Network Flows and Cuts

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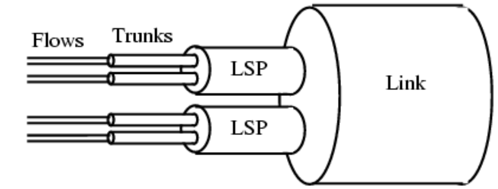
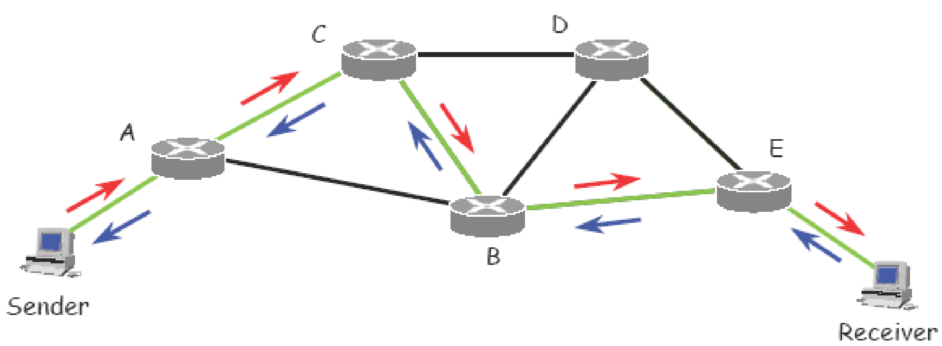
Example: IP network



- IP networks are constituted by routers connected by optical fibers
  - Packets cross the network entering and exiting through *edge routers*
  - Internal nodes are called *label switch router* (*LSR*)

IP network

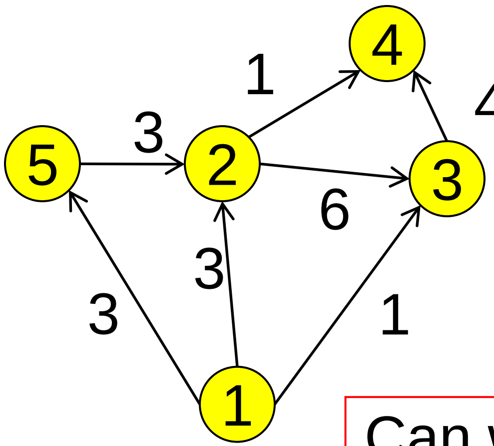


- Each connection (e.g. *Voice over IP*) needs to be assigned a given amount of bandwidth (*capacity*)

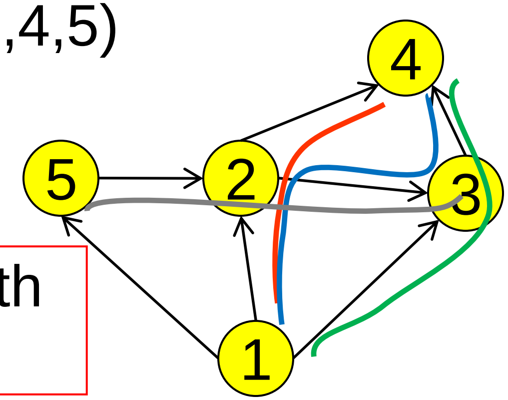
Example

K	band	origin	dest.
1	4Mb	1	4
2	3Mb	5	3

Example: two connections



Capacitated Network with 5 vertices and 4 edge routers (1,3,4,5)



Can we satisfy traffic demand with the given capacities?

