,2,\boldsymbol{\tau}}(x)\Big| \leq \max_{\mu-2 \leq i \leq \mu} |\lambda_i(f)| \\ &\leq 3 \max_{\mu-2 \leq i \leq \mu} \|f\|_{\infty,[\tau_{i+1},\tau_{i+2}]} = 3\|f\|_{\infty,[\tau_{\mu-1},\tau_{\mu+2}]}, \end{aligned}$$

where we used Lemma 9.4. This completes the proof. ∎

The following lemma shows that on one knot interval the spline $P_2 f$ approximates f almost as well as the best quadratic polynomial over a slightly larger interval. The proof depends on a standard trick that we will also use in the general case.

Lemma 9.6. *Let $[\tau_\mu, \tau_{\mu+1})$ be a subinterval of $[\tau_3, \tau_{n+1})$. On this interval the error $f - P_2 f$ is bounded by*

$$\|f - P_2 f\|_{\infty,[\tau_\mu,\tau_{\mu+1}]} \leq 4 \operatorname{dist}_{\infty,[\tau_{\mu-1},\tau_{\mu+2}]}(f, \pi_2). \tag{9.9}$$

Proof. Let $p \in \pi_2$ be any quadratic polynomial. Since $P_2 p = p$ and P_2 is a linear operator, application of (9.8) to $f - p$ yields

$$\begin{aligned} \big|f(x) - (P_2 f)(x)\big| &= \big|f(x) - p(x) - \big((P_2 f)(x) - p(x)\big)\big| \\ &\le \big|f(x) - p(x)\big| + \big|P_2(f-p)(x)\big| \\ &\le (1+3)\|f-p\|_{\infty,[\tau_{\mu-1},\tau_{\mu+2}]}. \end{aligned} \tag{9.10}$$

Since p is arbitrary we obtain (9.9). ∎

Proof of Theorem 9.2 for $d = 2$. Theorem 9.1 with $d = 2$ states that

$$\operatorname{dist}_{\infty,[a,b]}(f, \pi_2) \le C_2 h^3 \|D^3 f\|_{\infty,[a,b]},$$

where $h = b - a$ and $C_2 = 1/(2^3\, 3!)$. Specialising this estimate to the interval $[a, b] = [\tau_{\mu-1}, \tau_{\mu+2}]$ and combining with (9.9) we obtain (9.3) and hence (9.2) with $D_2 = 1/12$. ∎

9.2.3 The general case

The general case is analogous to the quadratic case, but the details are more involved. The crucial part is to find a sufficiently good local approximation operator. The operator P_2 is a quasi interpolant that is based on local interpolation with quadratic polynomials at the three points $x_{i,k} = \tau_{i+1} + k(\tau_{i+2} - \tau_{i+1})/2$ for $k = 0,\ 1,\ 2$. Those points are located symmetrically in the middle subinterval of the support of the B-spline $B_{i,2}$.

We will follow the same strategy for general degree. The resulting quasi-interpolant will be a special case of the one given in Theorem 8.7. The challenge is to choose the local interpolation points in such a way that the B-spline coefficients of the approximation can be bounded independently of the knots, as in Lemma 9.4. The key is to let all the $d + 1$ points be uniformly distributed in the *largest* subinterval $[a_i, b_i] = [\tau_\mu, \tau_{\mu+1}]$ of $[\tau_{i+1}, \tau_{i+d}]$,

$$x_{i,k} = a_i + \frac{k}{d}(b_i - a_i), \quad \text{for } k = 0,\ 1,\ \ldots,\ d. \tag{9.11}$$

Given $f \in \mathbb{C}_{\boldsymbol{\Delta}}[a, b]$ we define $P_d f \in \mathbb{S}_{d,\boldsymbol{\tau}}$ by

$$P_d f(x) = \sum_{i=1}^{n} \lambda_i(f) B_{i,d}(x), \quad \text{where} \quad \lambda_i(f) = \sum_{k=0}^{d} w_{i,k} f(x_{i,k}). \tag{9.12}$$

In this situation Theorem 8.7 specialises to the following.

Lemma 9.7. *Suppose that the functionals λ_i in (9.12) are given by $\lambda_i(f) = f(\tau_{i+1})$ if $\tau_{i+d} = \tau_{i+1}$, while if $\tau_{i+d} > \tau_{i+1}$ the coefficients of $\lambda_i(f)$ are given by*

$$w_{i,k} = \gamma_i(p_{i,k}), \quad \text{for } k = 0,\ 1,\ \ldots,\ d, \tag{9.13}$$

where $\gamma_i(p_{i,k})$ is the ith B-spline coefficient of the polynomial

$$p_{i,k}(x) = \prod_{\substack{j=0 \\ j \ne k}}^{d} \frac{x - x_{i,j}}{x_{i,k} - x_{i,j}}. \tag{9.14}$$

Then the operator P_d in (9.12) satisfies $P_d f = f$ for all $f \in \mathbb{S}_{d,\boldsymbol{\tau}}$.

We really only need reproduction of polynomials of degree d, but since all the interpolation points lie in one knot interval we automatically get reproduction of all of $\mathbb{S}_{d,\boldsymbol{\tau}}$.

The first challenge is to find a formula for the B-spline coefficients of $p_{i,k}$. Blossoming makes this easy.

Lemma 9.8. *Suppose the spline space $\mathbb{S}_{d,\boldsymbol{\tau}}$ is given together with the numbers $v_1, \ldots, v_d$. The ith B-spline coefficient of the polynomial $p(x) = (x - v_1) \ldots (x - v_d)$ can be written*

$$\gamma_i(p) = \frac{1}{d!} \sum_{(j_1,\ldots,j_d)\in\Pi_d} (t_{i+j_1} - v_1) \cdots (t_{i+j_d} - v_d), \tag{9.15}$$

where Π_d is the set of all permutations of the integers $\{1, 2, \ldots, d\}$.

