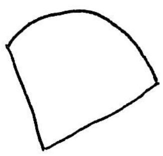
Convex sets

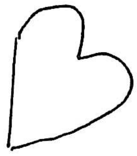
A set $C \subseteq \mathbb{R}^n$ is called <u>convex</u> if

$(1-\lambda)x + \lambda y \in C$ whenever $x, y \in C$ and $0 \leq \lambda \leq 1$

Geometrically: C contains the line segment between each pair of points



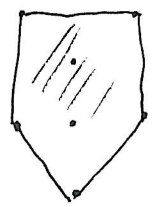
convex



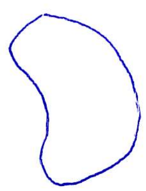
non-convex

- For any $X \subseteq \mathbb{R}^n$, define its <u>convex hull</u> as the smallest convex set containing X.

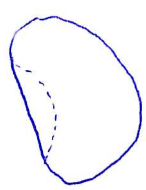
X



conv. hull(X)



X



conv. hull(X)

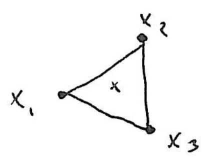
- Consider vectors $x_1, \dots, x_t \in \mathbb{R}^n$ and numbers $\lambda_j \geq 0$, $j = 1, \dots, t$, such that $\sum_{j=1}^{t} \lambda_j = 1$.

  Then the vector $x = \sum_{j=1}^{t} \lambda_j x_j$ is called a <u>convex combination</u> of $x_1, \dots, x_t$.

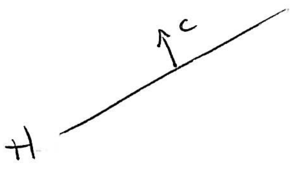
<u>Proposition</u>

conv. hull (X) is equal to the set of all convex combinations of points in X.

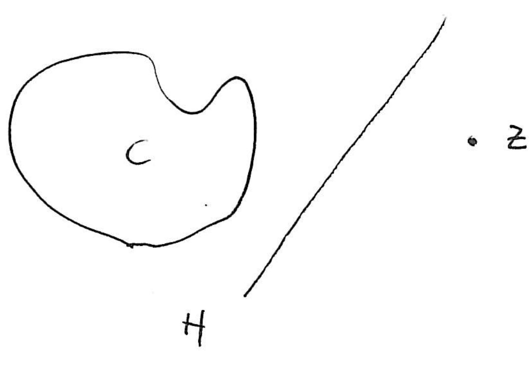
Proof: exercise



- <u>Hyperplane</u>: $H = \{x \in \mathbb{R}^n : c^T x = \delta\}$



- H <u>separates</u> a point z and a set C if z and c are in different components of $\mathbb{R}^n \setminus H$

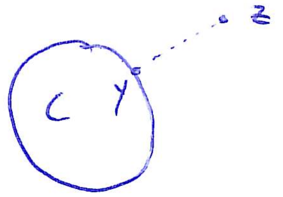


Theorem

Let C be a closed convex set in $\mathbb{R}^n$ and let $z \notin C$. Then there exists a hyperplane separating z and C.

Proof

Assume $C \neq \emptyset$. Then there exists a vector $y \in C$ that is nearest to z.



(Minimization of a continuous function over a compact set, Weierstrass: Extreme value theorem)

Define $c := z - y$ and $\delta := \frac{1}{2}\left(\|z\|^2 - \|y\|^2\right)$

Want to show that: $c^T z > \delta$

and $c^T x < \delta \quad \forall x \in C$

- $c^T z = (z-y)^T z > (z-y)^T z - \frac{1}{2}\underbrace{\|z-y\|^2}_{= \|z\|^2 - 2z^T y + \|y\|^2} = \delta$

- Assume there exists $x \in C$ with $c^T x \geq \delta$

From $c^T y < c^T y + \frac{1}{2}\|c\|^2 = \delta$, we have

$c^T (x - y) > 0$.