Commodity 1: flows

1Mb 1,(1,2),2,(2,4),4

1Mb 1,(1,3),3,(3,4),4

2Mb 1,(1,2),2,(2,3),3,(3,4),4

Commodity 2: flows

3Mb 5,(5,2),2,(2,3),3

Network Flow

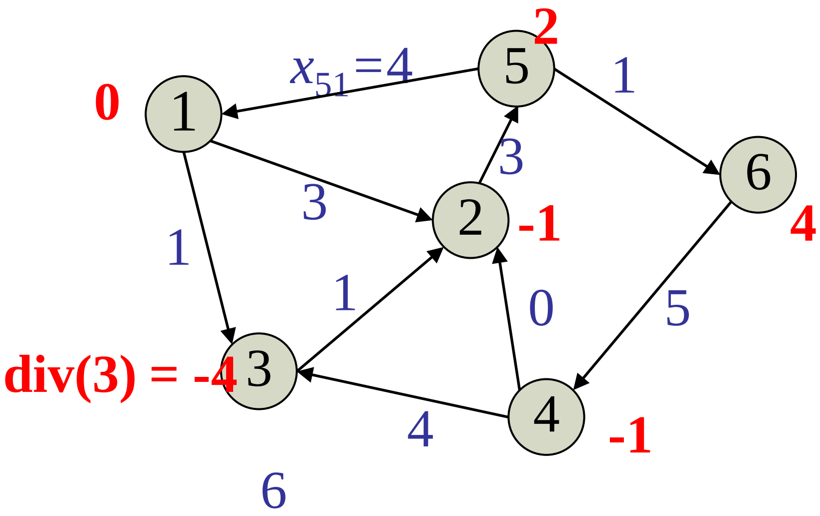
- Given a directed graph $D = (V,E)$

A ***FLOW*** is a function $x : E \to R$, *i.e.* $x \in R^E$

- Typically flows are required to be non-negative *i.e.* $x \in R_+^E$

The *divergence* of a flow x is the function $\text{div}_x : V \to R$, *given by*

$$\text{div}_x(v) = \sum_{e\in\delta^+(v)} x(e) - \sum_{e\in\delta^-(v)} x(e)$$



- Remark: $\sum_{v\in V} \text{div}_x(v) = 0$

Prove it!

$x(e) = x_e$: *flow sent on edge e* $(e \in E)$

Circulations

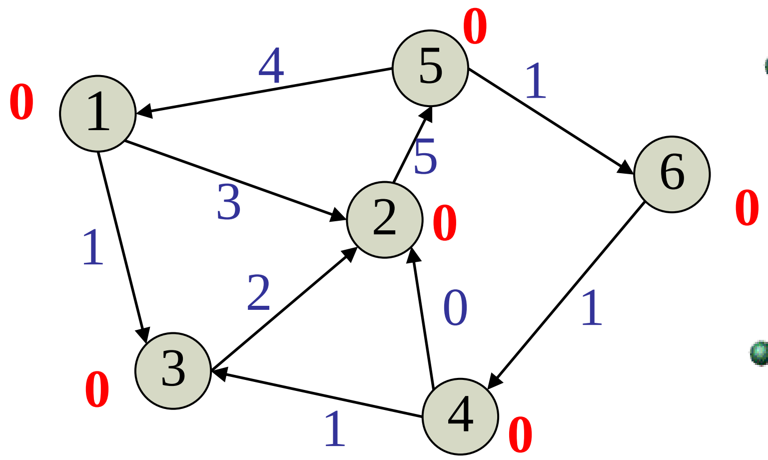
- We are interested in flows with given divergence $b \in R^V$

$$\mathrm{div}_x(v) = b(v) \qquad (v \in V)$$

$$\sum_{e\in\delta^+(v)} x(e) - \sum_{e\in\delta^-(v)} x(e) = b(v) \qquad (v \in V)$$ *flow balance equations*

