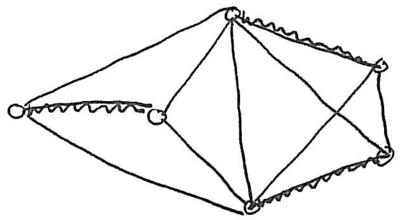
NON BIPARTITE MATCHING



How to find a maximum cardinality matching?

Theorem

$G=(V,E)$, graph, and M is a matching in G.

M is a matching of maximum cardinality

$\Leftrightarrow$ there exists no M-augmenting path.

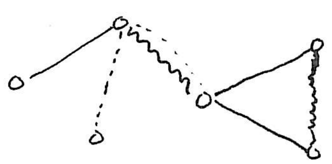
M-augmenting path:

v_0, v_t missing M

Can not find M-augmenting paths by the methods used in the bipartite case.

Alternative: Look for an M-alternating walk

Problem: Loops that cannot be deleted

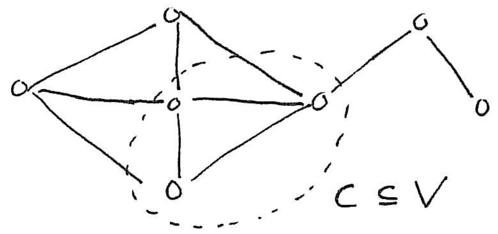


Odd cycles

Problem solved by Edmonds.

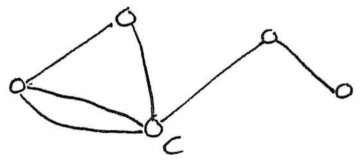
<u>Shrinking</u>:

$G = (V, E)$



$\Downarrow$

$G/C = (V/C, E/C)$



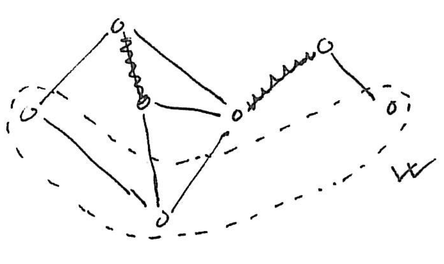
V/C : delete all vertices in C, add a new vertex called C

E/C : $e/C = e$ if e is disjoint from C

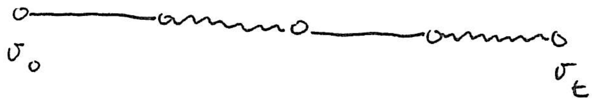
$e/C = uC$ if $e = uv$ with $u \notin C$, $v \in C$

- $G = (V, E)$ graph, M: matching in G

  W : set of vertices missed by M

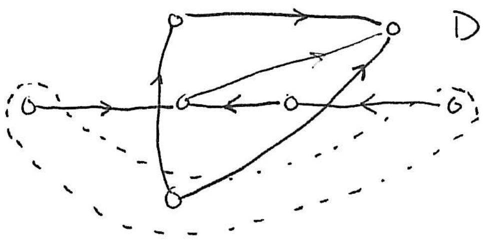


M-alternating walk :



Shortest M-alternating W-W walks can be found in an auxiliary directed graph; D:

- Include (w, w') if $wx \in E$ and $xw' \in M$ for any $x \in V$



M-alternating W-W walks corresponds to directed walks in D from a vertex in W to a vertex that is adjacent to ~~add~~ at least one vertex in W.

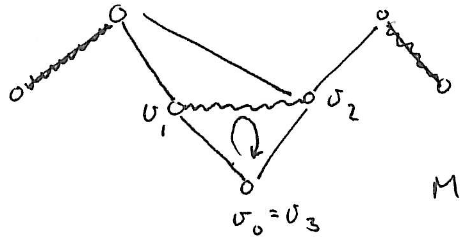
· M-augmenting path = M-alternating W-W walk of positive length with distinct vertices.

- **M-blossom** :

M-alternating walk $P = (v_0, v_1, \dots, v_t)$ where

- $v_0, \dots, v_{t-1}$ are distinct
- v_0 is missed by M
- $v_t = v_0$

Ex.

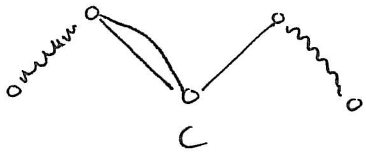


M- blossom

Theorem

Let C be an M-blossom in G.
Then M has maximum size in G if and only if M/C has maximum size in G/C.

Ex.



Clearly ~~not~~ of maximum size in G/C,
so M ~~does not~~ have maximum size in G.

