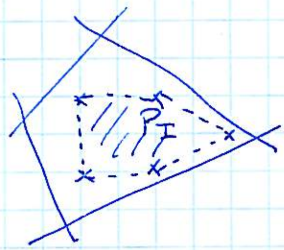
CUTTING PLANES

* Integer program

$$\max\{c^T x : Ax \le b, x \in \mathbb{Z}^n\}$$

Let $S = \{Ax \le b, x \in \mathbb{Z}^n\}$, $P = \{Ax \le b, x \in \mathbb{R}^n\}$



Let P_I = conv. hull (S) be the integer hull,
since the obj. function is linear

$$\max\{c^T x : x \in S\} = \max\{c^T x : x \in \text{conv. hull}(S)\}$$

$$= \max\{c^T x : x \in P_I\}$$

P_I is a polyhedron $\Rightarrow$ $P_I = \{x \in \mathbb{R}^n : \bar{A}x \le \bar{b}\}$

$$\max\{c^T x : x \in S\} = \max\{c^T x : \bar{A}x \le \bar{b}\}$$

The original integer program can be solved as a linear program.

Problem: $\bar{A}x \le \bar{b}$ may be both large and difficult to find.

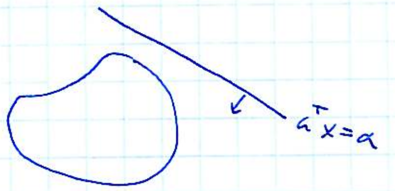
Two questions:

1) How to find additional inequalities?
2) How can the inequalities be used in algorithms?

* <u>Finding valid inequalities</u>

Let $S \subseteq \mathbb{R}^n$, $a^T x \le \alpha$ $(a \neq 0)$ is <u>valid</u> for S

if $S \subseteq \{ x \in \mathbb{R}^n : a^T x \le \alpha \}$



- <u>Chvatal-Gomory procedure</u>

$P = \cancel{\{ x \in \mathbb{R}^n : Ax \le b \}} = \{ Ax \le b ,\ x \in \mathbb{R}^n \}$

Each inequality in $Ax \le b$ is valid for P_I

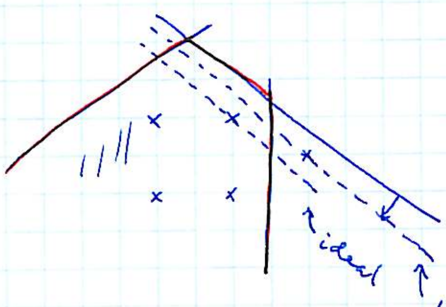
Basic idea: $x \le \alpha$, x integer

$$\Downarrow$$

$$x \le \lfloor \alpha \rfloor$$

integer rounding

Want to push the inequalities in P closer to P_I:



Ex. $\quad x_1 + 2x_2 \le \frac{10}{3}$ valid for P

$$\Downarrow$$

$x_1 + 2x_2 \le 3$ valid for P_I

- General procedure : (assume $P \subseteq \mathbb{R}^n_+$)

Multiply the i'th inequality by $\lambda_i \ge 0$ and sum the inequalities

$$\cancel{\left(\sum \lambda_i a_{i,\cdot}^T\right) x \le \sum \lambda_i b_i}$$

Let $\quad A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$

(i) The inequality

$$\sum_{j=1}^{n} \lambda^T a_j x_j \le \lambda^T b$$

is valid for P ($\lambda \ge 0$ and $\sum_{j=1}^{n} a_j x_j \le b$)

(ii) The inequality

$$\sum \lfloor \lambda^T a_j \rfloor x_j \le \lambda^T b$$

is valid for P ($x \geq 0$)

(iii) The inequality

$$\sum \lfloor \lambda^T a_j \rfloor x_j \le \lfloor \lambda^T b \rfloor$$

is valid for P_I (x is integer)

Theorem

Every valid inequality for P_I can be obtained by applying the Chvátal-Gomory procedure a finite number of times.

Ex Matching

$$P = \{ x_e : x_e \geq 0 , \sum_{e \in \delta(v)} x_e \leq 1 , \forall v \in V \}$$

Fractional matching polytope

P_I : matching polytope

~~Give~~ ~~Consider~~ Let $S \subset V$ of odd size, $|S| = 2k+1$ and consider the ~~and the corresponding~~ inequalities $\sum_{e \in \delta(v)} x_e \leq 1$ for $v \in S$

and $-x_e \leq 0$, $e \in \delta(S)$.