- Remark: The set of flows with given divergence is a polyhedron

A *circulation* is a flow x with $\mathrm{div}_x(v) = 0 \quad (v \in V)$



- Often an upper bound (*capacity function*) $c : E \to R$ is defined

$$0 \le x(e) \le c(e) \quad (e \in E)$$

- Sometimes a lower bound function $l : E \to R$ is defined and

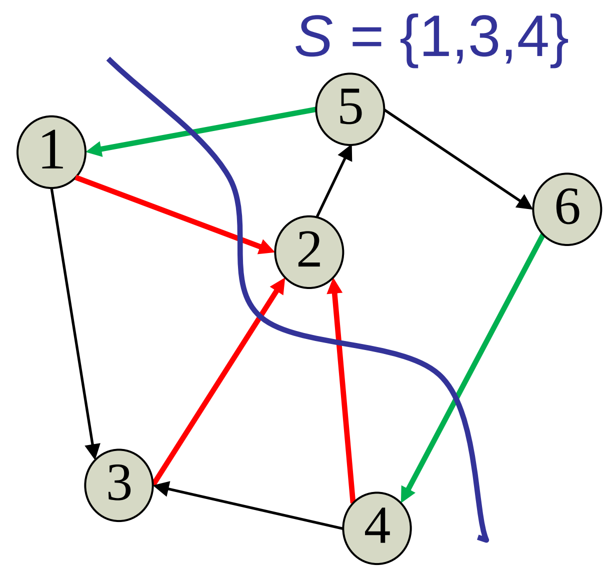
$$l(e) \le x(e) \le c(e) \quad (e \in E)$$

Cuts

- Let $S \subseteq V$.

CUT (of S) *set of edges leaving S*

$\delta^+(S) = \{(v,w) \in E : v \in S, w \notin S\}$



$\delta^-(S)$ *set of edges entering S*

$\delta^-(S) = \{(v,w) \in E : v \notin S, w \in S\}$

$\delta^+(S) = \{(1,2),(3,2),(4,2)\}$

$\delta^-(S) = \{(5,1),(6,4)\}$

- Remark: $\delta^-(S) =: \delta^+(V/S)$ (the cut of V/S)

Cuts and Divergence

Theorem (flow through a cut)

Let $\delta^+(S)$ be a cut of of $D = (V,E)$. *Then*

$$\sum_{e\in\delta^+(S)} x(e) - \sum_{e\in\delta^-(S)} x(e) = \sum_{v\in S} \operatorname{div}(v)$$

flow through cut $\delta^+(S)$

- Proof: excercise.

Corollary (circulation through a cut)

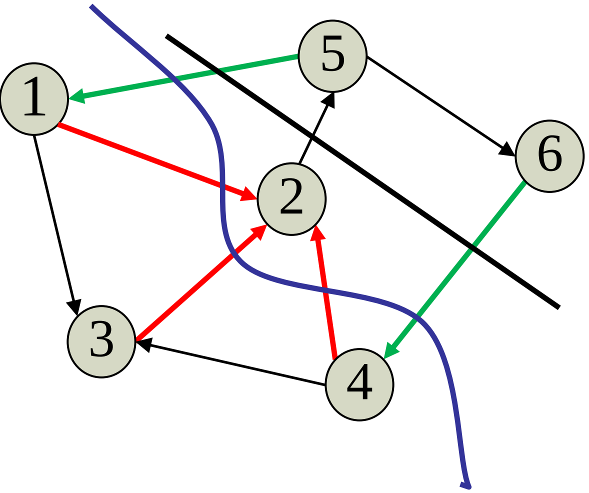
Let x be a circulation of $D = (V,E)$. *Then*

$$\sum_{e\in\delta^-(S)} x(e) = \sum_{e\in\delta^+(S)} x(e) \qquad (S \subseteq V)$$