Proof. By Theorem 4.16 we have

$$\gamma_i(p) = \mathcal{B}[p](\tau_{i+1}, \ldots, \tau_{i+d}),$$

where $\mathcal{B}[p]$ is the blossom of p. It therefore suffices to verify that the expression (9.15) satisfies the three properties of the blossom. This is simple and is left to the reader. ∎

Let us consider the special case $d = 2$ as an example. The set of all permutations of $\{1, 2\}$ are $\Pi_2 = \{(1, 2), (2, 1)\}$ and therefore

$$\gamma_i\big((x - v_1)(x - v_2)\big) = \frac{1}{2}\Big((\tau_{i+1} - v_1)(\tau_{i+2} - v_2) + (\tau_{i+2} - v_1)(\tau_{i+1} - v_2)\Big).$$

The next and most difficult step is to obtain a bound for $\lambda_i(f)$.

Theorem 9.9. *Let $P_d(f) = \sum_{i=1}^{n} \lambda_i(f) B_{i,d}$ be the operator in Lemma 9.7. Then*

$$|\lambda_i(f)| \le K_d \|f\|_{\infty,[\tau_{i+1},\tau_{i+d}]}, \quad i = 1, \ldots, n, \tag{9.16}$$

where

$$K_d = \frac{2^d}{d!}\big(d(d-1)\big)^d \tag{9.17}$$

depends only on d.

Proof. Fix an integer i. We may as well assume that $\tau_{i+1} < \tau_{i+d}$ since otherwise the result is obvious. From Lemma 9.8 we have

$$w_{i,k} = \sum_{(j_1,\ldots,j_d)\in\Pi_d} \prod_{r=1}^{d} \left(\frac{\tau_{i+j_r} - v_r}{x_{i,k} - v_r}\right)/d!, \tag{9.18}$$

where $(v_r)_{r=1}^{d} = (x_{i,0}, \ldots, x_{i,k-1}, x_{i,k+1}, \ldots, x_{i,d})$. and Π_d denotes the set of all permutations of the integers $\{1, 2, \ldots, d\}$. Since the numbers τ_{i+j_r} and v_r belongs to the interval $[\tau_{i+1}, \tau_{i+d}]$ for all r we have the inequality

$$\prod_{r=1}^{d} (\tau_{i+j_r} - v_r) \le (\tau_{i+d} - \tau_{i+1})^d. \tag{9.19}$$

We also note that $x_{i,k} - v_r = (k-q)(b_i - a_i)/d$ for some q in the range $1 \le q \le d$ but with $q \ne k$. Taking the product over all r we therefore obtain

$$\begin{aligned}\prod_{r=1}^{d} |x_{i,k} - v_r| &= \prod_{\substack{q=0\\ q\neq k}}^{d} \frac{|k-q|}{d}(b_i - a_i)\\ &= k!(d-k)!\left(\frac{b_i - a_i}{d}\right)^d \ge k!(d-k)!\left(\frac{\tau_{i+d} - \tau_{i+1}}{d(d-1)}\right)^d\end{aligned} \tag{9.20}$$

for all values of k and r since $[a_i, b_i]$ is the largest subinterval of $[\tau_{i+1}, \tau_{i+d}]$. The sum in (9.18) contains $d!$ terms which means that

$$\sum_{k=0}^{d} |w_{i,k}| \le \frac{[d(d-1)]^d}{d!}\sum_{k=0}^{d}\binom{d}{k} = \frac{2^d}{d!}[d(d-1)]^d = K_d$$

and therefore

$$\big|\lambda_i(f)\big| \le \|f\|_{\infty,[\tau_{i+1},\tau_{i+d}]}\sum_{k=0}^{d}|w_{i,k}| \le K_d\|f\|_{\infty,[\tau_{i+1},\tau_{i+d}]} \tag{9.21}$$

which is the required inequality. ∎

Theorem 9.9 is the central ingredient in the proof of Theorem 9.2, but it has many other consequences as well, some of which we will consider later in this chapter. In fact Theorem 9.9 gives one of the key properties of B-splines. If $f = \sum_{i=1}^{n} c_i B_{i,d,\boldsymbol{\tau}}$ is a spline in $\mathbb{S}_{d,\boldsymbol{\tau}}$ we know that $\lambda_i(f) = c_i$. The inequality (9.16) therefore states that a B-spline coefficient is at most K_d times larger than the spline it represents, where the constant K_d is *independent* of the knots. A similar conclusion holds for $d \le 2$, see Lemma 9.4 and the definition of P_0 and P_1 in (9.4) and (9.5). For later reference we record this in a corollary.

Corollary 9.10. *For any spline $f = \sum_{i=1}^{n} c_i B_{i,d}$ in $\mathbb{S}_{d,\boldsymbol{\tau}}$ the size of the B-spline coefficients is bounded by*

$$|c_i| \le K_d\|f\|_{\infty,[\tau_{i+1},\tau_{i+d}]},$$

where the the constant K_d depends only on d.

From the bound on $\lambda_i(f)$ we easily obtain a similar bound for the norm of $P_d f$.

Theorem 9.11. *Let f be a function in the space $C_{\boldsymbol{\Delta}}[a,b]$. On any subinterval $[\tau_\mu, \tau_{\mu+1})$ of $[\tau_{d+1}, \tau_{n+1})$ the approximation $P_d f$ is bounded by*

$$\|P_d f\|_{\infty,[\tau_\mu,\tau_{\mu+1}]} \le K_d\|f\|_{\infty,[\tau_{\mu-d+1},\tau_{\mu+d}]}, \tag{9.22}$$

where K_d is the constant in Theorem 9.9.