Hence there exists a λ with $0 \le \lambda \le 1$ and

$$\lambda < \frac{2c^T(x-y)}{\|x-y\|^2}$$

Define $\quad w := \lambda x + (1-\lambda) y \in C \quad$ (conv. comb.)

$$\|w-z\|^2 = \|\lambda(x-y) + (y-z)\|^2 = \|\lambda(x-y) - c\|^2$$

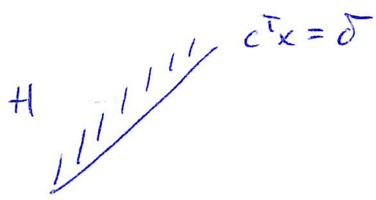
$$= \underbrace{\lambda^2 \|x-y\|^2 - 2\lambda c^T(x-y)}_{<0} + \|c\|^2$$

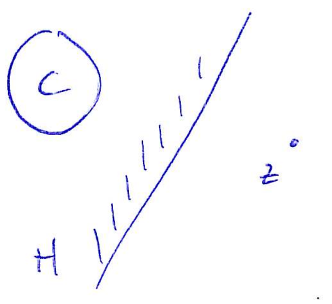
$$< \|c\|^2 = \|y-z\|^2$$

i.e. $\|w-z\| < \|y-z\|$, a contradiction to the fact that y is a point in C nearest to z.

$\square$

- <u>Half space</u>: $\quad H = \{x \in \mathbb{R}^n : c^T x \le \delta\}$





Theorem

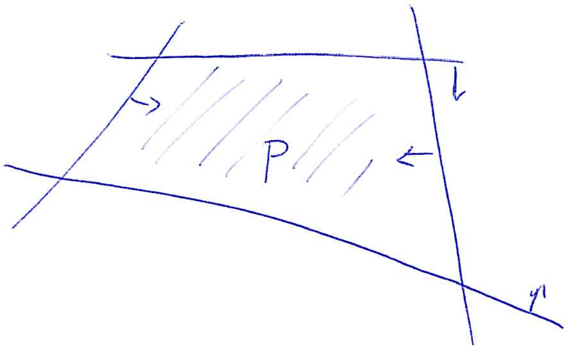
$\Downarrow$

$C \subseteq H, \quad z \notin H$

Polytopes and Polyhedra

A polyhedron is the intersection of a finite number of halfspaces.

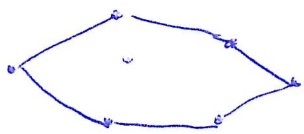
$$P = \{x \in \mathbb{R}^n : Ax \le b\}$$



Exercise: show that a polyhedron is convex

- $P \subseteq \mathbb{R}^n$ is called a polytope if P is the convex hull of a finite number of vectors.

  $P = \text{conv. hull}\{x_1, \dots, x_t\}$ for some vectors $x_1, \dots, x_t \in \mathbb{R}^n$

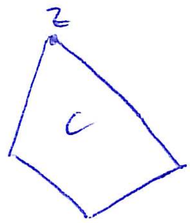


polytope

- Let C be a convex set.

  If $z \in C$ can not be written as a convex combination of two other points in C, then z is called a vertex.

ex



- <u>Characterization of vertices</u>

Let $P = \{x : Ax \leq b\}$ be a polyhedron and let $z \in P$.

A_z is the submatrix of A with the rows where $a_i z = b_i$

<u>Theorem</u>

z is a vertex of P if and only if $\text{rank}(A_z) = n$.

<u>Proof</u>

$\Rightarrow$ Let z be a vertex and suppose $\text{rank}(A_z) < n$. Then there exists $c \neq 0$ with $A_z c = 0$, and a $\delta > 0$ such that

$$a_i(z + \delta c) \leq b_i$$
$$a_i(z - \delta c) \leq b_i$$

for every row i not in A_z $\quad (a_i z < b)$

Combining this with $A_z c = 0$ and $Az \leq b$, we get

$$A(z + \delta c) \leq b \quad \text{and} \quad A(z - \delta c) \leq b$$

So $\underset{\Uparrow \atop P}{}$ $z = \frac{1}{2}\underbrace{(z+\delta c)}_{\Uparrow \; P} + \frac{1}{2}\underbrace{(z-\delta c)}_{\Uparrow \; P}$ is a convex

combination of points in P, a contradiction.

