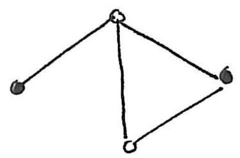
# Bipartite Matching

$G = (V, E)$, graph

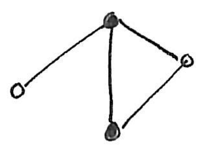
- Stable set : $C \subseteq V$ such that $e \not\subseteq C$ for each edge of $G$

Ex.



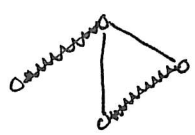
- Vertex cover : $W \subseteq V$ such that $e \cap W \neq \emptyset$ for each edge of $G$

Ex.



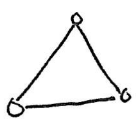
- Matching : $M \subseteq E$ such that $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$

Ex.



A matching is perfect if it covers all vertices.

Ex.



$K_3$ has no perfect matching

- M-augmenting paths

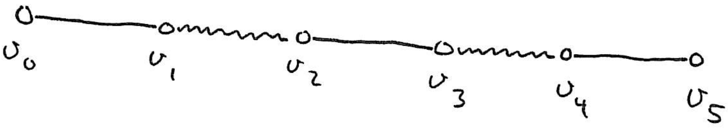
$M$ : matching in a graph $G=(V,E)$

A path $P=(v_0, v_1, \dots, v_t)$ is called M-augmenting if

(i) $t$ is odd

(ii) $v_1v_2, v_3v_4, \dots, v_{t-2}v_{t-1} \in M$

(iii) $v_0, v_t \notin \bigcup M$




Let $M' := M \,\Delta\, EP$ ← edges in $P$  ( $A \,\Delta\, B = (A\setminus B) \cup (B \setminus A)$ )

Then $|M'| = |M| + 1$

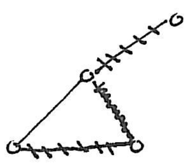
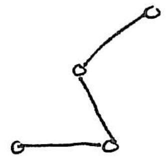
Theorem

$M$ is a matching of maximum cardinality $\iff$ there exists an M-augmenting path.

Proof

If $M$ is a matching of max. cardinality there is no M-augmenting path (see above).

If $M'$ is a matching larger than $M$, consider $G' := (V, M \cup M')$

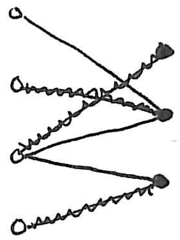
Ex.  $G' =$  (Each node has a max degree of 2)

Each component of $G'$ is either a path or a circuit. Since $|M'| > |M|$, at least one of these components should contain more edges of $M'$ than of $M$. Such a component forms an $M$-augmenting path. $\square$

**Theorem** (König's matching theorem)

For any bipartite graph $G=(V,E)$ the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover.

Ex

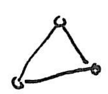


Proof

Let $\nu(G) := \max\{|M| : M \text{ is a matching}\}$

$\tau(G) := \min\{|W| : W \text{ is a vertex cover}\}$

We have that $\nu(G) \leq \tau(G)$

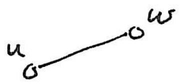
(  $\nu(G) < \tau(G)$ )

Assume that $G$ has at least one edge.

Will show that:

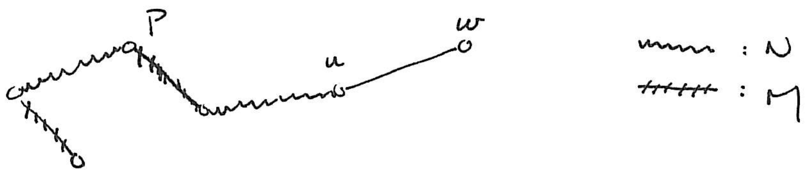
$G$ has a vertex $u$ covered by each maximum-size matching $\quad (*)$

Let $e = uw$ be any edge of $G$




$M$: matching of max-size missing $u$

$N$: matching of max-size missing $w$




$P$: the component of $M \cup N$ containing $u$

$P$ is a path with end vertex $u$

$P$ is not $M$-augmenting $\Rightarrow$ $P$ has even length

$\Rightarrow$ $P$ does not traverse $v$.

{ A bipartite graph does not contain odd-length cycles }

So $P \cup e$ would form an $N$-augmenting path, and we have a contradiction.

$(*)$ implies that for $G' := G - u$ we have $\nu(G') = \nu(G) - 1$

Then $C \cup \{e\}$ is a vertex cover of $G$ of size

$$\nu(G') + 1 = \nu(G)$$

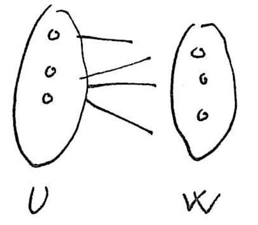
$\square$

- Cardinality bipartite matching algorithm

How to find M-augmenting paths?