Proof. Fix an x in some interval $[\tau_\mu, \tau_{\mu+1})$. Since the B-splines are nonnegative and form a partition of unity we have by Theorem 9.9

$$\begin{aligned}\big|P_d f(x)\big| &= \Big|\sum_{i=\mu-d}^{\mu}\lambda_i(f)B_{i,d,\boldsymbol{\tau}}(x)\Big| \le \max_{\mu-d\le i\le\mu}\big|\lambda_i(f)\big|\\ &\le K_d \max_{\mu-d\le i\le\mu}\|f\|_{\infty,[\tau_{i+1},\tau_{i+d}]} = K_d\|f\|_{\infty,[\tau_{\mu-d+1},\tau_{\mu+d}]}\end{aligned}$$

This completes the proof. ∎

The following corollary shows that $P_d f$ locally approximates f essentially as well as the best polynomial approximation of f of degree d.

Corollary 9.12. *On any subinterval $[\tau_\mu, \tau_{\mu+1})$ the error $f - P_d f$ is bounded by*

$$\|f - P_d f\|_{\infty,[\tau_\mu,\tau_{\mu+1}]} \leq (1 + K_d)\,\mathrm{dist}_{\infty,[\tau_{\mu-d+1},\tau_{\mu+d}]}(f, \pi_d), \tag{9.23}$$

where K_d is the constant in Theorem 9.9

Proof. We argue exactly as in the quadratic case. Let $p \in \pi_d$ be any polynomial in π_d. Since $P_d\, p = p$ and P_d is a linear operator we have

$$\begin{aligned}\bigl|f(x) - (P_d f)(x)\bigr| &= \bigl|f(x) - p(x) - \bigl((P_d f)(x) - p(x)\bigr)\bigr| \\ &\leq \bigl|f(x) - p(x)\bigr| + \bigl|P_d(f - p)(x)\bigr| \\ &\leq (1 + K_d)\|f - p\|_{\infty,[\tau_{\mu-d+1},\tau_{\mu+d}]}.\end{aligned}$$

Since p is arbitrary we obtain (9.23). ∎

Proof of Theorem 9.2 for general d. By Theorem 9.1 we have for any interval $[a, b]$

$$\mathrm{dist}_{\infty,[a,b]}(f, \pi_d) \leq C_d h^{d+1}\|D^{d+1} f\|_{\infty,[a,b]},$$

where $h = b - a$ and C_d only depends on d. Combining this estimate on $[a, b] = [\tau_{\mu-d+1}, \tau_{\mu+d}]$ with (9.23) we obtain (9.3) and hence (9.2) with $D_d = (K_d + 1)C_d$. ∎

We have accomplished our task of estimating the distance from a function in $\mathbb{C}^{d+1}_{\boldsymbol{\Delta}}[a, b]$ to an arbitrary spline space $\mathbb{S}_{d,\boldsymbol{\tau}}$. However, there are several unanswered questions. Perhaps the most obvious is whether the constant K_d is the best possible. A moment's thought will make you realise that it certainly is not. One reason is that we made use of some rather coarse estimates in the proof of Theorem 9.9. Another reason is that we may obtain better estimates by using a different approximation operator.

In fact, it is quite easy to find a better operator which is also a quasi-interpolant based on local interpolation. Instead of choosing the local interpolation points uniformly in the largest subinterval of $[\tau_{i+1}, \tau_{i+d}]$, we simply choose the points uniformly in $[\tau_{i+1}, \tau_{i+d}]$,

$$x_{i,k} = \tau_{i+1} + \frac{k}{d}(\tau_{i+d} - \tau_{i+1}), \quad \text{for } k = 0, 1, \ldots, d.$$

It is easy to check that the bound (9.19) on the numerator still holds while the last estimate in the bound on the denominator (9.20) is now unnecessary so we have

$$\prod_{r=1}^{d} |x_{i,k} - v_r| = \prod_{\substack{q=0\\ q\neq k}}^{d} \frac{|k - q|}{d}(\tau_{i+d} - \tau_{i+1}) = \frac{k!(d - k)!}{d^d}(\tau_{i+d} - \tau_{i+1})^d.$$

This gives a new constant

$$\tilde{K}_d = \frac{2^d d^d}{d!}.$$

Note that the new approximation operator will not reproduce the whole spline space for $d > 2$. This improved constant can therefore not be used in Corollary 9.10.

The constant can be improved further by choosing the interpolation points to be the extrema of the Chebyshev polynomial, adjusted to the interval $[\tau_{i+1}, \tau_{i+d}]$.

9.3 Stability of the B-spline basis

In order to compute with polynomials or splines we need to choose a basis to represent the functions. If a basis is to be suitable for computer manipulations it should be reasonably insensitive to round-off errors. In particular, functions with 'small' function values should have 'small' coefficients and vice versa. A basis with this property is said to be *well conditioned* or *stable* and the stability is measured by the *condition number* of the basis. In this section we will study the condition number of the B-spline basis.

9.3.1 A general definition of stability

The stability of a basis can be defined quite generally. Instead of considering polynomials we can consider a general linear vector space where we can measure the size of the elements through a norm; this is called a *normed linear space.*

Definition 9.13. *Let $\mathbb{V}$ be a normed linear space. A basis (ϕ_j) for $\mathbb{V}$ is said to be stable with respect to a vector norm $\|\cdot\|$ if there are small positive constants C_1 and C_2 such that*

$$C_1^{-1}\big\|(c_j)\big\| \leq \Big\|\sum_j c_j\phi_j\Big\| \leq C_2\big\|(c_j)\big\|, \tag{9.24}$$

for all sets of coefficients $\boldsymbol{c} = (c_j)$. Let C_1^* and C_2^* denote the smallest possible values of C_1 and C_2 such that (9.24) *holds. The condition number of the basis is then defined to be $\kappa = \kappa((\phi_i)_i) = C_1^* C_2^*$.*

