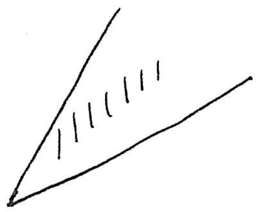
Convex Cones

$C \subseteq \mathbb{R}^n$ is a convex cone if

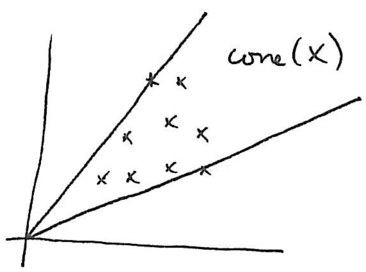
$\lambda x + \mu y \in C$ for any $x, y \in C$ and $\lambda, \mu \geq 0$

Ex



- cone (X), conical hull, smallest cone containing X
- A cone C is called finitely generated if $C = \text{cone}(X)$ for some finite set X.

Ex



- Finitely generated cones are closed
- A cone is polyhedral if $C = \{x \in \mathbb{R}^n : Ax \leq 0\}$

FARKAS' LEMMA

A : $m \times n$ - matrix , $A \in \mathbb{R}^{m \times n}$

$b \in \mathbb{R}^m$

Fredholm's alternative :

$Ax = b$ has a solution $\Longleftrightarrow$ there is no y with $y^T A = 0$ and $y^T b = -1$

Generalization to linear systems :

Farkas' lemma

The linear system $Ax = b$, $x \geq 0$ has a solution if and only if there is no $y \in \mathbb{R}^m$ with $y^T A \geq 0$ and $y^T b < 0$.

Proof

$\Rightarrow$ Suppose $Ax = b$ has a solution $x_0 \geq 0$ and there exists y_0 with $y_0^T A \geq 0$ and $y_0^T b < 0$.

Then

$$0 > y_0^T b = y_0^T (A x_0) = (y_0^T A) x_0 \geq 0$$

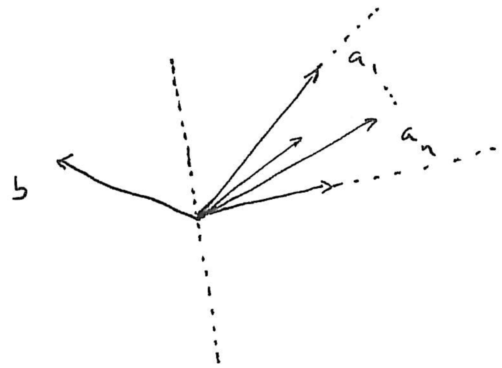
A contradiction.

$\Leftarrow$ Suppose $Ax = b$ has no solution $x \geq 0$

Let $A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$

So $b \notin C := \text{cone}\{a_1, \dots, a_n\}$

closed and convex



b and C can be separated by a hyperplane through the origin, i.e. there exists a vector c such that $c^T b < 0$ while $c^T x \geq 0$ for each $x \in C$.

In particular $c^T a_j \geq 0$, $j = 1, \dots, n$

So $y := c$ satisfies $y^T A \geq 0$ and $y^T b < 0$.

$\square$

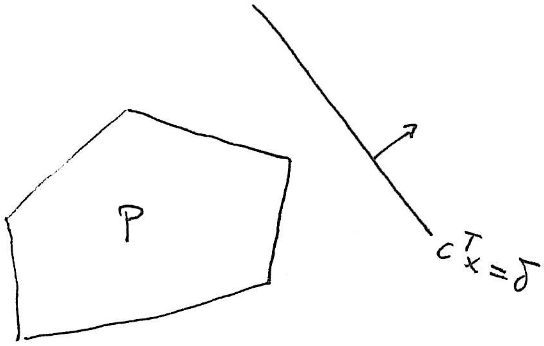
- Alternative version(s)

$Ax \leq b$ has a solution $\Leftrightarrow$ there is no y with $y \geq 0$, $y^T A = 0$ and $y^T b < 0$

Corollary

Suppose $Ax \leq b$ has at least one solution.
Then for every solution x of $Ax \leq b$ one has $c^T x \leq \delta$ if and only if there exists a vector $y \geq 0$ such that $y^T A = c^T$ and $y^T b \leq \delta$

$P = \{x : Ax \leq b\}$

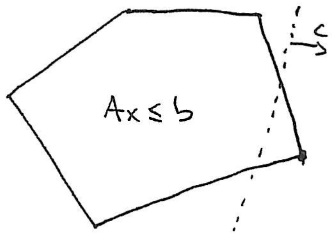


Proved using Farkas' lemma. Self study.

