Exact Methods

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Exact methods for CO

- Combinatorial optimization problem solved to optimality

  (Q) $\max \{w^T x : x \in S \subseteq \{0,1\}^n\} = v(Q)$ 0-1 *linear program*

- Consider $T \supseteq S$ and let (R) $\max \{w^T x : x \in T\} \geq v(Q)$

- Suppose the <u>optimal solution</u> x_R to (R) can be found efficiently

  e.g. $T = P$ polyhedron and $S = P \cap \{0,1\}^n$

- Let $x^* \in S$ (incumbent solution) and $LB = w^T x^* \leq v(Q)$

- $LB = -\infty$ if no incumbent solution is known

Solving (R) = Solving (Q)?

- Only if some conditions are verified.

Solving (R) instead of (Q)

1. *Infeasibility*. T is empty $\rightarrow$ S is empty

2. *Optimality*. $x_R \in S \rightarrow x_R$ optimal for (Q).

$$v(Q) \geq w^T x_R = v(R) \geq v(Q)$$

3. *Value Dominance*. $w^T x_R \leq LB = w^T x^* \rightarrow x^*$ optimal for (Q).

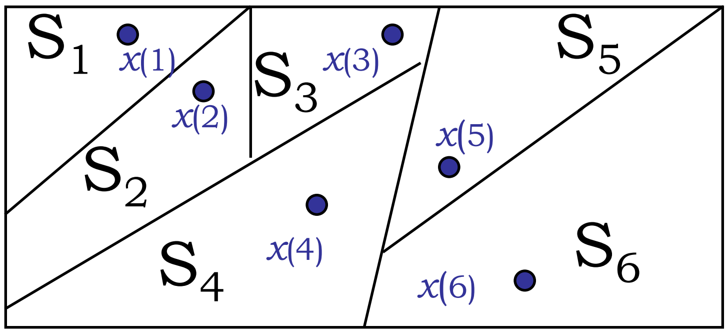
$$w^T x^* \geq v(R) \geq v(Q) \geq w^T x^*$$

- If none of this conditions is satisfied we do *divide-et-impera*
- Partition S and decompose (Q) into a number of smaller

subproblems (Q(u)) $\max \{w^T x : x \in S(u)\}$, $\bigcup_u S(u) = S$

Solving Q(*u*)

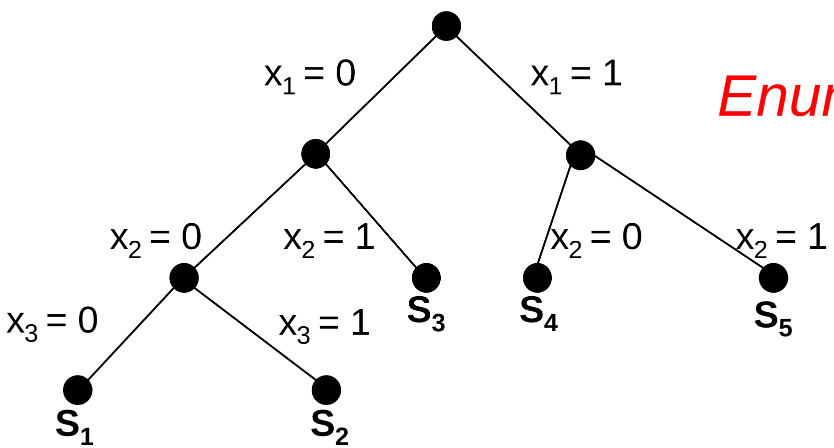
- (Q(*u*)) max $\{w^T x : x \in S(u)\}$ still difficult but "smaller" than *Q*



(Q(*u*)) max $\{w^T x : x \in S(u)\}$

$v(Q) = \max_u v(Q(u))$

- Partitions can be built recursively by fixing variables to 0 or 1.