At the risk of confusion we have used the same symbol both for the norm in $\mathbb{V}$ and the vector norm of the coefficients. In our case $\mathbb{V}$ will be some spline space $\mathbb{S}_{d,\boldsymbol{t}}$ and the basis (ϕ_j) will be the B-spline basis. The norms we will consider are the p-norms which are defined by

$$\|f\|_p = \|f\|_{p,[a,b]} = \left(\int_a^b |f(x)|^p dx\right)^{1/p} \quad \text{and} \quad \|\boldsymbol{c}\|_p = \left(\sum_j |c_j|^p\right)^{1/p}$$

where p is a real number in the range $1 \leq p < \infty$. Here f is a function on the interval $[a, b]$ and $\boldsymbol{c} = (c_j)$ is a real vector. For $p = \infty$ the norms are defined by

$$\|f\|_\infty = \|f\|_{\infty,[a,b]} = \max_{a\leq x\leq b} |f(x)| \quad \text{and} \quad \|\boldsymbol{c}\|_\infty = \big\|(c_j)\big\|_\infty = \max_j |c_j|,$$

In practice, the most important norms are the 1-, 2- and ∞-norms.

In Definition 9.13 we require the constants C_1 and C_2 to be 'small', but how small is 'small'? There is no unique answer to this question, but it is typically required that C_1 and C_2 should be independent of the dimension n of $\mathbb{V}$, or at least grow very slowly with n. Note that we always have $\kappa \geq 1$, and $\kappa = 1$ if and only if we have equality in both inequalities in (9.24).

A stable basis is desirable for many reasons, and the constant $\kappa = C_1C_2$ crops up in many different contexts. The condition number κ does in fact act as a sort of derivative of the basis and gives a measure of how much an error in the coefficients is magnified in a function value.

Proposition 9.14. *Suppose (ϕ_j) is a stable basis for $\mathbb{V}$. If $f = \sum_j c_j \phi_j$ and $g = \sum_j b_j \phi_j$ are two elements in $\mathbb{V}$ with $f \neq 0$, then*

$$\frac{\|f-g\|}{\|f\|} \leq \kappa \frac{\|\boldsymbol{c}-\boldsymbol{b}\|}{\|\boldsymbol{c}\|}, \tag{9.25}$$

where κ is the condition number of the basis as in Definition 9.13.

Proof. From (9.24), we have the two inequalities $\|f-g\| \leq C_2 \|(c_j - b_j)\|$ and $1/\|f\| \leq C_1/\|(c_j)\|$. Multiplying these together gives the result. ∎

If we think of g as an approximation to f then (9.25) says that the relative error in $f-g$ is bounded by at most κ times the relative error in the coefficients. If κ is small a small relative error in the coefficients gives a small relative error in the function values. This is important in floating point calculations on a computer. A function is usually represented by its coefficients relative to some basis. Normally, the coefficients are real numbers that must be represented inexactly as floating point numbers in the computer. This round-off error means that the computed spline, here g, will differ from the exact f. Proposition 9.14 shows that this is not so serious if the perturbed coefficients of g are close to those of f and the basis is stable.

Proposition 9.14 also provides some information as to what are acceptable values of C_1^* and C_2^*. If for example $\kappa = C_1^* C_2^* = 100$ we risk losing 2 decimal places in evaluation of a function; exactly how much accuracy one can afford to lose will of course vary.

One may wonder whether there are any unstable polynomial bases. It turns out that the power basis $1, x, x^2, \ldots,$ on the interval $[0,1]$ is unstable even for quite low degrees. Already for degree 10, one risks losing as much as 4 or 5 decimal digits in the process of computing the value of a polynomial on the interval $[0,1]$ relative to this basis, and other operations such as numerical root finding is even more sensitive.

9.3.2 Stability of the B-spline basis, $p = \infty$

Since splines and B-splines are defined via the knot vector, it is quite conceivable that the condition number of the B-spline basis could become arbitrarily large for certain knot configurations, for example in the limit when two knots merge into one. One of the key features of splines is that this cannot happen.

Theorem 9.15. *There is a constant K_d which depends only on the polynomial degree d, such that for all spline spaces $\mathbb{S}_{d,\boldsymbol{t}}$ and all splines $f = \sum_{i=1}^n c_i B_{i,d} \in \mathbb{S}_{d,\boldsymbol{t}}$ with B-spline coefficients $\boldsymbol{c} = (c_i)_{i=1}^n$, the two inequalities*

$$K_d^{-1} \|\boldsymbol{c}\|_\infty \leq \|f\|_{\infty,[t_1,t_{n+d}]} \leq \|\boldsymbol{c}\|_\infty \tag{9.26}$$

hold.

Proof. We have already proved variants of the second inequality several times; it follows since B-splines are nonnegative and sum to (at most) 1.

The first inequality is a consequence of Corollary 9.10. The value of the constant K_d is $K_0 = K_1 = 1$, $K_2 = 3$ while it is given by (9.17) for $d > 2$. ∎

The condition number of the B-spline basis on the knot vector $\boldsymbol{\tau}$ with respect to the ∞-norm is usually denoted $\kappa_{d,\infty,\boldsymbol{\tau}}$. By taking the supremum over all knot vectors we obtain the knot independent condition number $\kappa_{d,\infty}$,

$$\kappa_{d,\infty} = \sup_{\boldsymbol{\tau}} \kappa_{d,\infty,\boldsymbol{\tau}}.$$

Theorem 9.15 shows that $\kappa_{d,\infty}$ is bounded above by K_d.

Although K_d is independent of the knots, it grows quite quickly with d and seems to indicate that the B-spline basis may well be unstable for all but small values of d. However, by using different techniques it is possible to find better estimates for the condition number, and it is indeed known that the B-spline basis is very stable, at least for moderate values of d. It is simple to determine the condition number for $d \leq 2$; we have $\kappa_{0,\infty} = \kappa_{1,\infty} = 1$ and $\kappa_{2,\infty} = 3$. For $d \geq 3$ it has recently been shown that $\kappa_{d,\infty} = O(2^d)$. The first few values are known to be approximately $\kappa_{3,\infty} \approx 5.5680$ and $\kappa_{4,\infty} \approx 12.088$.

