Totally unimodular matrices

A matrix A is called <u>totally unimodular</u> if each square submatrix of A has determinant equal to 0 or ± 1.

In particular: each entry is 0 or ± 1.

<u>Ex</u>

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{totally unimodular}$$

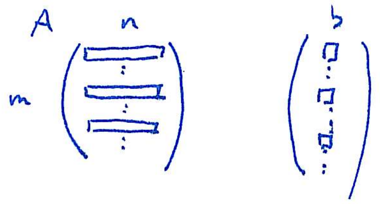
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \det(A) = 2$$

<u>Theorem</u>

$A \in \mathbb{R}^{m \times n}$, totally unimodular and $b \in \mathbb{Z}^m$.
Then each vertex of the polyhedron $P = \{x : Ax \leq b\}$ is an integer vector.

<u>Proof</u>

Let z be a vertex of P, then the submatrix A_z has rank n. So A_z has a non singular $n \times n$ submatrix A'.
Let b' be the corresponding part of b.



We know that $A'z = b'$, and hence $z = (A')^{-1}b$.

Since $|\det A'| = 1$, it follows from Cramer's rule that all entries of the matrix $(A')^{-1}$ are integer.

Therefore, z is an integer vector. $\square$

$$x_i = \frac{\sum \pm b_i \det(B_i)}{\det(A)}$$

$(n-1) \times (n-1)$

- Polyhedra in general
  - need not have vertices

A polyhedron is integer if for each vector c for which $\max\{c^T x : x \in P\}$ is finite, the maximum is attained by some integer vector.

Corollary 1

A : $m \times n$ matrix, totally unimodular, $b \in \mathbb{Z}^m$.
Then the polyhedron $P = \{x : Ax \leq b\}$ is an integer polyhedron.

Proof

Let x^* be an optimal solution of $\max\{c^T x : x \in P\}$.
Can choose $d', d'' \in \mathbb{Z}^n$ such that $d' \leq x^* \leq d''$

Consider $Q = \{x : Ax \leq b, d' \leq x, x \leq d''\}$ which is a bounded polyhedron.

Rewriting the system gives

$$\begin{pmatrix} A \\ -I \\ I \end{pmatrix} x \leq \begin{pmatrix} b \\ -d' \\ d'' \end{pmatrix}$$

Totally unimodular

By the previous theorem Q is an integer polytope.

$\max\{c^T x : x \in Q\}$ attained by some integer vector $\hat{x}$.

$\hat{x} \in P$ and $c^T \hat{x} \geq c^T x^* = \max\{c^T x : Ax \leq b\}$, so $\hat{x}$ is an optimum solution $\square$

<u>Corollary 2</u>

A : $m \times n$-matrix, totally unimodular, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$.

Then both problems in the LP-duality equation:

$$\max\{c^T x : Ax \leq b\} = \min\{y^T b : y \geq 0, \ y^T A = c^T\}$$

have integer optimum solutions (if the optima are finite).

- The above properties ~~totally~~ characterizes total unimodularity.

Hoffman-Kruskal : The reverse implication holds in Corollary 1.

- TU matrices and bipartite graphs

Theorem

A graph is bipartite if and only if its incidence matrix is totally unimodular.

Proof

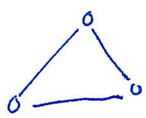
$\Leftarrow$) Let A be totally unimodular. Assume G not bipartite then G contains an odd cycle. The submatrix of A which corresponds to the odd cycle has determinant 2; a contradiction.

$\Rightarrow$) Let G be bipartite. Consider a $t \times t$ submatrix of A. Proof by induction on t. ($t=1$ follows from the def. of the incidence matrix.)

$\square$

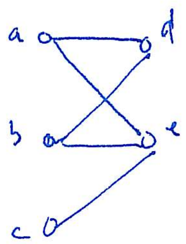
Ex

- K_3



$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

, not TU

- 

Bipartite

$$A = \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

TU

- <u>König's matching theorem</u>

Let G be bipartite. Then the cardinality of a maximal matching is equal to the cardinality of a minimal vertex cover. $(\nu(G) = \tau(G))$

Proof

Let A be the incidence matrix of G.

Both optima in

$$\max\{\mathbb{1}^T x : x \geq 0, Ax \leq 1\} = \min\{y^T \mathbb{1} : y \geq 0, y^T A \geq 1\}$$

are attained by integer optimum solutions x^* and y^*.

x^* is the ~~indicator~~ incidence vector for a matching M with $|M| = \mathbb{1}^T x^*$.

y^* is also a $\{0,1\}$ vector. Let W be the set of vertices for which $y^*_v = 1$. Then W is a vertex cover with $|W| = y^{*T}\mathbb{1}$, and $|M| = |W|$.

$\square$

- This approach can also be extended to weighted versions.

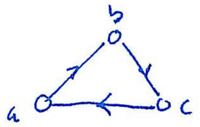
- Incidence matrix of a directed graph

Let $D = (V, A)$ be a directed graph.

The $V \times A$ incidence matrix M of D is defined by:

$$M_{v,a} = \begin{cases} 1, & \text{if } a \text{ leaves } v \\ -1, & \text{if } a \text{ enters } v \\ 0, & \text{otherwise} \end{cases}$$

Ex



$$M = \begin{array}{c} a \\ b \\ c \end{array} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

Theorem

The incidence matrix M of a directed graph D is totally unimodular.

Proof

Let B be a $t \times t$ submatrix of M.

Proof by induction on t ($t=1$ is trivial)

Three cases:

1) B has a zero column $\Rightarrow \det(B) = 0$

2) B has a column with exactly one nonzero. Calculate $\det(B)$ using this column and use the induction assumption.

3) Every column of B has one 1 and one -1. The row vectors of B add up to the zero vector $\Rightarrow \det(B) = 0$. $\square$

- Using the above result, we can prove both Hoffman's circulation theorem and the max-flow min-cut theorem.

- <u>Interval matrices</u>

An interval matrix has 0-1 entries and each row is of the form

$$(0, \dots, 0, \underbrace{1, \dots, 1}_{\text{consecutive 1's}}, 0, \dots, 0)$$

<u>Theorem</u>

Each interval matrix is totally unimodular.

<u>Proof</u>

Let M be an interval matrix and let B be a $t \times t$ submatrix of M.

Let N be the $t \times t$ matrix given by

$$N = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ & & \ddots & \ddots & \\ \cdots & & & & 1 \end{pmatrix}$$

Note that $\det(N) = 1$, and NB^T is a submatrix of the incidence matrix of some directed graph.

So $\det(NB^T) \in \{0, \pm 1\}$, and therefore

$$\det(B) = \det(B^T) \in \{0, \pm 1\}$$

$\square$