# Second assignment 2010
# Part 1: flow decomposition

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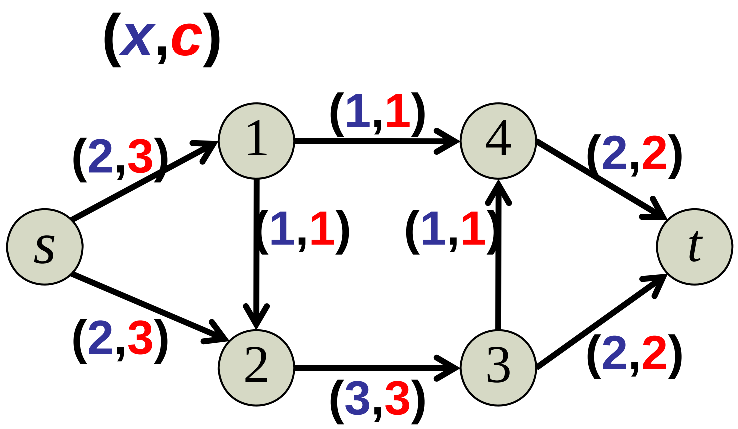
University of Oslo, INF-MAT5360 - Autumn 2010 (Mathematical optimization)

# st-flow

- Given a directed graph $D = (V,E)$
- 2 distinct vertices $s,t \in V$
- s source: no edges entering $s$
- $t$ sink: no edges outgoing from $t$
- Edge capacity function $c : E \to R_+$




An *st-flow is a function* $x: E \to R$, *satisfying*

$$\sum_{e \in \delta^+(v)} x(e) = \sum_{e \in \delta^-(v)} x(e) \quad (v \in V \setminus \{s,t\})$$ flow conservation constraints

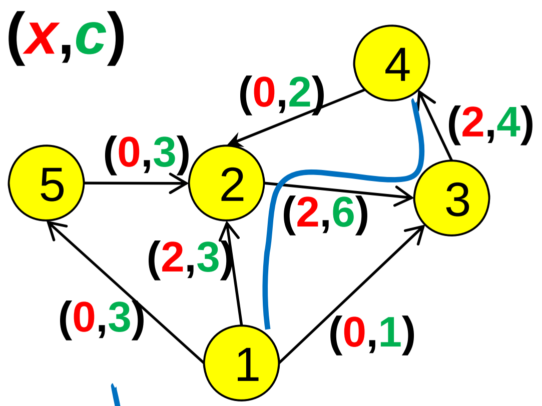
$$0 \le x \le c$$ non-negativity and capacity constraints

- Obs: $\mathrm{div}_x(v) = 0$ *for all* $v \neq s,t$.

# Flow on paths and cycles

- Given a directed graph $D = (V,E)$
- $P = \{u=u_1,(u_1\ , u_2),u_2\ , \ldots,(u_{k-1},u_k), u_k = v)$ *uv*-path
- $y^P \in \{0,1\}^E$ incidence vector of $P$

> A flow $x \in R^A$ is called *flow on the path* $P$ if $x = k \cdot y^P$, with $k > 0$
> ($x$ has positive components only on the edges of $P$)

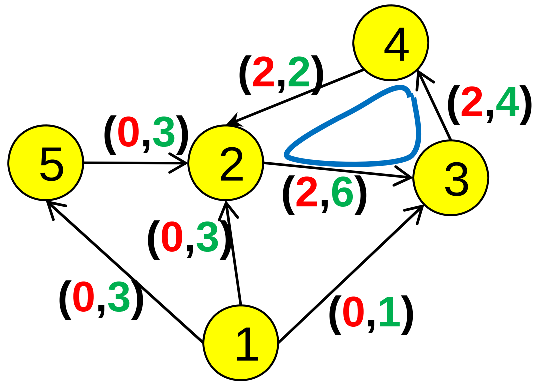



$$\begin{matrix}12\\13\\15\\23\\34\\42\\52\end{matrix}\begin{pmatrix}1\\0\\0\\1\\1\\0\\0\end{pmatrix} = y^P$$