9.3.3 Stability of the B-spline basis, $p < \infty$

In this section we are going to generalise Theorem 9.15 to any p-norm. This is useful in some contexts, especially the case $p = 2$ which is closely related to least squares approximation. The proof uses standard tools from analysis, but may seem technical for the reader who is not familiar with the techniques.

Throughout this section p is a fixed real number in the interval $[1,\infty)$ and q is a related number defined by the identity $1/p + 1/q = 1$. A classical inequality for functions that will be useful is the Hölder inequality

$$\int_a^b \bigl|f(x)g(x)\bigr| dx \leq \|f\|_p \|g\|_q.$$

We will also need the Hölder inequality for vectors which is given by

$$\sum_{i=1}^n |b_i c_i| \leq \|(b_i)_{i=1}^n\|_p \|(c_i)_{i=1}^n\|_q.$$

In addition to the Hölder inequalities we need a fundamental inequality for polynomials. This states that for any polynomial $g \in \pi_d$ and any interval $[a,b]$ we have

$$\bigl|g(x)\bigr| \leq \frac{C}{b-a} \int_a^b \bigl|g(z)\bigr|\, dz, \quad \text{for any } x \in [a,b], \tag{9.27}$$

where the constant C only depends on the degree d. This is a consequence of the fact that all norms on a finite dimensional vector space are equivalent.

In order to generalise the stability result (9.26) to arbitrary p-norms we need to introduce a different scaling of the B-splines. We define the p-norm B-splines to be identically zero if $\tau_{i+d+1} = \tau_i$ and

$$B^p_{i,d,\boldsymbol{t}} = \left(\frac{d+1}{\tau_{i+d+1} - \tau_i}\right)^{1/p} B_{i,d,\boldsymbol{t}}, \tag{9.28}$$

otherwise. We can then state the p-norm stability result for B-splines.

Theorem 9.16. *There is a constant K that depends only on the polynomial degree d, such that for all $1 \le p \le \infty$, all spline spaces $\mathbb{S}_{d,\boldsymbol{t}}$ and all splines $f = \sum_{i=1}^n c_i B_{i,d}^p \in \mathbb{S}_{d,\boldsymbol{t}}$ with p-norm B-spline coefficients $\boldsymbol{c} = (c_i)_{i=1}^n$ the inequalities*

$$K^{-1}\|\boldsymbol{c}\|_p \le \|f\|_{p,[\tau_1,\tau_{m+d}]} \le \|\boldsymbol{c}\|_p \tag{9.29}$$

hold.

Proof. We first prove the upper inequality. Let $\gamma_i = (d+1)/(\tau_{i+d+1} - \tau_i)$ denote the pth power of the scaling factor in (9.28) for $i = 1, \ldots, n$ and set $[a,b] = [\tau_1, \tau_{n+d+1}]$. Remembering the definition of $B_{i,d,\boldsymbol{\tau}}^p$ and the identity $1/p + 1/q = 1$ and applying the Hölder inequality for sums we obtain

$$\sum_i \left|c_i B_{i,d}^p\right| = \sum_i \left|c_i \gamma_i^{1/p} B_{i,d}^{1/p}\right| B_{i,d}^{1/q} \le \Big(\sum_i |c_i|^p \gamma_i B_{i,d}\Big)^{1/p} \Big(\sum_i B_{i,d}\Big)^{1/q}.$$

Raising both sides of this inequality to the pth power and recalling that B-splines sum to (at most) 1 we obtain the inequality

$$\Big|\sum_i c_i B_{i,d}^p(x)\Big|^p \le \sum_i |c_i|^p \gamma_i B_{i,d}(x) \quad \text{for any } x \in \mathbb{R}. \tag{9.30}$$

It can be shown that the integral of a B-spline is given by

$$\int_{\tau_i}^{\tau_{i+d+1}} B_{i,d}(x)dx = \frac{\tau_{i+d+1} - \tau_i}{d+1} = \frac{1}{\gamma_i}.$$

Making use of this and (9.30) we find

$$\|f\|_{p,[a,b]}^p = \int_a^b \Big|\sum_i c_i B_{i,d}^p(x)\Big|^p dx \le \sum_i |c_i|^p \gamma_i \int_a^b B_{i,d}(x)\,dx = \sum_i |c_i|^p.$$

Taking pth roots on both sides proves the upper inequality.

Consider now the lower inequality. The spline f is given as a linear combination of p-norm B-splines, but can very simply be written as a linear combination of the usual B-splines,

$$f = \sum_i c_i B_{i,d}^p = \sum_i c_i \gamma_i^{1/p} B_{i,d}.$$

From the first inequality in (9.26) we then obtain for each i

$$\Big(\frac{d+1}{\tau_{i+d+1} - \tau_i}\Big)^{1/p} |c_i| \le K_d \max_{\tau_{i+1} \le x \le \tau_{i+d}} |f(x)|,$$

where the constant K_d only depends on d. Extending the maximum to a larger subinterval and applying the inequality (9.27) we find

$$\begin{aligned}
|c_i| &\le K_d (d+1)^{-1/p} \big(\tau_{i+d+1} - \tau_i\big)^{1/p} \Big| \max_{\tau_i \le x \le \tau_{i+d+1}} |f(x)| \\
&\le C K_d (d+1)^{-1/p} \big(\tau_{i+d+1} - \tau_i\big)^{-1+1/p} \int_{\tau_i}^{\tau_{i+d+1}} |f(y)|\,dy.
\end{aligned}$$

Next, we apply the Hölder inequality for integrals to the product $\int_{\tau_i}^{\tau_{i+d+1}} |f(y)|\, 1\, dy$ and obtain

$$|c_i| \le CK_d(d+1)^{-1/p}\Bigl(\int_{\tau_i}^{\tau_{i+d+1}} |f(y)|^p\, dy\Bigr)^{1/p}.$$

