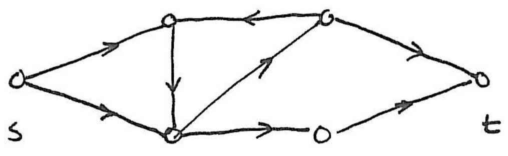
NETWORK FLOWS

$D = (V, A)$, directed graph

$s, t \in V$



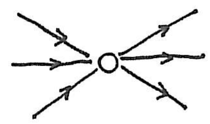
- $f : A \to \mathbb{R}$ is called an <u>s-t flow</u> if

  (i) $f(a) \geq 0 \quad \forall a \in A$

  (ii) $$\sum_{a \in \delta^{in}(u)} f(a) = \sum_{a \in \delta^{out}(u)} f(a) \quad , \quad \forall u \in V \setminus \{s, t\}$$

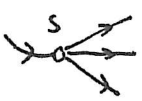
  $\uparrow$

  Flow conservation law



- <u>Value</u> of an s-t flow

$$\text{value}(f) := \sum_{a \in \delta^{out}(s)} f(a) - \sum_{a \in \delta^{in}(s)} f(a)$$



net flow leaving s

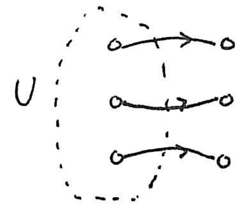
- Capacity function , $c : A \to \mathbb{R}^{+}$

$f(a) \leq c(a) \quad , \quad \forall a \in A$

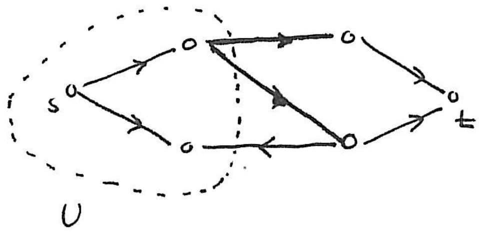
Maximum flow problem:

> Find an s-t flow under c of maximum value.

- <u>Cut</u> : $\delta^{out}(U) := \{vw \in A : v \in U, w \notin U\}$



- <u>s-t cut</u> : a cut with $s \in U$, $t \notin U$



The capacity of a cut : $c(\delta^{out}(U)) := \sum_{a \in \delta^{out}(U)} c(a)$

<u>Proposition</u>

For every s-t flow under c and every s-t cut $\delta^{out}(U)$ one has:

$$\text{value}(f) \le c(\delta^{out}(U))$$

Proof

$$\text{value}(f) = \sum_{a \in \delta^{out}(s)} f(a) - \sum_{a \in \delta^{in}(s)} f(a) + \sum_{v \in U \setminus \{s\}} \Big(\sum_{a \in \delta^{out}(v)} f(a) - \sum_{a \in \delta^{in}(v)} f(a) \Big)$$

$$= \sum_{v \in U} \Big(\sum_{a \in \delta^{out}(v)} f(a) - \sum_{a \in \delta^{in}(v)} f(a) \Big)$$

$$= \sum_{a \in \delta^{out}(U)} f(a) - \sum_{a \in \delta^{in}(U)} f(a)$$

$$\leq \sum_{a \in \delta^{out}(U)} f(a) \leq \sum_{a \in \delta^{out}(U)} c(a) = c(\delta^{out}(U))$$

$\square$

Note that equality holds if and only if

$$f(a) = 0 \quad \forall a \in \delta^{in}(U)$$

$$f(a) = c(a) \quad \forall a \in \delta^{out}(U)$$

Theorem, Max-flow min-cut theorem

[Ford, Fulkerson, 1956]

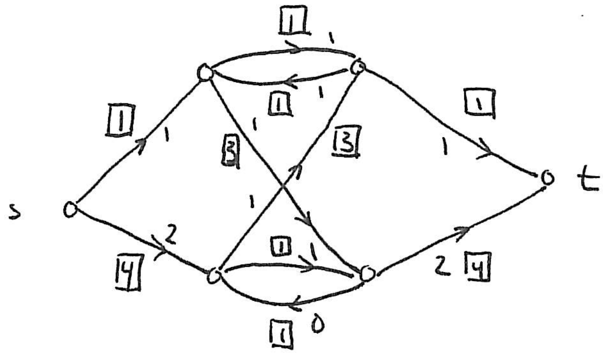
The maximum value of an s-t flow under c is equal to the minimum capacity of an s-t cut.

$$\max_{f:\ \text{s-t flow}} \text{value}(f) = \min_{\delta^{out}(U):\ \text{s-t cut}} c(\delta^{out}(U))$$

We will prove the theorem and present the algorithm at the same time.

