FORELESNING 11

Topic: Branch and Bound

* Divide and conquer

Consider the problem: $z = \max\{c^T x : x \in S\}$

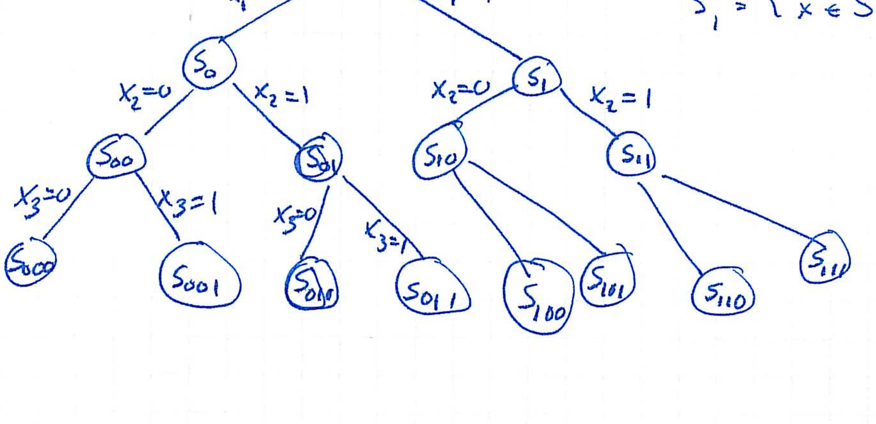
¡How can we break the problem into smaller subproblems and put the information together again?!

Proposition

Let $S = S_1 \cup \dots \cup S_K$ be a decomposition of S into smaller sets, and let $z^k = \max\{c^T x : x \in S_k\}$, $k = 1, \dots, K$. Then $z = \max_k z^k$

* Often represented using an enumeration tree:

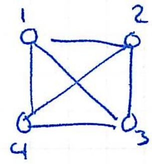
Ex $S \subseteq \{0,1\}^3$



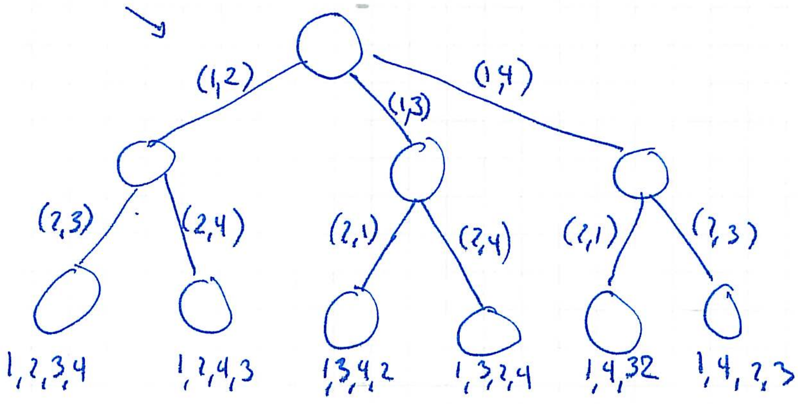
$S_0 = \{x \in S : x_1 = 0\}$

$S_1 = \{x \in S : x_1 = 1\}$

Ex TSP , 4 cities



All tours containing the edge (1,2)



Multiway branching

* ## Implicit enumeration

Complete enumeration is totally impossible for most problems

Use bounds ~~on~~ on the values of $\{z^k\}$

Proposition

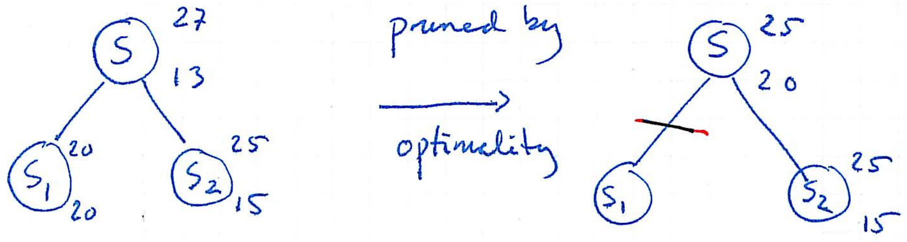
Let $S = S_1 \cup \dots \cup S_K$ be a decomposition of S into smaller sets, and let $z^k = \max\{c^T x : x \in S_k\}$, $k = 1, \dots, K$, $\bar{z}^k$ be an upper bound on z^k and $\underline{z}^k$ be a lower bound on z^k. Then

$\bar{z} = \max_k \bar{z}^k$ is an upper bound on z.