Raising both sides to the pth power and summing over i we obtain

$$\sum_i |c_i|^p \le C^p K_d^p (d+1)^{-1} \sum_i \int_{\tau_i}^{\tau_{i+d+1}} |f(y)|^p\, dy \le C^p K_d^p \|f\|_{p,[a,b]}^p.$$

Taking pth roots we obtain the lower inequality in (9.29) with $K = CK_d$. ∎

9.4 Convergence of the control polygon for spline functions

Recall that for a spline function $f(x) = \sum_i c_i B_{i,d,\boldsymbol{\tau}}$ the control polygon is the piecewise linear interpolant to the points (τ_i^*, c_i), where $\tau_i^* = (\tau_{i+1} + \cdots + \tau_{i+d})/d$ is the ith knot average. In this section we are going to prove that the control polygon converges to the spline it represents when the knot spacing approaches zero. The main work is done in Lemma 9.17 which shows that a corner of the control polygon is close to the spline since c_i is close to $f(\tau_i^*)$, at least when the spacing in the knot vector is small. The proof of the lemma makes use of the fact that the size of a B-spline coefficient c_i can be bounded in terms of the size of the spline on the interval $[\tau_{i+1}, \tau_{i+d+1}]$, which we proved in Theorem 9.9 and Lemma 9.4 (and Section 9.2.1),

$$|c_i| \le K_d \|f\|_{[\tau_{i+1},\tau_{i+d}]}. \tag{9.31}$$

The norm used here and throughout this section is the ∞-norm.

Lemma 9.17. *Let f be a spline in $\mathbb{S}_{d,\boldsymbol{\tau}}$ with coefficients (c_i). Then*

$$|c_i - f(\tau_i^*)| \le K(\tau_{i+d} - \tau_{i+1})^2 \|D^2 f\|_{[\tau_{i+1},\tau_{i+d}]}, \tag{9.32}$$

where $\tau_i^* = (\tau_{i+1} + \cdots + \tau_{i+d})/d$, the operator D^2 denotes (one-sided) differentiation (from the right), and the constant K only depends on d.

Proof. Let i be fixed. If $\tau_{i+1} = \tau_{i+d}$ then we know from property 5 in Lemma 2.6 that $B_{i,d}(\tau_i^*) = 1$ so $c_i = f(\tau_i^*)$ and there is nothing to prove. Assume for the rest of the proof that the interval $J = (\tau_{i+1}, \tau_{i+d})$ is nonempty. Since J contains at most $d-2$ knots, it follows from the continuity property of B-splines that f has at least two continuous derivatives in J. Let x_0 be a number in the interval J and consider the spline

$$g(x) = f(x) - f(x_0) - (x - x_0)Df(x_0)$$

which is the error in a first order Taylor expansion of f at x_0. This spline lies in $\mathbb{S}_{d,\boldsymbol{\tau}}$ and can therefore be written as $g = \sum_i b_i B_{i,d,\boldsymbol{\tau}}$ for suitable coefficients (b_i). More specifically we have

$$b_i = c_i - f(x_0) - (\tau_i^* - x_0)Df(x_0).$$

Choosing $x_0 = \tau_i^*$ we have $b_i = c_i - f(\tau_i^*)$ and according to the inequality (9.31) and the error term in first order Taylor expansion we find

$$|c_i - f(\tau_i^*)| = |b_i| \le K_d \|g\|_J \le \frac{K_d(\tau_{i+d} - \tau_{i+1})^2}{2} \|D^2 f\|_J.$$

The inequality (9.32) therefore holds with $K = K_d/2$ and the proof is complete. ∎

Lemma 9.17 shows that the corners of the control polygon converge to the spline as the knot spacing goes to zero. This partly explains why the control polygon approaches the spline when we insert knots. What remains is to show that the control polygon as a whole also converges to the spline.

Theorem 9.18. *Let $f = \sum_{i=1}^{n} c_i B_{i,d}$ be a spline in $\mathbb{S}_{d,\boldsymbol{\tau}}$, and let $\Gamma_{d,\boldsymbol{\tau}}(f)$ be its control polygon. Then*

$$\big\|\Gamma_{d,\boldsymbol{\tau}}(f) - f\big\|_{[\tau_1^*,\tau_n^*]} \leq Kh^2 \|D^2 f\|_{[\tau_1,\tau_{n+d+1}]}, \tag{9.33}$$

where $h = \max_i\{\tau_{i+1} - \tau_i\}$ and the constant K only depends on d.

Proof. As usual, we assume that $\boldsymbol{\tau}$ is $d+1$-regular (if not we extend it with $d+1$-tuple knots at either ends and add zero coefficients). Suppose that x is in $[\tau_1^*, \tau_m^*]$ and let j be such that $\tau_j^* \leq x < \tau_{j+1}^*$. Observe that since the interval $J^* = (\tau_j^*, \tau_{j+1}^*)$ is nonempty we have $\tau_{j+1} < \tau_{j+d+1}$ and J^* contains at most $d-1$ knots. From the continuity property of B-splines we conclude that f has a continuous derivative and the second derivative of f is at least piecewise continuous in J^*. Let

$$g(x) = \frac{(\tau_{j+1}^* - x) f(\tau_j^*) + (x - \tau_j^*) f(\tau_{j+1}^*)}{\tau_{j+1}^* - \tau_j^*}$$

be the linear interpolant to f on this interval. We will show that both $\Gamma = \Gamma_{d,\boldsymbol{\tau}}(f)$ and f are close to g on J^* and then deduce that Γ is close to f because of the triangle inequality

$$\big|\Gamma(x) - f(x)\big| \leq \big|\Gamma(x) - g(x)\big| + \big|g(x) - f(x)\big|. \tag{9.34}$$

Let us first consider the difference $\Gamma - g$. Note that

$$\Gamma(x) - g(x) = \frac{(\tau_{j+1}^* - x)(b_j - f(\tau_j^*)) + (x - \tau_j^*)(b_{j+1} - f(\tau_{j+1}^*))}{\tau_{j+1}^* - \tau_j^*}$$

for any x in J^*. We therefore have

$$\big|\Gamma(x) - g(x)\big| \leq \max\Big\{\big|b_j - f(\tau_j^*)\big|, \big|b_{j+1} - f(\tau_{j+1}^*)\big|\Big\},$$

for $x \in J^*$. From Lemma 9.17 we then conclude that

$$|\Gamma(x) - g(x)| \leq K_1 h^2 \|D^2 f\|_J, \quad x \in J^*, \tag{9.35}$$

where $J = [\tau_1, \tau_{m+d+1}]$ and K_1 is a constant that only depends on d.