Algorithm

- Start with a matching $M$
- Orient each edge $e = uw$ as follows




$V = U \cup W$

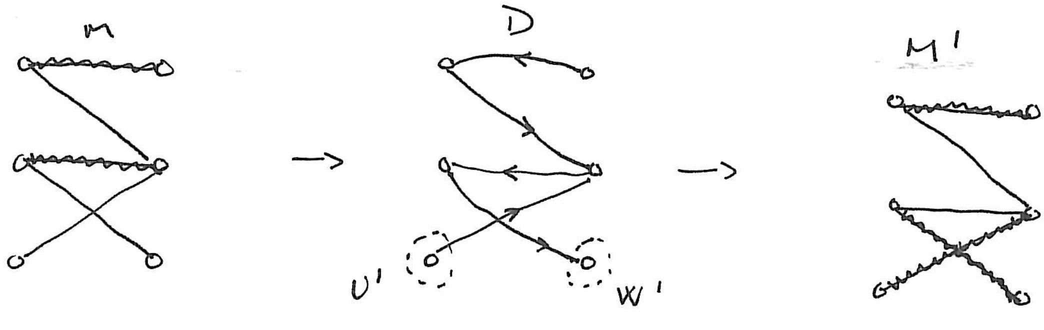
- $e \in M$ : $u \leftarrow w$
- $e \notin M$ : $u \rightarrow w$

This gives a directed graph $D$.

Consider sets $U' := U \setminus \cup M$ $\quad W' := W \setminus \cup M$

A directed path in $D$ from a vertex in $U'$ to a vertex in $W'$ gives an M-augmenting path.

Ex




- A maximum-size matching in a bipartite graph can be found in time $O(|V|\,|E|)$

# The matching polytope

$G = (V, E)$, graph

$M$: matching

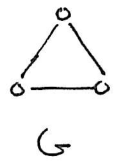
- The **incidence vector** of $M$ is defined by

$$\chi^M(e) = \begin{cases} 1, & e \in M \\ 0, & e \notin M \end{cases} \qquad \text{vector in } \mathbb{R}^E$$

- The **matching polytope** of $G$

$$P_M(G) := \text{conv. hull}\,\{\chi^M : M \text{ is a matching in } G\}$$

Ex

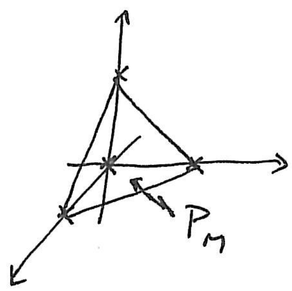



$\chi^1 = (0,0,0)$

$\chi^2 = (1,0,0)$

$\chi^3 = (0,1,0)$

$\chi^4 = (0,0,1)$




- A polytope is a bounded polyhedron and can be described by linear inequalities

- Consider perfect matchings first

$$P_{PM}(G) := \text{conv. hull}\,\{\chi^M : M \text{ is a perfect matching in } G\}$$

## Theorem

$G = (V, E)$, bipartite. Then the perfect matching polytope is equal to the set of vectors $x \in \mathbb{R}^E$ satisfying

(i) $x_e \geq 0$, $\forall e \in E$

(ii) $\sum_{e \in \delta(v)} x_e = 1$, $\forall v \in V$

## Proof

Let $P := \{ x \in \mathbb{R}^E : x_e \geq 0, \forall e \in E, \sum_{e \in \delta(v)} x_e = 1, \forall v \in V \}$,

and let $x \in P_{PM}(G)$. Then $x \in P$ since each $\chi^M \in P$.

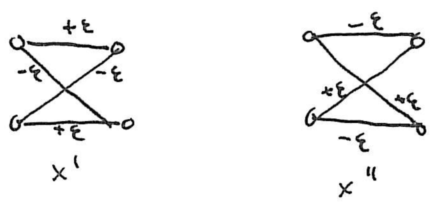
; $P$ is convex and contains each $\chi^M$, so $P_{PM} \subseteq P$;

Assume $P \neq P_{PM}$. Then $P$ has a vertex that is not integral.

Let $G_x = (V, E_x)$ be the graph with only fractional valued edges.

Then $G_x$ must contain a cycle of even length.

Construct new $x'$ and $x''$ by adding/subtracting $\varepsilon$ on alternating edges:




Then $x = \frac{1}{2}(x' + x'')$ and $x$ is not a vertex.

$\square$

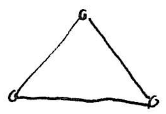
Corollary

$G=(V,E)$, bipartite. Then the matching polytope is equal to the set of vectors $x \in \mathbb{R}^E$ satisfying

(i) $x_e \geq 0$, $\forall e \in E$

(ii) $\sum_{e \in \delta(v)} x_e \leq 1$, $\forall v \in V$

Ex Bipartiteness is essential

$K_3$ 

$x_1 \geq 0$ $\quad x_1 + x_2 \leq 1$

$x_2 \geq 0$ $\quad x_1 + x_3 \leq 1$

$x_3 \geq 0$ $\quad x_2 + x_3 \leq 1$

$x_1 = x_2 = x_3 = \frac{1}{2}$ is a solution, but does not belong to the matching polytope.