$\underline{z} = \max_k \underline{z}^k$ is a lower bound on z.

- How can bound information be put to use?

Ex 1
(Max)

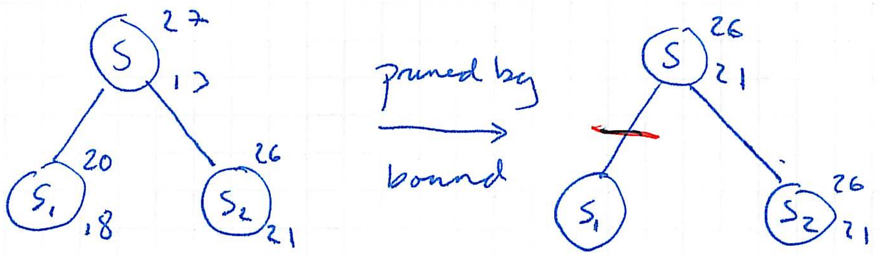


$\bar{z} = \max_k \bar{z}^k = \max\{20, 25\} = 25$

$\underline{z} = \max_k \underline{z}^k = \max\{20, 15\} = 20$

$z' = 20 \Rightarrow$ ~~optim~~ no need to examine S_1 further

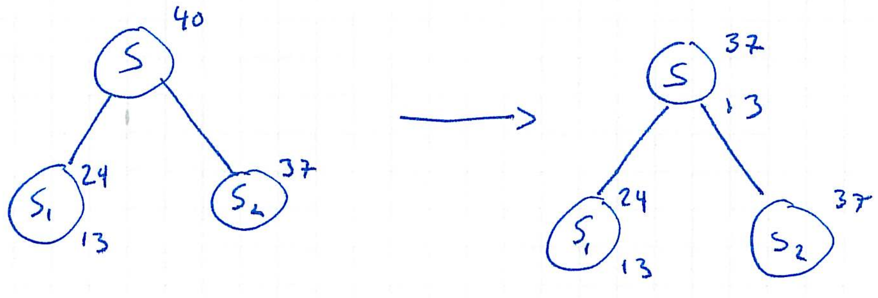
Ex 2
(Max)



$\bar{z} = \max\{20, 26\} = 26$

$\underline{z} = \max\{18, 21\} = 21$

Ex.3



$\bar{z} = \max\{24, 37\} = 37$

$\underline{z} = \max\{13, -\} = 13$

No pruning possible

Can use pruning to enumerate a large number of solutions implicitely

(i) Pruning by optimality : $z^t = \max\{c^T x : x \in S_t\}$ has been solved

(ii) Pruning by bound : $\bar{z}^t \leq \underline{z}$

(iii) Pruning by infeasibility : $S_t = \emptyset$

- Primal bounds (lower) : provided by feasible solutions
  Dual bounds (upper): provided by relaxation or duality

+ 4*

- Several questions have to be answered before we have a well defined algorithm

  * What relaxation (or dual problem) solved for dual bound?
  * How should the feasible region be separated into smaller regions?
  * In what order should the subproblems be examined?

* The most common way to solve integer programs is to use implicit enumeration, or branch and bound, using LP relaxations to provide bounds

- Relaxation

The optimization problem

$$\max \{ g(x) : x \in B \}$$

is a relaxation of $\max \{ c^T x : x \in S \}$

if (i) ~~$f(x)$~~ $g(x) \geq c^T x$ for each $x \in S$

(ii) $S \subseteq B$

Proposition

Let $z = \max \{ c^T x : x \in S \}$ and $z^R = \max \{ g(x) : x \in B \}$.

Then $z^R \geq z$.

Proof

Let x^* be an optimal sol. to the original problem.

Then $x^* \in S \subseteq B$ and $z = c(x^*) \leq g(x^*)$.