Enumeration Tree

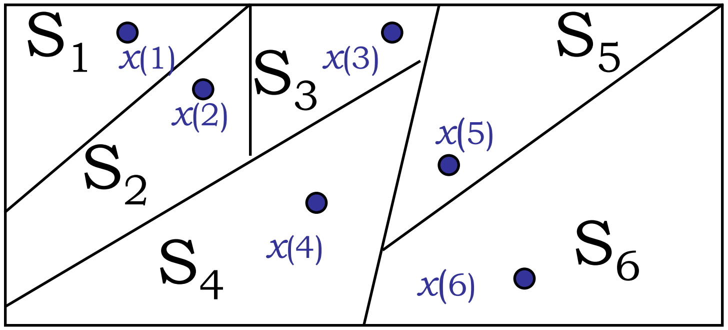
$S_3 = \{x \in S: x_1 = 0 \wedge x_2 = 1\}$

- Each leaf of the <u>complete</u> tree corresponds to a specific 0,1 vector!

Divite-et-impera

- We can try to solve each (Q(u)) by solving a relaxation (R(u))

(Q(u)) max $\{w^T x : x \in S(u)\}$ (R(u)) max $\{w^T x : x \in T(u) \supseteq S(u)\}$



$x(u)$ optimal solution to $R(u)$

$z(u) = w^T x(u)$ optimal value of $R(u)$

x^* overall incumbent, $z_L = w^T x^*$

1. *Infeasibility*. $T(u)$ is empty $\rightarrow$ S(u) is empty

3. *Value Dominance*. $z(u) \leq$ LB $= w^T x^*$ no use to solve $Q(u)$

   $w^T x^* \geq z(u) \geq v(Q(u))$ no solution in $S(u)$ better than incumbent x^*

2. *Optimality*. $x_{R(u)} \in S(u) \rightarrow x_{R(u)}$ optimal for (Q(u)).

   if $w^T x_{R(u)} >$ LB we can set the incumbent $x^* = x_{R(u)}$

Branch-and-bound

Branch-and-Bound algorithm for 0,1 programming

- *Step 1. (Initialization) Let* $V_n = \{v_r\}$, $z_L = -\infty$.
- *Step 2. (Termination.) If* $V_n = \emptyset$, *terminate (*x^* *is optimal).*
- *Step 3. (Node selection and solution)*
  Select u *in* V_n *. Set* $V_n := V_n \setminus \{u\}$. *Solve the LP relaxation* (R(u)).
- *Step 4. (Pruning.)*
  (i) If (R(u)) *is infeasible Goto 2.*
  (ii) If $x(u) \in S(u)$ and $w^T x(u) > z_L$, *let* $x^* = x(u)$, $z_L = w^T x(u)$. *Goto 2.*
  (iii) If $z(u) = w^T x(u) \leq z_L$, *Goto 2.*

Step 5. (Branching.) Choose a variable x_i .
Let $S(u_0) = \{x \in S \cap \{x_i = 0\}\}$. $S(u_1) = \{x \in S \cap \{x_i = 1\}\}$.
Add u_0 and u_1 to $V_{n.}$ *Goto 2.*

Example

$$\max -5x_1 - 9x_2 - 7x_3 - 5x_4$$

$$\begin{cases} 4x_1 + 5x_2 + 3x_3 + 2x_4 \geq 7 \\ x \in \{0,1\}^4 \end{cases}$$

$$\hat{x}_1 = 1$$

$$\hat{x}_2 = 3/5 = 0,6$$

$$\hat{x}_3 = 0$$

$$\hat{x}_4 = 0$$

UB = -10,4

LB = -14 ⇨ $\begin{cases} \hat{x}_1 = \hat{x}_2 = 1 \\ \hat{x}_3 = \hat{x}_4 = 0 \end{cases}$

Example

$$\max -5x_1 - 9x_2 - 7x_3 - 5x_4$$

$$\begin{cases} 4x_1 + 5x_2 + 3x_3 + 2x_4 \geq 7 \\ x \in \{0,1\}^4 \end{cases}$$

$$\hat{x}_1 = 1$$

$$\hat{x}_2 = 3/5 = 0,6$$

$$\hat{x}_4 = 0$$

$$\hat{x}_3 = 0$$

$x_2 = 0$

$$\max -5x_1 - 7x_3 - 5x_4$$

$$4x_1 + 3x_3 + 2x_4 \geq 7$$

$x_2 = 1$

$$-9 + \max -5x_1 - 7x_3 - 5x_4$$

$$4x_1 + 3x_3 + 2x_4 \geq 2$$

Example

$x_2 = 0$

$\max -5x_1 - 7x_3 - 5x_4$

$4x_1 + 3x_3 + 2x_4 \geq 7$

$x_2 = 1$

$-9 + \max -5x_1 - 7x_3 - 5x_4$

$4x_1 + 3x_3 + 2x_4 \geq 2$

$$\begin{cases} \hat{x}_2 = 1 \\ \hat{x}_1 = 1/2 = 0,5 \\ \hat{x}_3 = \hat{x}_4 = 0 \end{cases}$$