$k = 2$ $\quad P = \{1,(1,2),2,(2,3),3,(3,4),4\}$

$x = 2 \cdot y^P$

*flow on a path*




$$\begin{matrix}12\\13\\15\\23\\34\\42\\52\end{matrix}\begin{pmatrix}0\\0\\0\\1\\1\\1\\0\end{pmatrix} = y^C$$

$k = 2$ $\quad C = \{2,(2,3),3,(3,4),4,(4,2),2\}$

$x = 2 \cdot y^C$

*flow on a cycle*

# Summing flows

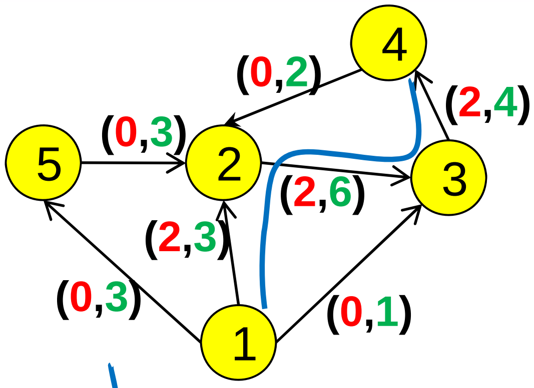
*Theorem*

Let $x$ be an *st*-flow and let $x'$ *be a flow on an st-path or on a cycle*. *If* $z = x + x'$ *satisfies* $0 \leq z \leq c$, *then* $z$ is an *st-flow*.

*If* $z = x - x'$ *satisfies* $0 \leq z \leq c$, *then* $z$ is an *st-flow*.

Show it.

# Example




$$\begin{matrix}12\\13\\15\\23\\34\\42\\52\end{matrix}\begin{pmatrix}1\\0\\0\\1\\1\\0\\0\end{pmatrix} = y^P$$

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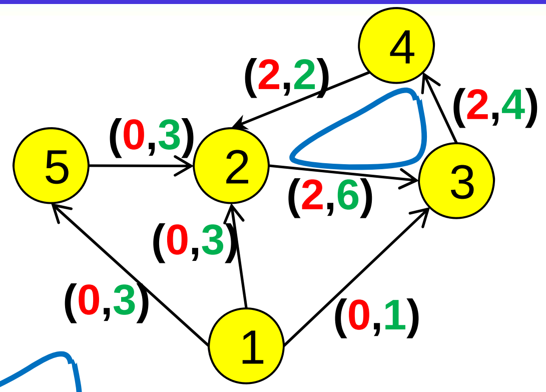


$$\begin{matrix}12\\13\\15\\23\\34\\42\\52\end{matrix}\begin{pmatrix}0\\0\\0\\1\\1\\1\\0\end{pmatrix} = y^C$$

$k = 2$ $P =$ {1,(1,2),2,(2,3),3,(3,4),4}

$x = 2 \cdot y^P$

$k = 2$ $C =$ {2,(2,3),3,(3,4),4,(4,2),2}

$x = 2 \cdot y^C$

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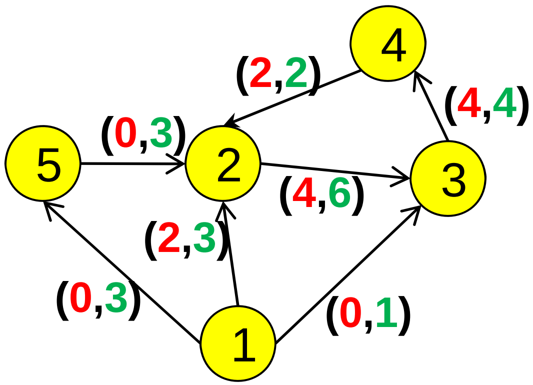
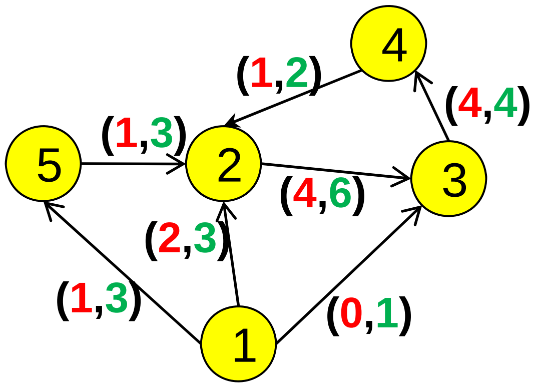



