INTEGER PROGRAMMING

~~Ck~~ Problem classes:

1) Mixed-Integer Programming Problem (MIP):

$$\max\{c^T x + h^T y : Ax + Gy \leq b, \; x \in \mathbb{Z}^n, \; y \in \mathbb{R}^p\}$$

~~2) Binary Integer Program (BIP): $x \in \mathbb{B}^n$~~

~~3) Pure~~

2) Pure Integer Programming

$$\max\{c^T x : Ax \leq b, \; x \in \mathbb{Z}^n\}$$

3) Binary Integer Programming (BIP)

$$\max\{c^T x : Ax \leq b, \; x \in \mathbb{B}^n\}$$

4) Mixed 0-1 Programming

$$\max\{c^T x + h^T y : Ax + Gy \leq b, \; x \in \mathbb{B}^n, \; y \in \mathbb{R}^p\}$$

5) Linear programming

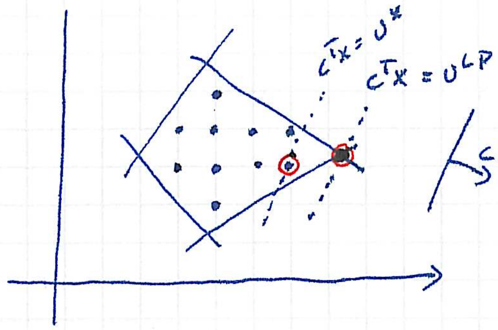
$$\max\{h^T y : Gy \leq b, \; y \in \mathbb{R}^p\}$$

- Linear programming relaxation
  - remove integer requirements

$$v^* = \max\{c^T x : Ax \leq b,\ x \in \mathbb{Z}^n\}$$

$$v^* \leq v^{LP}$$

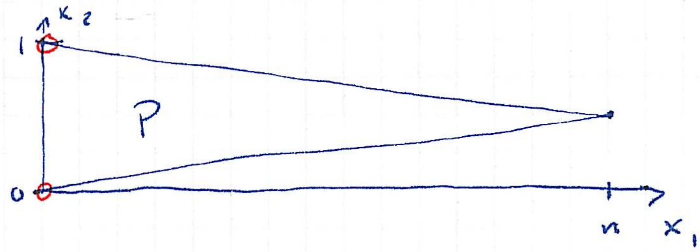
$$v^{LP} = \max\{c^T x : Ax \leq b,\ x \in \mathbb{R}^n\}$$



Rounding to a feasible integer solution may be difficult or impossible

The optimal solution to the LP relaxation can be arbitrarily far away from the optimal solution to the MIP.

Ex



$$v^{LP} = \max\{x_1 : x \in P\} = n$$

$$v^* = \max\{x_1 : x \in P,\ x \in \mathbb{B}^2\} = 0$$

* Combinatorial optimization Problem (COP)

Consists of :

- A finite <u>ground set</u> N
- A set $\mathcal{F} \subseteq 2^N$ of feasible solutions
- Costs c_j $\forall j \in N$

The cost of $F \in \mathcal{F}$ is $c(F) = \sum_{j \in F} c_j$

COP : $\max \{ c(F) : F \in \mathcal{F} \}$

Many COPs can be written as BIPs or MIPs.

<u>Ex</u> Max. cardinality matching

Graph, $G = (V, E)$

- $N = E$
- $\mathcal{F}$ = Matchings in G
- $c_e = 1$ $\forall e \in E$

Can be formulated as an integer program

$$\max \sum_{e \in E} x_e$$

$$\text{subject to} \quad \sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V \qquad (*)$$

$$x_e \geq 0 \quad \forall e \in E$$

$$x_e \in \mathbb{Z} \quad \forall e \in E$$

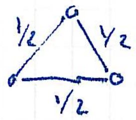
Let A be the $V \times E$ incidence matrix of G:

$$A_{v,e} = \begin{cases} 1, & v \in e \\ 0, & v \notin e \end{cases} \qquad \begin{array}{c|c} & e_1 \quad \cdots \quad e_m \\ \hline \begin{array}{c} v_1 \\ \vdots \\ v_n \end{array} & \left[\begin{array}{ccc} 1 & & \vdots \\ \vdots & & 1 \\ 1 & & 1 \end{array}\right] \end{array}$$

The integer program $(*)$ is the same as

$$\max \{ \mathbb{1}^T x : x \geq 0, \; Ax \leq \mathbb{1}, \; x \text{ integer} \}$$

For K_3



$$v^* = 1$$
$$v^{LP} = \frac{3}{2}, \quad \text{i.e.} \quad v^* < v^{LP}$$

- Duality relation

$$\max \{ c^T x : Ax \leq b, \; x \in \mathbb{Z}^n \} \leq \max \{ c^T x : Ax \leq b, \; x \in \mathbb{R}^n \}$$
$$\overset{\text{LP-dualitet}}{=} \min \{ y^T b : y \geq 0, \; y^T A = c^T, \; y \in \mathbb{R}^m \}$$
$$\leq \min \{ y^T b : y \geq 0, \; y^T A = c^T, \; y \in \mathbb{Z}^m \}$$

