Lecture 1 - INF-MAT 5360

- Practical information
- Optimization - overview
- Modeling - linear programming

Practical information

- Lecturer: Truls Flatberg, Truls.Flatberg@sintef.no
- Course web page: http://www.uio.no/studier/emner/matnat/ifi/INF-MAT5360/h08/
- Literature:
  - A. Schrijver: A course in Combinatorial Optimization
  - G. Dahl: An introduction to convexity, polyhedral theory and combinatorial optimization
  - C. Gueret et al: Applications of optimization with Xpress-MP
- 3 obligatory projects
- Oral exam (December)

Optimization

An optimization problem has the following basic ingredients:

- Objective function to be minimized or maximized
- Variables that influence the objective function
- Constraints that restricts the legal values of the variables

Problem: Find values for the variables that minimizes (alt. maximizes) the objective function and obeys the constraints.

Mathematical formulation of an optimization problem

$$\min f(x)$$
$$\text{subject to } x \in X$$

- $f : X \longrightarrow \mathbb{R}$: objective function
- x: variables
- X: set of feasible solutions (constraints)

Example:

$$\begin{aligned} \min \ & (x_1 - 2)^2 + (x_2 - 1)^2 \\ \text{s.t.} \ & x_1^2 - x_2 \leq 0 \\ & x_1 + x_2 \leq 2 \end{aligned}$$

- $f(x) = (x_1 - 2)^2 + (x_2 - 1)^2$
- $x = (x_1, x_2)^T$
- $X = \{x \in \mathbb{R}^2 : x_1^2 - x_2 \leq 0, x_1 + x_2 \leq 2\}$

Classification

Objective function

- none - feasibility problem
- multiple - multi-criteria optimization

Variables

- continuous - continuous optimization
- discrete values - discrete optimization

Constraints

- none - unconstrained optimization
- finite set of feasible solutions - combinatorial optimization

Special cases:

- Linear programming - objective and constraints are linear functions
- Integer programming - as for LP, but with integer valued variables
- Nonlinear optimization - objective and constraints are nonlinear functions
- Convex optimization - convex objective and the set of feasible solutions is convex

Uncertainty in parameters and constraints:

- Stochastic optimization
- Robust optimization

Main topics of this course:

- Modeling, OPL studio
- Combinatorial optimization
  - Spanning tree
  - Shortest path
  - Matching
- Network flow
- Polyhedral theory
- Integer optimization

Linear programming is assumed known (INF-MAT 3370).

Modeling

Developing linear and integer programming models

1. Choosing variables
2. Modeling constraints
3. Objective function

Linear program

- Optimize a linear function
- Constraints are linear equations or inequalities

$$
\begin{array}{lrl}
\text{maximize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \leq b_1 \\
 & & \vdots \\
 & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \leq b_m \\
 & x_1, x_2, \ldots, x_n & \geq 0
\end{array}
$$

- Matrix form

$$
\begin{array}{ll}
\max & c^T x \\
\text{s.t.} & Ax \leq b \\
 & x \geq 0
\end{array}
$$

Example: Chess set problem

- A small company produces chess sets of two different sizes
- The small set requires 3 hours of work and 1 kg wood
- The large set requires 2 hours of work and 3 kg wood
- 4 persons working 40 hours a week
- Only 200 kg wood per week can be obtained
- Large set yield profit of $20 while the small has a profit of $5

How many of each set should be produced each week to maximize profit?

Chess set problem: model

1. Choosing variables

$$x_s : \text{ the number of small sets to make}$$
$$x_l : \text{ the number of large sets to make}$$

2. Modeling constraints

$$3x_s + 2x_l \leq 160$$
$$x_s + 3x_l \leq 200$$
$$x_s, x_l \geq 0$$

3. Objective function

$$\max 5x_s + 20x_l$$

Limitations of the model

A model will always be a simplification of the real world.

Explicit assumptions

- All chess sets could be sold
- No changeover time between sizes of set

Implicit assumptions

- No machine will break down
- All workers will turn up
- Are fractional chess sets meaningful?

Typical LP model constructs

- Simple upper and lower bounds

$$x \geq 4, y \leq 1600$$

- Flow constraints
  Divisible elements (e.g. water, electricity) that come together or are divided

$$x_1 + x_2 + x_3 = 1000$$

- Resource constraints
  Limitations of available resources

$$x + y \leq 500$$

Typical LP model constructs

- Material balance constraints
  What comes in must come out

$$x_1 + x_2 + x_3 = y_1 + y_2$$

- Quality requirements
  Ensuring quality when blending product

$$x_1 + 2x_2 + 1.5x_3 >= 5$$

- Accounting constraints
  Help variables create quasi-independent pieces

$$x_{total} = x_1 + x_2 + x_3 + x_4$$

Soft constraints

- Hard constraints : physical, accounting, definitional constraints
- Soft constraints : constraints that can be violated at a cost

$$\min \; c_1 x_1 + c_2 x2 + Cy$$
$$\text{subject to } x_1 + x_2 \geq 500 - y$$

- y is called a slack variable (or panic variable)
- Can use slack variables to ensure the existence of feasible solutions

Objective functions

- Minimax objective functions

$$\min \max(x_1, x_2, \ldots, x_n)$$

Equivalent with

$$\begin{aligned} \min\ & s \\ \text{subject to}\ & s \geq x_1 \\ & s \geq x_2 \\ & \vdots \\ & s \geq x_n \end{aligned}$$

- Ratio objective function

$$\min f = \frac{\sum_j a_j x_j}{\sum_j b_j x_j}$$

Introduce new variables $d = \frac{1}{\sum_j b_j x_j}$ and $y_j = dx_j$.

$$f = \sum_j d a_j x_j = \sum_j a_j y_j$$

Replace x_j with y_j/d in constraints.

A scheduling example

A construction company is hired to renovate the university campus. The work can be broken down into a series of n tasks. Each task i has a duration d_i and can only begin after other tasks have been completed, i.e. for each task we have a set of predecessor tasks P_i.

What is the shortest time to complete the renovation?

1. Choosing variables

$$x_i = \text{ the starting time of task } i$$

2. Modeling constraints
   Assume the clock starts at the beginning of the project

$$x_i \geq 0$$

   A task must start after its predecessors

$$x_i \geq x_j + d_j \text{ for all } j \in P_i$$

3. Objective function

$$\min \max(x_1 + d_1, x_2 + d_2, \ldots, x_n + d_n)$$