Multiply each ineq. by ½ and add together

$$x(E(S)) \leq k + \frac{1}{2}$$

Thus

$$x(E(S)) \leq k \quad \text{is valid for } P_I$$

¡ P + ~~ineq~~ odd set ineq. ⇒ complete description !

- **Boolean implications**

Knapsack problem

$$\max \{ \sum_{j=1}^{n} c_j x_j : \sum_{j=1}^{n} a_j x_j \leq b , x_j \in \{0,1\} \}$$

$$P = \{ \sum_{j=1}^{n} a_j x_j \leq b , 0 \leq x \leq 1 \}$$

P_I : knapsack polytope

Ex: $n=3$, $a_1 = 3$, $a_2 = 3$, $a_3 = 2$ $b = 7$

Let $C = \{1,2,3\}$. $a(C) = \sum_{j\in C} a_j = 8 > b$

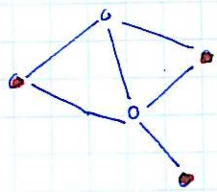
Can not have all variables in C equal to 1.

$\Rightarrow \quad x(C) \le |C| - 1$

A set C with $a(C) \geq b$ is called a cover and the valid inequality $x(C) \le |C|-1$ is called a cover inequality.

- Combinatorial implications

Ex. Node packing, stable set



$$\max \{ \textstyle\sum x_u : x_u + x_v \le 1 \text{ for } uv \in E, \; x_u \in \{0,1\} \}$$

$$P = \{ x_u + x_v \le 1 \text{ for } uv\in E, \; 0 \le x_u \le 1 \}$$

$P = P_I$ only for bipartite graphs

A clique is a ~~subset~~ complete subgraph, i.e. a subset V_0 such that $uv \in E$ for all distinct $u, v \in V_0$.

Then $\sum_{v\in V_0} x_v \le 1$ is a valid inequality

$\uparrow$

clique inequality

Cutting-plane algorithms (6.4)

Idea: add cutting planes (i.e. valid inequalities) to a linear program, in order to approximate the underlying integer program

Consider an integer program

$v^* = \max \{ c^T x : x \in P, \ x \text{ integral} \}$

where $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$

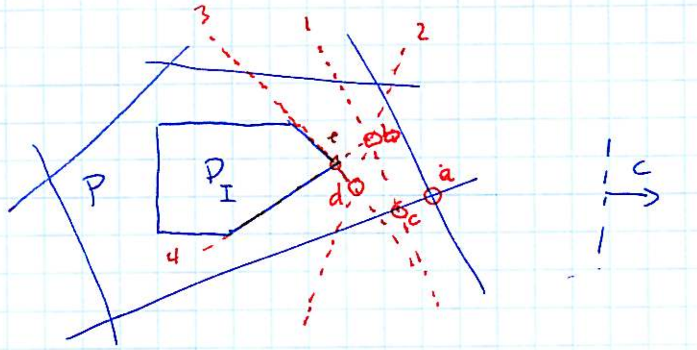
~~The algorithm~~ Let Π be a finite class of valid ineq. for P_I.

Fractional cutting-plane algorithm

1. Initialization. $A^0 = A$, $b^0 = b$, $k = 0$
2. Optimization. Solve $\max \{ c^T x : A^k x \leq b^k \}$ and let x^k be the optimal solution. If x^k is integral, stop.
3. Separation. Test if x^k satisfies all the inequalities in Π. If it does, stop. Otherwise ~~take~~ add one or more violated ineq. to obtain $A^{k+1} x \leq b^{k+1}$ and go to 2.

~~Unless~~

Ex



Unless Π defines P_I, we are not guaranteed to find an integer solution.

~~Measure of success : optimality gap $c^T x^N - v(A)$~~

Gives an upper bound on the optimal value:

$$v^* \leq c^T x^k$$

Combining with a heuristic that gives feasible solutions $\hat{x}^k \in S$, we have that $c^T \hat{x}^k \leq v^* \leq c^T x^k$ in each iteration.

+ Separation :

Check if $\pi^T x^k \leq \pi_0$ for all $(\pi, \pi_0) \in \Pi$ and if it does not, find at least one violated ineq.

May be viewed as an optimization problem

$$\max \{ \pi^T x^k - \pi_0 : (\pi, \pi_0) \in \Pi \}$$

! The difficulty will depend upon Π !

Combining this approach with branch and bound:

1) A priori addition of valid inequalities

2) Generate cutting planes throughout the branch and bound tree : branch-and-cut algorithm

(cplex , xpress-MP)