$x = 2 \cdot y^P + 2 \cdot y^C$

# Example

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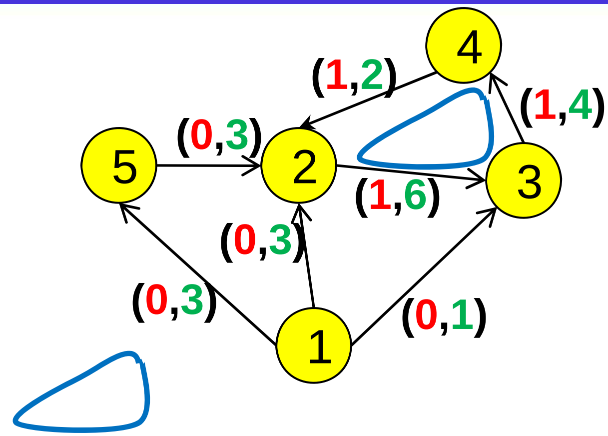


$$\begin{array}{r} 12 \\ 13 \\ 15 \\ 23 \\ 34 \\ 42 \\ 52 \end{array} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = y^C$$

$k = 1 \quad C = \{2,(2,3),3,(3,4),4,(4,2),2\}$

$x = 1 \cdot y^C$

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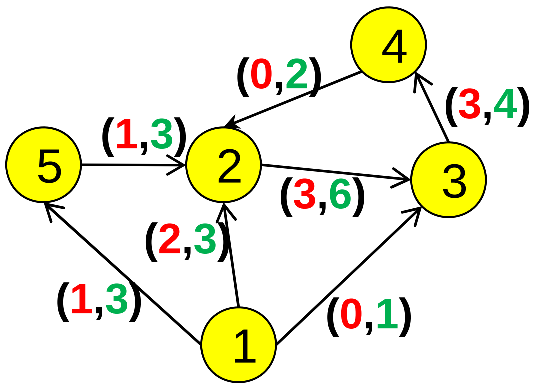

# The flow decomposition theorem

*Theorem:*

Let $x \in R^E$ be an st-flow of $D$. There exists a set $\Pi = \{P_1,\ldots\}$ of *st*-paths and a set $\Omega = \{C_1,\ldots\}$ of cycles such that: $x = \sum_{P\in\Pi} k^P y^P + \sum_{C\in\Omega} k^C y^C$

with $k^P y^P$ flow on path $P \in \Pi$ and $k^C y^C$ flow on cycle $C \in \Omega$

(*A flow can be decomposed into flows on st-path and flows on cycles.*

Prove it! Hints:

- if $x = 0$, then the theorem holds trivially. So we can assume $x_{uv} > 0$.
- if $u \in V$ has $div_x(u) > 0$ then there is a vertex vertex $w$ with $div_x(w) < 0$
- if $u \in V$ has $div_x(u) > 0$ then there is a vertex $v$ with $x_{uv} > 0$.
  - if $v$ has $div_x(v) < 0$ then we have *uv-path* with positive flow. Use it
  - otherwise there exists $w \in V$ with $x_{vw} > 0$. How can we use it?

# More hints

- Start with any vertex *u*. Take an outgoing edge *(u,v)*. Build the path *(u,(u,v),v)*. Take an outgoing edge *(v,w)*. Build the path *(u,(u,v),v,(v,w),w)…* sooner or later you meet every vertex, or a vertex you have already encountered.