As $x^* \in B$, $g(x^*)$ is a lower bound on z^R,

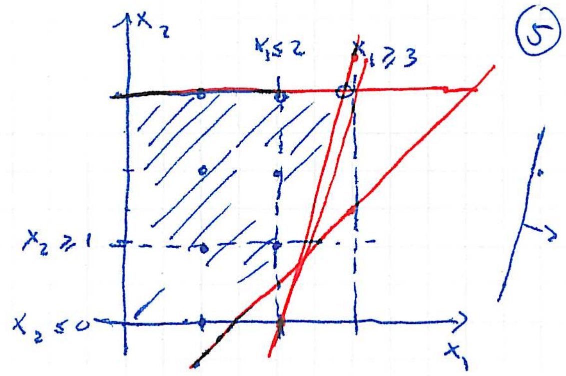
and so $z \leq g(x^*) \leq z^R$. $\square$

Ex Linear programming relaxation

- $g(x) = c^T x$
- $S = \{ Ax \leq b \} \cap \mathbb{Z}^n$

$B = \{ Ax \leq b \}$

Ex



$$z = \max \; 4x_1 - x_2$$

$$\text{s.t.} \quad 7x_1 - 2x_2 \le 14$$
$$x_2 \le 3$$
$$2x_1 - 2x_2 \le 3$$
$$x \in \mathbb{Z}^2_+$$

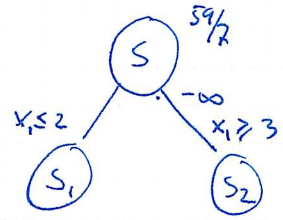
Bounding : LP relaxation : $\bar{z} = \frac{59}{7}$

$(x_1^*, x_2^*) = (\frac{20}{7}, 3)$

Lower bound : $\underline{z} = -\infty$

Branching : Split into two about a fractional value in the LP-solution

$S_1 = S \cap \{x : x_1 \le 2\}$ $\quad$ $S_2 = S \cap \{x : x_1 \ge 3\}$

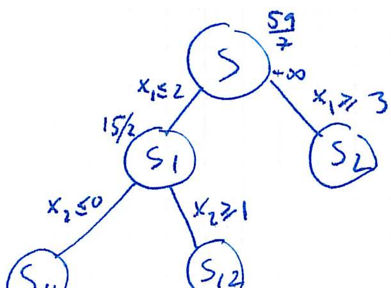


Choosing a node : Arbitrarily S_1

Bounding : $\bar{z}_1 = \frac{15}{2}$ $\quad (x_1^*, x_2^*) = (2, \frac{1}{2})$

Branching : $S_{11} = S_1 \cap \{x : x_2 \le 0\}$

$S_{12} = S_1 \cap \{x : x_2 \ge 1\}$



Choosing a node: S_2

Bounding:
~~No feasible~~ LP is infeasible $\bar{z}_2 = -\infty$

S_2 pruned by infeasibility

Choosing a node: S_{12}

Bounding $z^{12} = 7$ $\bar{x}^{12} = (2,1)$

S_{12} ~~branch~~ pruned by optimality

~~$\underline{z} = \max\{\underline{z}, 7\}$~~ $\underline{z} = \max\{\underline{z}, 7\} = 7$

Choosing a node: S_{11}

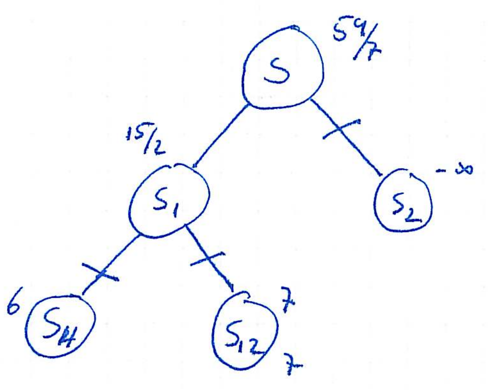
Bounding: $\bar{x}^{11} = (\frac{3}{2}, 0)$ $\bar{z}_{11} = 6$

$\bar{z}_{11} = 6 < 7 = \underline{z}$

S_{11} pruned by bound

No more nodes to examine

$x = (2,1)$ with value 7 is optimal



* LP-based branch and bound

Storing the tree : only active nodes

How to bound : LP relaxation

How to branch : - most fractional value
(i.e. close to ½)

How to choose a node : - depth-first strategy
(find feasible solutions)

- best node first
(largest upper bound)

(or a combination)

* Branch and bound systems

- Preprocessor
- Simplex algorithm
- Choice of branching and node selection
- Priorities

Other elements : - SOS sets
- Strong branching
- Reduced cost fixing
- Primal heuristics