$\Leftarrow$ Suppose $\operatorname{rank}(A_z) = n$, z not a vertex.

$z = \frac{1}{2}(x+y)$ for $x \neq y \neq z$, $x, y \in P$

For every row a_i of A_z:

$$a_i x \leq b_i = a_i z \Rightarrow a_i(x-z) \leq 0$$
$$a_i y \leq b_i = a_i z \Rightarrow a_i(y-z) \leq 0$$

$$y - z = -(x-z) \Rightarrow a_i(x-z) = 0$$

and hence $A_z(x-z) = 0$ with $x - z \neq 0$

□

- Consequence:

  A polyhedron has only a finite number of vertices.

  At most 2^m collections of subrows

  $\Downarrow$

  At most 2^m vertices

Theorem

Let P be a bounded polyhedron with vertices $x_1, \dots, x_t$. Then $P = \text{conv. hull}\{x_1, \dots, x_t\}$.

Proof

$\text{conv. hull}\{x_1, \dots, x_t\} \subseteq P$ ($x_1, \dots, x_t \in P$, P convex)

Reverse inclusion : $z \in P \Rightarrow z \in \text{conv. hull}\{x_1, \dots, x_t\}$

By induction on $n - \text{rank}(A_z)$

* $n - \text{rank}(A_z) = 0$: $\text{rank}(A_z) = n \Rightarrow z$ is a vertex of P $\Rightarrow z \in \text{conv. hull}\{x_1, \dots, x_t\}$

* $n - \text{rank}(A_z) > 0$: There exists $c \neq 0$ such that $A_z c = 0$.

Define : $\mu_0 := \max\{\mu : z + \mu c \in P\}$
$\nu_0 := \min\{\nu : z - \nu c \in P\}$

Let $x := z + \mu_0 c$, $y := z - \nu_0 c$

Now $\mu_0 = \min\left\{ \frac{b_i - a_i z}{a_i c} : a_i \text{ row of } A ; a_i c > 0 \right\}$

Follows from
$$a_i(z + \mu c) \leq b_i$$
$$\mu \leq \frac{b_i - a_i z}{a_i c}$$

Let the minimum be obtained for i_0, i.e.

$$\mu_0 = \frac{b_{i_0} - a_{i_0} z}{a_{i_0} c}$$

Therefore: $A_z x = A_z z + \mu_0 A_z c = A_z z$

$$a_{i_0} x = a_{i_0}(z + \mu_0 c) = b_{i_0}$$

A_x contains all rows in A_z + a_{i_0}

$$\left.\begin{array}{l} A_z c = 0 \\ a_{i_0} c \neq 0 \end{array}\right. \Rightarrow \text{rank}(A_x) > \text{rank}(A_z)$$

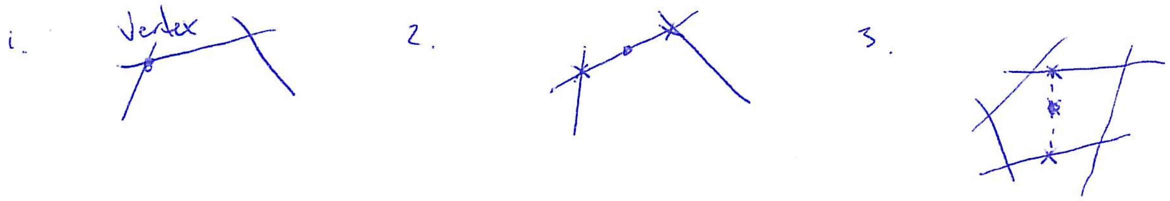
By the induction hypothesis $x \in$ conv. hull $\{x_1, \dots, x_t\}$

Similarly, $y \in$ conv. hull $\{x_1, \dots, x_t\}$

$$z = \frac{\nu_0}{\nu_0 + \mu_0} x + \frac{\mu_0}{\nu_0 + \mu_0} y \in \text{conv. hull } \{x_1, \dots, x_t\}$$

$\square$

; Show that a convex combination of points from the convex hull is in the convex hull ;



Theorem

Each polytope is a bounded polyhedron

Proof

Self study