The second difference $f(x) - g(x)$ in (9.34) is the error in linear interpolation to f at the endpoints of J^*. For this process we have the standard error estimate

$$\big|f(x) - g(x)\big| \leq \frac{1}{8}(\tau_{j+1}^* - \tau_j^*)^2 \|D^2 f\|_{J^*} \leq \frac{1}{8} h^2 \|D^2 f\|_J, \quad x \in J^*. \tag{9.36}$$

If we now combine (9.35) and (9.36) as indicated in (9.34), we obtain the Theorem with constant $K = K_1 + 1/8$. ∎

Because of the factor h^2 in Theorem 9.18 we say (somewhat loosely) that the control polygon converges quadratically to the spline.

Exercises for Chapter 9

9.1 In this exercise we will study the order of approximation by the Schoenberg Variation Diminishing Spline Approximation of degree $d \geq 2$. This approximation is given by

$$V_d f = \sum_{i=1}^{n} f(\tau_i^*) B_{i,d}, \quad \text{with} \quad \tau_i^* = \frac{\tau_{i+1} + \cdots \tau_{i+d}}{d}.$$

Here $B_{i,d}$ is the ith B-spline of degree d on a $d+1$-regular knot vector $\boldsymbol{\tau} = (\tau_i)_{i=1}^{n+d+1}$. We assume that $\tau_{i+d} > \tau_i$ for $i = 2, \ldots, n$. Moreover we define the quantities

$$a = \tau_1, \quad b = \tau_{n+d+1}, \quad h = \max_{1 \leq i \leq n} \tau_{i+1} - \tau_i.$$

We want to show that $V_d f$ is an $O(h^2)$ approximation to a sufficiently smooth f.

We first consider the more general spline approximation

$$\tilde{V}_d f = \sum_{i=1}^{n} \lambda_i(f) B_{i,d}, \quad \text{with} \quad \lambda_i(f) = w_{i,0} f(x_{i,0}) + w_{i,1} f(x_{i,1}).$$

Here $x_{i,0}$ and $x_{i,1}$ are two distinct points in $[\tau_i, \tau_{i+d}]$ and $w_{i,0}, w_{i,1}$ are constants, $i = 1, \ldots, n$.

Before attempting to solve this exercise the reader might find it helpful to review Section 9.2.2

a) Suppose for $i = 1, \ldots, n$ that $w_{i,0}$ and $w_{i,1}$ are such that

$$\begin{aligned} w_{i,0} + w_{i,1} &= 1 \\ x_{i,0} w_{i,0} + x_{i,1} w_{i,1} &= \tau_i^* \end{aligned}$$

Show that then $\tilde{V}_d p = p$ for all $p \in \pi_1$. (Hint: Consider the polynomials $p(x) = 1$ and $p(x) = x$.)

b) Show that if we set $x_{i,0} = \tau_i^*$ for all i then $\tilde{V}_d f = V_d f$ for all f, regardless of how we choose the value of $x_{i,1}$.

In the rest of this exercise we set $\lambda_i(f) = f(\tau_i^*)$ for $i = 1, \ldots, n$, i.e. we consider $V_d f$. We define the usual uniform norm on an interval $[c, d]$ by

$$\|f\|_{[c,d]} = \sup_{c \leq x \leq d} |f(x)|, \quad f \in C_{\boldsymbol{\Delta}}[c, d].$$

c) Show that for $d + 1 \leq l \leq n$

$$\|V_d f\|_{[\tau_l, \tau_{l+1}]} \leq \|f\|_{[\tau_{l-d}^*, \tau_l^*]}, \quad f \in C_{\boldsymbol{\Delta}}[a, b].$$

d) Show that for $f \in C_{\boldsymbol{\Delta}}[\tau_{l-d}^*, \tau_l^*]$ and $d + 1 \leq l \leq n$

$$\|f - V_d f\|_{[\tau_l, \tau_{l+1}]} \leq 2 \operatorname{dist}_{[\tau_{l-d}^*, \tau_l^*]}(f, \pi_1).$$

e) Explain why the following holds for $d+1 \le l \le n$

$$\mathrm{dist}_{[\tau^*_{l-d},\tau^*_l]}(f,\pi_1) \le \frac{(\tau^*_l - \tau^*_{l-d})^2}{8} \|D^2 f\|_{[\tau^*_{l-d},\tau^*_l]}.$$

f) Show that the following $O(h^2)$ estimate holds

$$\|f - V_d f\|_{[a,b]} \le \frac{d^2}{4} h^2 \|D^2 f\|_{[a,b]}.$$

(Hint: Verify that $\tau^*_l - \tau^*_{l-d} \le hd$.)