The inequality can be strict (e.g. K_3 example)

- A polytope is called <u>integer</u> if each of the vertices is an integer vector

If a polytope for $P = \{ x : Ax \leq b \}$ is integer, then

$$\max \{ c^T x : Ax \leq b \}$$

has an integer optimal solution,

i.e. $\max \{ c^T x : Ax \leq b, \; x \in \mathbb{Z}^n \} = \max \{ c^T x : Ax \leq b \}$

Polyhedral combinatorics:

Study combinatorial problems by using
* convex analysis
* polyhedral theory
* linear programming

Basic approach:

~~Translate~~

1. Represent each $F \in \mathcal{F}$ by its incidence vector χ^F in $\mathbb{R}^N$.
   Let P be the polytope defined by the convex hull of the incidence vectors;

$$P = \text{conv.hull}(\chi^F : F \in \mathcal{F})$$

2. Find a complete linear description of P.
   (Polytope = bounded polyhedron)
3. Apply LP duality to obtain a minimax theorem
4. Develop an algorithm using the minimax theorem as a stopping criterion.

· Difficulty: Step 2, often only a partial description is known
~~Still of great~~

Illustration: Maximum weight forest

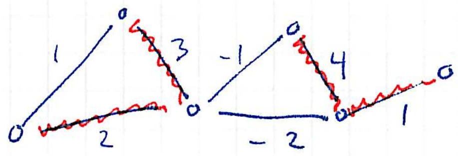
$G = (V, E)$ connected graph

w : weight function

Find a forest of max. weight

(eq with minimum spanning tree if $w_e \geq 0 \ \forall e \in E$)

Ex



Greedy algorithm:

- Extend a forest by adding the maximum weight edge that do not introduce a cycle
- Stop when the next edge has non-positive weight

We shall prove the correctness of the algorithm as part of the ~~following~~ theorem below.

- Forest polytope $F(G) = $ conv. hull $(\chi^F :$ F is a forest in G$)$

Theorem 5.1

$F(G) \subseteq \mathbb{R}^E$ is the solution set of the following linear system

(i) $x_e \geq 0 \quad \forall e \in E$

(ii) $x(E(S)) \leq |S| - 1 \quad \forall S \subseteq V$

$E(S)$: edges with both endpoints in S


S

Proof

Let $Q = \{ x \in \mathbb{R}^E : x_e \geq 0, \ \forall e \in E, \ x(E(S)) \leq |S|-1 \ \forall S \subseteq V \}$

Shall prove that $F(G) = Q$.

Integral vectors in Q are incidence vectors of forests ; (ii) prevents cycles ;

Convexity implies that $F(G) \subseteq Q$.

Reverse inclusion : $Q \subseteq F(G)$?

Let $\bar{x}$ be a vertex of Q. Want to show that $\bar{x}$ is integral.

$\bar{x}$ is the unique optimal solution of an LP problem $\max \{ c^T x : x \in Q \}$ for suitable c.

The LP dual of this problem

$$\min \sum_{S \subseteq V} y_S \cdot (|S|-1)$$

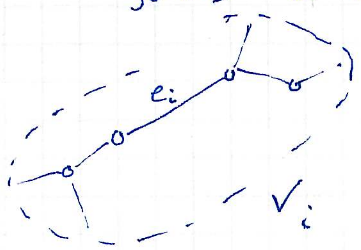
subject to

$$\sum_{S : e \in E(S)} y_S \geq c_e \qquad \forall e \in E$$

$$y_S \geq 0 \qquad \forall S \subseteq V$$

Let $F = \{e_1, \dots, e_m\}$ be the result of the greedy algorithm i'th iteration : e_i is added

forms a new component V_i



Define dual solution y_S as follows

$$y_S = 0 \quad \text{if} \quad S \notin \{V_i,\ i=1,\dots,r\}$$

For V_i, $i = 1, \dots, r$

Start with

$$y_{V_r} = c(e_r)$$

Complementary slackness : $x'_{e_r} > 0 \Rightarrow \sum_{S:\, e_r \in E(S)} y_S = c(e_r)$

Must have $y_S = 0$ for all other S with both endnodes of e_r.

$$y_{V_j} = c(e_j) - \sum_{i \in I(j)} y_{V_i} \quad , \; j = r-1, r-2, \dots, 1$$

with $I(j) = \{i : j+1 \le i \le r$ both endnodes of e_j are in $V_i\}$

y dual feasible, x' primal feasible and complementary slackness holds $\Rightarrow$ both solutions are optimal.

Uniqueness of $\bar{x}$ implies that $\bar{x} = x'$

Therefore every vertex of Q is integral and we have $Q = F(G)$.

$\square$