Basic idea: increase the flow by increasing the flow on a path from s to t.

<u>Ex</u>

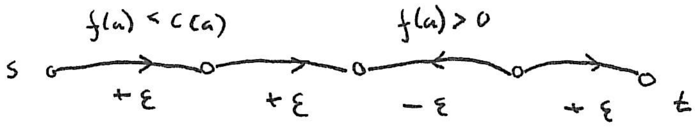


No s-t path with $f(a) < c(a)$, $\forall a \in P$, however the flow is not maximum

- Slight generalization

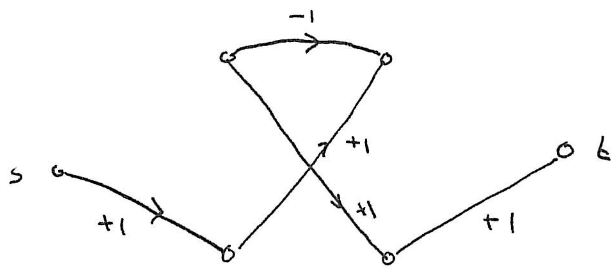
An s-t path is called <u>flow</u> <u>augmenting</u> if

- $f(a) < c(a)$ for every forward arc
- $f(a) > 0$ for every reverse arc



Can increase flow by increasing flows on forward arcs and reducing flow on reverse arcs.

Ex



<u>Proof</u> (Max-flow min-cut)

Need only to show that there exists an s-t flow f and an s-t cut $\delta^{out}(U)$ such that

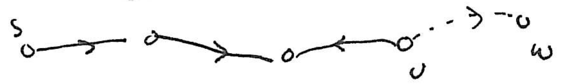
$$\text{value}(f) = c(\delta^{out}(U))$$

Let x be a flow of maximum value. Let U denote

$$\{ v \in V : \text{there exists a flow augmenting path from } s \text{ to } v \}$$

Clearly $s \in U$ and $t \notin U$.

For every arc $vw \in \delta^{out}(U)$ we must have $f(a) = c(a)$, since otherwise adding vw to the flow augmenting path to v, would yield such a path to w, but $w \notin U$.



Similarly, for every arc $a \in \delta^{in}(U)$ we have $f(a) = 0$.

So, by earlier note,

$$\text{value}(f) = c(\delta^{out}(U))$$

$\square$

Corollary

An s-t flow is of maximum value if and only if there is no flow augmenting path.

Corollary (Integrality theorem)

↓ If c is integer-valued, there exists an integer-valued maximum flow.

- Maximum flow algorithm (Ford - Fulkerson)

  - Start with any feasible flow (e.g. $f(a)=0$)
  - Repeatedly find a flow augmenting path P, and augment f by the maximum value permitted.

$$\min(\varepsilon_1, \varepsilon_2) \; : \quad \begin{array}{l} \varepsilon_1 = \min\,(c(a)-f(a) : a \text{ forward in } P) \\ \varepsilon_2 = \min\,(f(a) : a \text{ reverse in } P) \end{array}$$

Observation: when c is integral, f remains integral and each augmentation is of value at least 1.

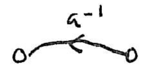
Theorem

If all capacities $c(a)$ are rational, the algorithm terminates.

How to find a flow augmenting path ?

Create an auxiliary graph $D_f(V, A_f)$ by the following rule :

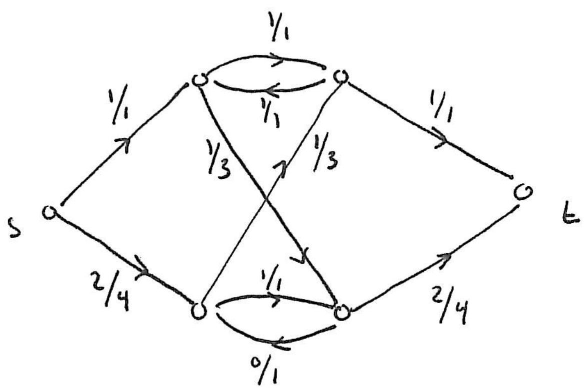
$f(a) < c(a) \Rightarrow a \in A_f$ 

$f(a) > 0 \Rightarrow a^{-1} \in A_f$ 

- s-t paths in D_f corresponds to flow augmenting paths

Ex

D



D_f