- Problem: given a matching M, find a matching N with $|N| = |M| + 1$ or conclude that M is a maximum-size matching

- Algorithm

Case 1: There is no M-alternating W-W walk

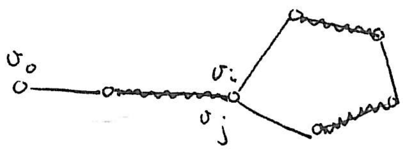
$\Rightarrow$ M has maximum size (no M-aug. path)

Case 2: There is an M-alternating W-W walk, P is the shortest such walk.

a) P is a path and M-augmenting, $N = M \Delta EP$

b) P is not a path

(Choose $i < j$ such that $v_i = v_j$ with j as small as possible



Switch on $v_0, \ldots, v_i$, then C is an M-blossom. Apply the algorithm recursively to $G' = G/C$ and $M' = M/C$.

- M'-augmenting paths can be transformed to M-augmenting paths

- M' maximum size in G' $\Rightarrow$ M maximum size in G.

Theorem

A maximum size matching can be found in time $O(|V|^2|E|)$

Proof: $M-M$ walk: $O(|E|)$

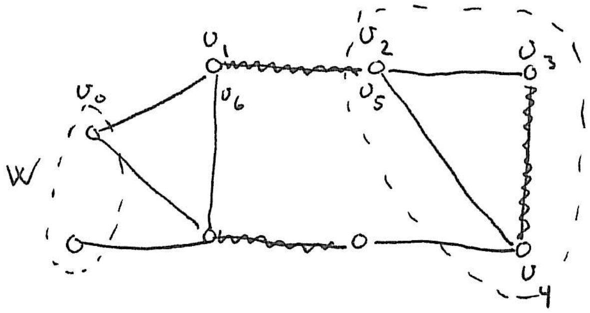
$G \setminus C$: $O(|E|)$

Recursion depth : $O(|V|)$

Applied at most $O(|V|)$ times $\Rightarrow$ $O(|V|^2|E|)$

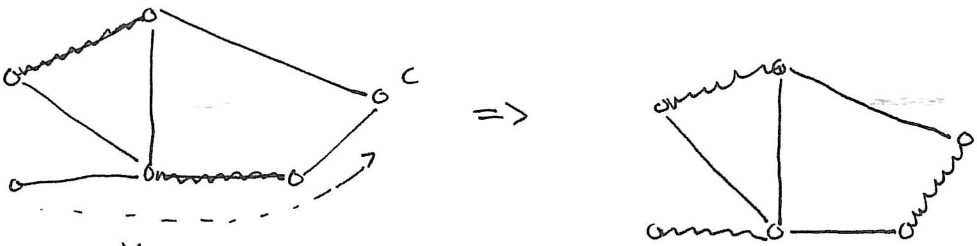
$\square$

Ex



Blossom, C

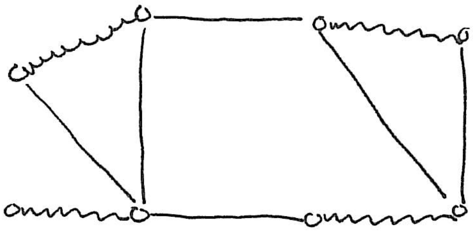
switching v_0v_1 and v_1v_2 and shrinking C



M - aug.

~~[illegible]~~:

Corresponds to:



The matching polytope

Perfect matching polytope

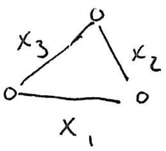
$$P_{PM}(G) = \text{conv. hull} \{ \chi^M \mid M \text{ perfect matching in } G \}$$

For bipartite graphs

$$P_{PM}(G) = \{ x \in \mathbb{R}^E : x_e \geq 0, \ \forall e \in E, \ \sum_{e \in \delta(v)} x_e = 1, \ \forall v \in V \}$$

- Does not hold for general graphs:

K_3



$$x_1 + x_2 = 1$$
$$x_1 + x_3 = 1$$
$$x_2 + x_3 = 1$$

$$x_1, x_2, x_3 \geq 0$$

$$x_1 = x_2 = x_3 = \frac{1}{2}$$

- Edmonds' perfect matching polytope theorem

The perfect matching of any graph $G = (V, E)$ is determined by

(i) $x_e \geq 0$, $\forall e \in E$

(ii) $\sum_{e \in \delta(v)} x_e = 1$, $\forall v \in V$

(iii) $\sum_{e \in \delta(U)} x_e \geq 1$, $\forall U \subseteq V$, $|U|$ odd

- Edmonds' matching polytope theorem

For any graph $G=(V,E)$ the matching polytope is determined by

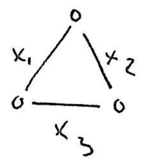
(i) $x_e \geq 0$, $\forall e \in E$

(ii) $\sum_{e \in \delta(v)} x_e \leq 1$, $\forall v \in V$

(iii) $\sum_{e \subseteq U} x_e \leq \lfloor \frac{1}{2}|U| \rfloor$ $\forall U \subseteq V$ with $|U|$ odd

Ex

K_3



$x_1, x_2, x_3 \geq 0$

$x_1 + x_2 \leq 1$

$x_1 + x_3 \leq 1$

$x_2 + x_3 \leq 1$

$x_1 + x_2 + x_3 \leq 1$