UB = -11,5

$x_1 = 0$

$-9 + \max -7x_3 - 5x_4$

$3x_3 + 2x_4 \geq 2$

Similar

$x_1 = 1$

$-14 + \max -7x_3 - 5x_4$

$3x_3 + 2x_4 \geq -2$

$\text{UB} \leq -14 = \text{LB}$

Example

$x_2 = 0$

$$\max -5x_1 - 7x_3 - 5x_4$$
$$4x_1 + 3x_3 + 2x_4 \geq 7$$

$$\begin{cases} \hat{x}_1 = \hat{x}_3 = 1 \\ \hat{x}_2 = \hat{x}_4 = 0 \end{cases}$$

LB = -12
x binary

update LB

Optimal

$x_2 = 1$

$$-9 + \max -5x_1 - 7x_3 - 5x_4$$
$$4x_1 + 3x_3 + 2x_4 \geq 2$$

$$\begin{cases} \hat{x}_2 = 1 \\ \hat{x}_1 = 1/2 = 0{,}5 \\ \hat{x}_3 = \hat{x}_4 = 0 \end{cases}$$

UB = -11,5

$x_1 = 0$

$$-9 + \max -7x_3 - 5x_4$$
$$3x_3 + 2x_4 \geq 2$$

$x_1 = 1$

$$-14 + \max -7x_3 - 5x_4$$
$$3x_3 + 2x_4 \geq -2$$

UB ≤ -14 = LB

Open choices in B&B

- Problem Selection: how to choose next problem?

  Depth First Search, Best Bound, …

- Branching: how to choose next branching variable?

  Most "fractional" variable

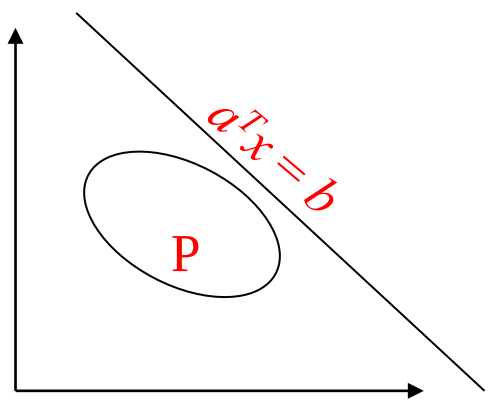
- Relaxation: how to generate good formulations?

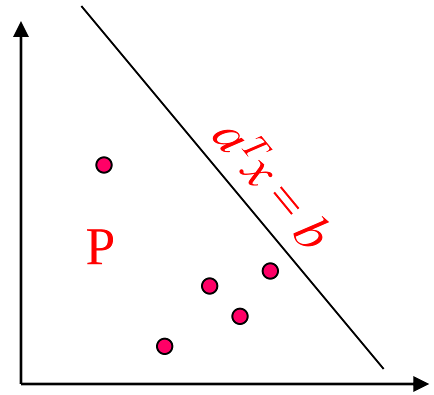
  Find new inequalities (cutting planes) *to add to the initial formulation to make it stronger*.

Valid Inequalities

Valid Inequality

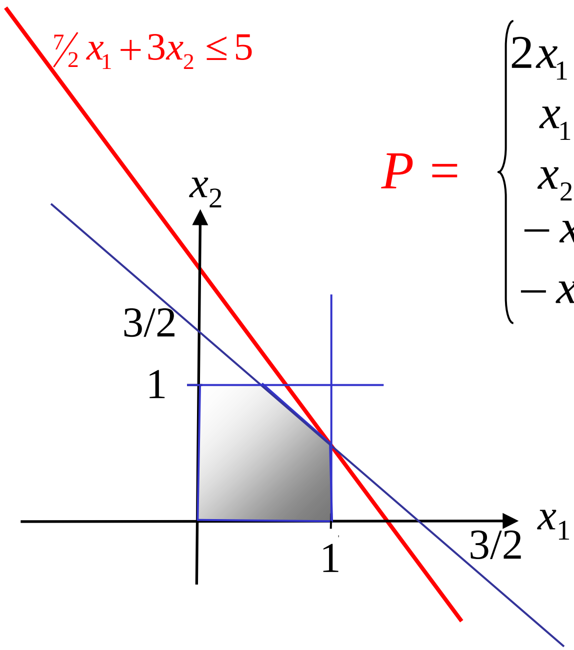
Let $P \subseteq R^n$, $a \in R^n$ e $b \in R$. The linear inequality $a^T x \leq b$ is *valid* for P if it is satisfied by every point of P