9.2 In this exercise we want to perform a numerical simulation experiment to determine the order of approximation by the quadratic spline approximations

$$V_2 f = \sum_{i=1}^{n} f(\tau_i^*) B_{i,2}, \quad \text{with} \quad \tau_i^* = \frac{\tau_{i+1} + \tau_{i+2}}{2},$$

$$P_2 f = \sum_{i=1}^{n} \big(-\frac{1}{2} f(\tau_{i+1}) + 2 f(\tau_i^*) - \frac{1}{2} f(\tau_{i+2}) \big) B_{i,2}.$$

We want to test the hypotheses $f - V_2 f = O(h^2)$ and $f - P_2 f = O(h^3)$ where $h = \max_i \tau_{i+1} - \tau_i$. We test these on the function $f(x) = \sin x$ on $[0,\pi]$ for various values of h. Consider for $m \ge 0$ and $n_m = 2 + 2^m$ the 3-regular knot vector $\boldsymbol{\tau}^m = (\tau_i^m)_{i=1}^{n_m+3}$ on the interval $[0,\pi]$ with uniform spacing $h_m = \pi 2^{-m}$. We define

$$V_2^m f = \sum_{i=1}^{n} f(\tau^m_{i+3/2}) B^m_{i,2}, \quad \text{with} \quad \tau_i^m = \frac{\tau^m_{i+1} + \tau^m_{i+2}}{2},$$

$$P_2^m f = \sum_{i=1}^{n} \big(-\frac{1}{2} f(\tau^m_{i+1}) + 2 f(\tau^m_{i+3/2}) - \frac{1}{2} f(\tau^m_{i+2}) \big) B^m_{i,2},$$

and $B^m_{i,2}$ is the ith quadratic B-spline on $\boldsymbol{\tau}^m$. As approximations to the norms $\|f - V_2^m f\|_{[0,\pi]}$ and $\|f - P_2^m f\|_{[0,\pi]}$ we use

$$E_V^m = \max_{0 \le j \le 100} |f(j\pi/100) - V_2^m f(j\pi/100)|,$$

$$E_P^m = \max_{0 \le j \le 100} |f(j\pi/100) - P_2^m f(j\pi/100)|.$$

Write a computer program to compute numerically the values of E_V^m and E_P^m for $m = 0,1,2,3,4,5$, and the ratios E_V^m / E_V^{m-1} and E_P^m / E_P^{m-1} for $1 \le m \le 5$. What can you deduce about the approximation order of the two methods?

Make plots of $V_2^m f$, $P_2^m f$, $f - V_2^m f$, and $f - P_2^m f$ for some values of m.

9.3 Suppose we have $m \ge 3$ data points $\big(x_i, f(x_i)\big)_{i=1}^m$ sampled from a function f, where the abscissas $\boldsymbol{x} = (x_i)_{i=1}^m$ satisfy $x_1 < \cdots < x_m$. In this exercise we want to derive a local quasi-interpolation scheme which only uses the data values at the x_i's and which has $O(h^3)$ order of accuracy if the y-values are sampled from a smooth function f. The method requires m to be odd.

From $\boldsymbol{x}$ we form a 3-regular knot vector by using every second data point as a knot

$$\boldsymbol{\tau} = (\tau_j)_{j=1}^{n+3} = (x_1, x_1, x_1, x_3, x_5, \ldots, x_{m-2}, x_m, x_m, x_m), \tag{9.37}$$

where $n = (m+3)/2$. In the quadratic spline space $\mathbb{S}_{2,\boldsymbol{\tau}}$ we can then construct the spline

$$Q_2 f = \sum_{j=1}^{n} \lambda_j(f) B_{j,2}, \tag{9.38}$$

where the B-spline coefficients $\lambda_j(f)_{j=1}^n$ are defined by the rule

$$\lambda_j(f) = \frac{1}{2}\Big(-\theta_j^{-1} f(x_{2j-3}) + \theta_j^{-1}(1+\theta_j)^2 f(x_{2j-2}) - \theta_j f(x_{2j-1}) \Big), \tag{9.39}$$

for $j = 1, \ldots, n$. Here $\theta_1 = \theta_n = 1$ and

$$\theta_j = \frac{x_{2j-2} - x_{2j-3}}{x_{2j-1} - x_{2j-2}}$$

for $j = 2, \ldots, n-1$.

a) Show that Q_2 simplifies to P_2 given by (9.6) when the data abscissas are uniformly spaced.

b) Show that $Q_2 p = p$ for all $p \in \pi_2$ and that because of the multiple abscissas at the ends we have $\lambda_1(f) = f(x_1)$, $\lambda_n(f) = f(x_m)$, so only the original data are used to define $Q_2 f$. (Hint: Use the formula in Exercise 1.

c) Show that for $j = 1, \ldots, n$ and $f \in \mathbb{C}_{\boldsymbol{\Delta}}[x_1, x_m]$

$$|\lambda_j(f)| \le (2\theta + 1)\|f\|_{\infty,[\tau_{j+1},\tau_{j+2}]},$$

where

$$\theta = \max_{1 \le j \le n} \{\theta_j^{-1}, \theta_j\}.$$

d) Show that for $l = 3, \ldots, n$, $f \in \mathbb{C}_{\boldsymbol{\Delta}}[x_1, x_m]$, and $x \in [\tau_l, \tau_{l+1}]$

$$|Q_2(f)(x)| \le (2\theta + 1)\|f\|_{\infty,[\tau_{l-1},\tau_{l+2}]}.$$

e) Show that for $l = 3, \ldots, n$ and $f \in \mathbb{C}_{\boldsymbol{\Delta}}[x_1, x_m]$

$$\|f - Q_2 f\|_{\infty,[\tau_l,\tau_{l+1}]} \le (2\theta + 2)\,\mathrm{dist}_{[\tau_{l-1},\tau_{l+2}]}(f, \pi_2).$$

f) Show that for $f \in \mathbb{C}^3_{\boldsymbol{\Delta}}[x_1, x_m]$ we have the $O(h^3)$ estimate

$$\|f - Q_2 f\|_{\infty,[x_1,x_m]} \le K(\theta)|\Delta x|^3 \|D^3 f\|\|\|_{\infty,[x_1,x_m]},$$

where

$$|\Delta x| = \max_j |x_{j+1} - x_j|$$

and the constant $K(\theta)$ only depends on θ.