LINEAR PROGRAMMING

maximize $c^T x$

subject to $Ax \leq b$



- Alternative : $\max \{ c^T x : Ax \leq b \}$

- Feasible set $P = \{ x \in \mathbb{R}^n : Ax \leq b \}$

  polyhedron (and convex)

- In an LP-problem with finite optimal value the set P^* of optimal solutions is a polyhedron and convex :

$$P^* = \{ x \in \mathbb{R}^n : Ax \leq b , \; c^T x = v^* \}$$

- If P is a non-empty polytope, then $\max \{ c^T x : Ax \leq b \}$ is attained at a vertex of P.

$$c^T x = \sum \lambda_i c^T v_i \leq c^T v_k^* \sum \lambda_i = c^T v_k^*$$

$$\left(\begin{array}{l} x = \sum \lambda_i v_i \\ \sum \lambda_i = 1 \end{array} \right) \qquad \text{where } c^T v_k^* \geq c^T v_i \quad \forall i$$

- Algorithms

  - Simplex : Finds an optimal vertex solution
    (Dantzig, 51) Exponential worst case running time.

  - Ellipsoid : Polynomial running time, not working
    (Khachiyan, 79) in practice

  - Inner point : Polynomial and comparable with
    (Karmarkar, 84) the simplex method in practice.

- To each LP problem $\max\{c^T x : Ax \le b\}$ we associate a dual problem

$$\begin{aligned} &\min \quad y^T b \\ &\text{subject to} \quad y^T A = c^T \\ &\qquad\qquad\quad y \ge 0 \end{aligned}$$

$\{\min\{y^T b : y \ge 0,\ y^T A = c^T\}\}$

- Duality theorem of linear programming

$$\boxed{\begin{array}{l} \max\{c^T x : Ax \le b\} = \min\{y^T b : y \ge 0,\ y^T A = c^T\} \\ \text{provided that both sets are non-empty} \end{array}}$$

Proof

$$c^T x = (y^T A)x = y^T (Ax) \le y^T b$$

$$\Rightarrow \quad \max\{c^T x : Ax \le b\} \le \min\{y^T b : y \ge 0,\ y^T A = c^T\}$$

{ Technical: max and min are attained }

Need to show equality.

Let $\delta := \max\{c^T x : Ax \le b\}$ Hence $Ax \le b \Rightarrow c^T x \le \delta$.

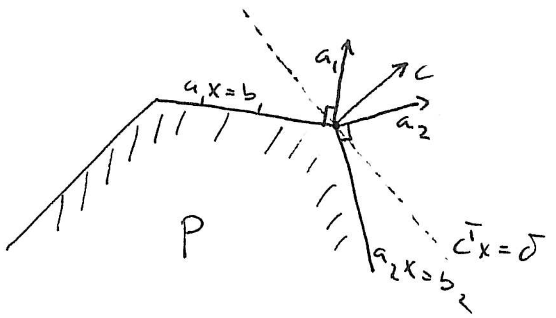
From the corollary to Farkas' lemma, it follows that there exists a vector y such that

$$y \ge 0 \;,\; y^T A = c^T \;,\; y^T b \le \delta$$

$$\Rightarrow \min\{y^T b : y \ge 0 \;,\; y^T A = c^T\} \le \delta$$

□

- Geometrically



$c^T x$ must be a non-negative linear combination of $a_1 x_1 = b_1, \ldots, a_k x_k = b_k$

$$\lambda_1 a_1 + \ldots + \lambda_k a_k = c^T$$

$$\lambda b_1 + \ldots + \lambda_k b_k = \delta \qquad \text{for } \lambda_1, \ldots, \lambda_k \ge 0$$

$y^* := (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0)$ is feasible for the dual problem

Therefore

$$\max\{c^T x : Ax \leq b\} = \delta = \lambda_1 b_1 + \dots + \lambda_k b_k$$
$$\geq \min\{y^T b : y \geq 0,\ y^T A = c^T\}$$

Weak duality : $\max\{c^T x : Ax \leq b\} \leq \min\{y^T b : y \geq 0,\ y^T A = c^T\}$

So y^* is an optimum solution of the dual problem.