Valid inequalities for polyhedra

- Let $P = \{x \in \boldsymbol{R}^n : Ax \leq b\}$ polyhedron ($A \in R^{m\times n}$, $b \in R^m$)
- Any conic combination of the constraints defining P *is* valid for P



$$P = \begin{cases} 2x_1 + 2x_2 \leq 3 \\ x_1 \leq 1 \\ x_2 \leq 1 \\ -x_1 \leq 0 \\ -x_2 \leq 0 \end{cases} \qquad u = \begin{pmatrix} 3/2 \\ 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$

$$u^{\mathrm{T}}Ax \leq u^{\mathrm{T}}b$$

$$\tfrac{3}{2}\cdot(2x_1 + 2x_2) + \tfrac{1}{2}\cdot(x_1) \leq \tfrac{3}{2}\cdot 3 + \tfrac{1}{2}\cdot 1$$

$$\Downarrow$$

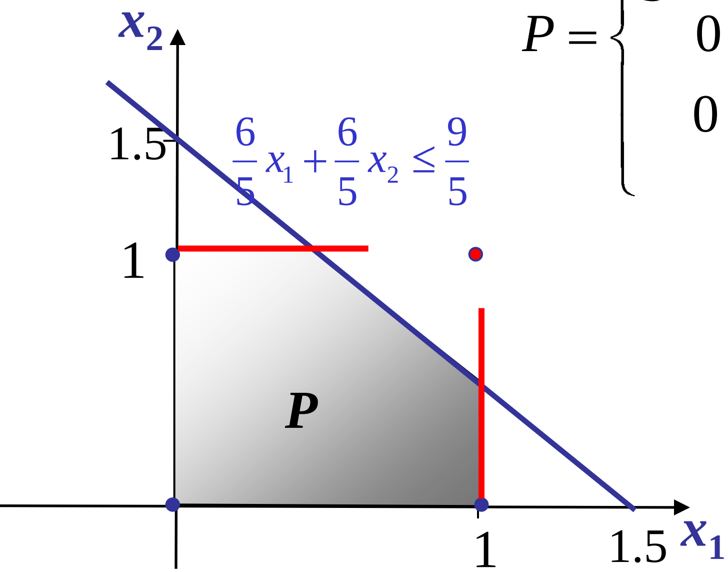
$$\tfrac{7}{2}x_1 + 3x_2 \leq 5$$

Gomory Cuts

Let $P \subseteq \{x \in R^n : Ax \leq b,\ A \in R^{m\times n},\ b \in R^m\}$ and $u \in R_+^m$, then $\lfloor u^{\mathrm{T}}A \rfloor x \leq \lfloor u^{\mathrm{T}}b \rfloor$ is a *Gomory-cut* and is valid for $P \cap \{0,1\}^n$

A Gomory cut

$$P = \begin{cases} \frac{6}{5}x_1 + \frac{6}{5}x_2 \le \frac{9}{5} \\ 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{cases}$$

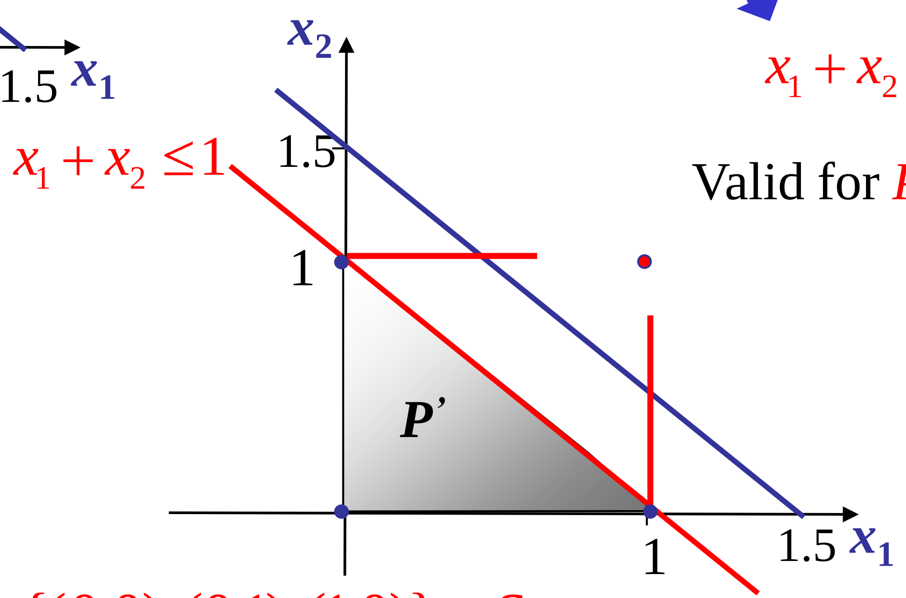


$$S = P \cap \{0,1\}^2 = \{(0,0), (0,1), (1,0)\}$$

$$\left\lfloor \frac{6}{5} \right\rfloor x_1 + \left\lfloor \frac{6}{5} \right\rfloor x_2 \le \left\lfloor \frac{9}{5} \right\rfloor$$

$$x_1 + x_2 \le 1$$

Valid for $P \cap \{0,1\}^2$



$$P' = \begin{cases} \frac{6}{5}x_1 + \frac{6}{5}x_2 \le \frac{9}{5} \\ x_1 + x_2 \le 1 \\ 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{cases}$$

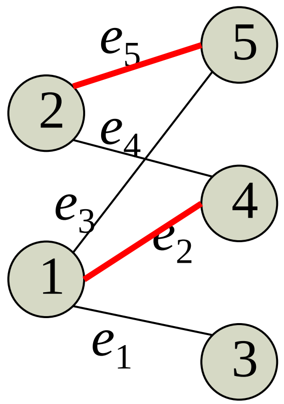
$$P' \cap \{0,1\}^2 = \{(0,0), (0,1), (1,0)\} = S$$

Example: matching

- $G = (V,E)$ undirected graph.

- Matching $M \subseteq E$: subset of edges *meeting* each vertex at most once

- $x \in \{0,1\}^E$ incidence vector of matching in $G \rightarrow x(\delta(v)) \leq 1$ for all $v \in V$



$$x_1 + x_2 + x_3 \leq 1 \quad \delta(1)$$
$$x_4 + x_5 \leq 1 \quad \delta(2)$$
$$x_1 \leq 1 \quad \delta(3)$$
$$x_2 + x_4 \leq 1 \quad \delta(4)$$
$$x_3 + x_5 \leq 1 \quad \delta(5)$$

$\Rightarrow$ $A_G x \leq 1$

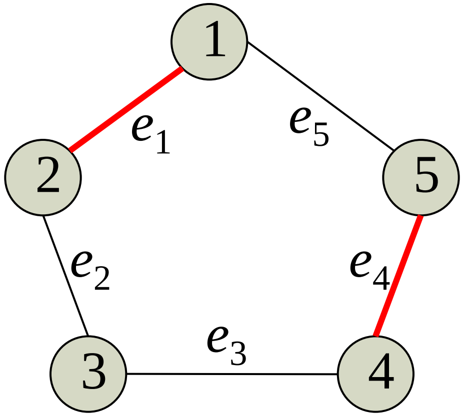
- Maximum Cardinality Matching = $\max\{1^T x : A_G x \leq 1,\ x \in \{0,1\}^E\}$

$$\|$$

G bipartite $\rightarrow A_G$ is TU $\Rightarrow$ $\max\{1^T x : A_G x \leq 1,\ x \geq 0\}$ LP!

Example: non-bipartite matching

- *What if* $G = (V,E)$ is non bipartite?



$x_1 + x_5 \le 1$ $\delta(1)$

$x_1 + x_2 \le 1$ $\delta(2)$

$x_2 + x_3 \le 1$ $\delta(3)$ $\Rightarrow$ $A_G x \le 1$

$x_3 + x_4 \le 1$ $\delta(4)$

$x_4 + x_5 \le 1$ $\delta(5)$

OBS: maximum cardinality matching value: 2 but …

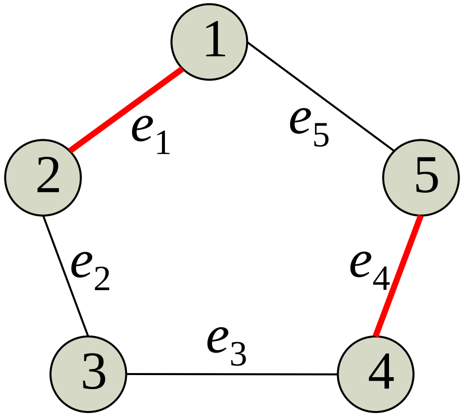
$$\max\{1^T x: A_G x \le 1, x \ge 0\} = 2.5 \quad (x_i = ½, i = 1, \ldots, 5)$$

S incidence vectors of matching of G

- A_G non totally unimodular $\Rightarrow$ $P = \{x \in R^E : A_G x \le 1, x \ge 0\} \neq \text{conv}(S)$

Odd-cycle inequalities

P ={$x \in R^E$: $A_G x \leq 1$, $x \geq 0$} $\neq$ conv(S)



$x_1 + x_5 \leq 1$ $\delta(1)$

$x_1 + x_2 \leq 1$ $\delta(2)$

$x_2 + x_3 \leq 1$ $\delta(3)$

$x_3 + x_4 \leq 1$ $\delta(4)$

$x_4 + x_5 \leq 1$ $\delta(5)$

$A_G x \leq 1$

C odd-cycle (odd number of vertices)
M matching

➡ $|M \cap C| \leq (|C|-1) / 2$

➡ $$\sum_{e \in C} x_e \leq \lfloor |C| / 2 \rfloor$$ must be satisfied by every incidence vector of a matching of G

➡ valid for conv(S) (but not for P! *Why?)*

Odd-cycle inequalities and Gomory cuts

$P = \{x \in R^E : A_G x \le 1, x \ge 0\} \neq \mathrm{conv}(S)$

Consider the edges of a cycle $C = \{e_1, \ldots, e_k\}$

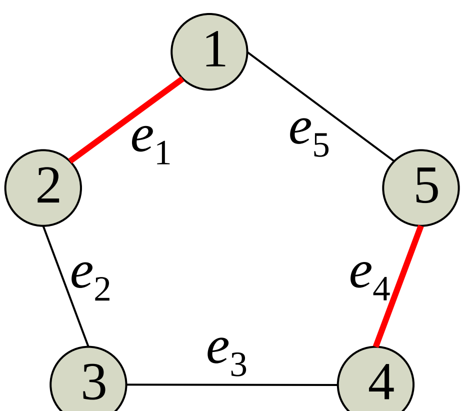
For every pair of consecutive edges e_r , e_{r+1}

$x_r + x_{r+1} \le 1$ is valid for P (why? Hint obtain it as a conic combination of inequalities defining P.)

Summing up all the $|C|$ inequalities associated to C

$$\sum_{e \in C} 2x_e \le |C|$$

Dividing by 2 and rounding down we finally obtain $\sum_{e \in C} x_e \le \lfloor |C|/2 \rfloor$



$$x_1 + x_5 \le 1$$
$$x_1 + x_2 \le 1$$
$$x_2 + x_3 \le 1$$
$$x_3 + x_4 \le 1$$
$$x_4 + x_5 \le 1$$

$$\Rightarrow \sum_{e \in C} 2x_e \le 5$$

$$\sum_{e \in C} x_e \le 2$$

Branch & Cut

- Branch-and-Bound + cutting planes = Branch-and-Cut
- A *cut* is an inequality which is valid for $P \cap \{0,1\}^n$ but not for P
- While solving current problem we try to strengthen the current relaxation by adding valid inequalities
- This is done in a dynamic-simplex-method fashion: the current solution is input to an oracle which tries to find a violated cut.
- Two types of cuts
  - Template cuts: cuts with a given pattern (e.g. odd-cycles)
  - General cuts.