u-v Cuts

- Let $S \subseteq V$ and let $u, v \in V$

$u \in S$ and $v \notin S$, then
$\delta^+(S)$ is *u-v cut*

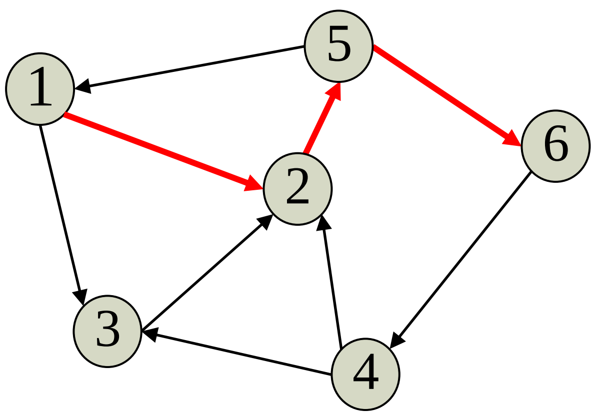


- Two 1-6 *cuts*

  $S^1 = \{1,3,4\}$
  $\delta^+(S^1) = \{(1,2),(3,2),(4,2)\}$

  $S^2 = \{1,2,3,4\}$
  $\delta^+(S^2) = \{(2,5)\}$

u-v path: *a directed path from u to v in $D = (V,E)$*



- A 1-6 *path* $P = \{1,(1,2),2,(2,5),5,(5,6),6\}$

u-v cuts and *u-v paths*

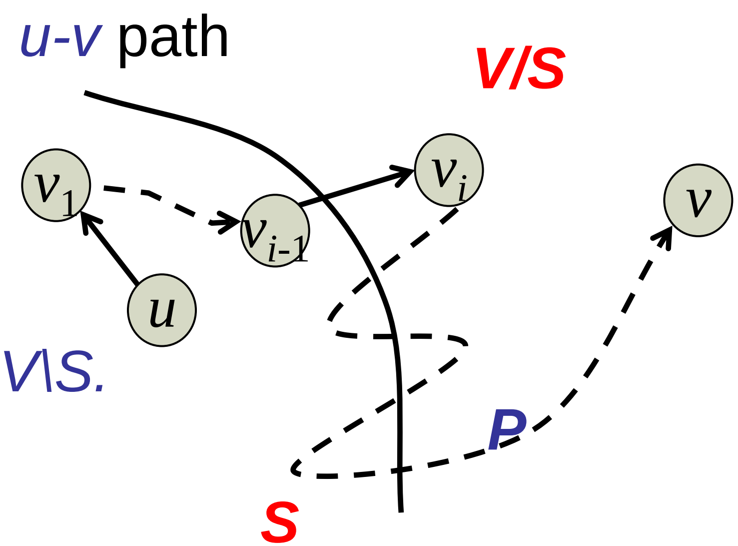
Theorem (every *u-v path meets every u-v cut*)

Let EP be the set of edges of a *u-v* path P and let $C \subseteq E$ be a *u-v* cut. Then $EP \cap C \neq \emptyset$.

$P = \{v_0,(v_0, v_1), v_1, \dots, v_{k-1}, (v_{k-1}, v_k), v_k\}$

with $u = v_0$ and $v_k = v$

Let $\delta^+(S)$ be a *u-v cut*, with $u \in S$ *and* $v \in V \setminus S$



- Let i be the smallest index such that $v_i \in V \setminus S$.
  - v_i *exists since* $v_k \in V \setminus S$.
  - *Since* $v_0 \in S$, *then* $i \geq 1$.
  - *Then* $v_{i-1} \in S$, $v_i \in V \setminus S$ ⟹ $(v_{i-1}, v_i) \in \delta^+(S)$

∎

u-v cuts and *u-v paths*

Theorem (connectivity and emtpy cuts)

Let *D be a directed graph. D* contains no *u-v* path if and only if *D* contains an empty *u-v* cut.

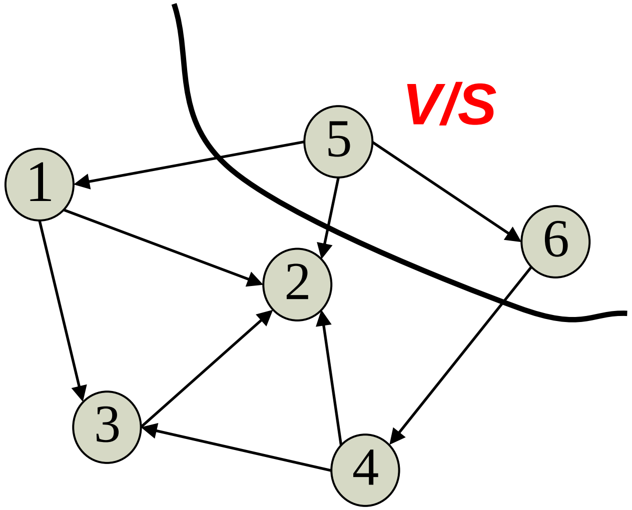
Sufficiency. (Empty *u-v* cut → no *u-v* path)

$\delta^+(S) = \emptyset$ (empty *u-v cut*), with $u \in S$ and $v \in V/S$

Let EP be the set of edges of a *u-v* path *P*.

$\delta^+(S) \cap EP = \emptyset$, a contradiction.

■



S = {1,2,3,4}

$\delta^+(S) = \emptyset$

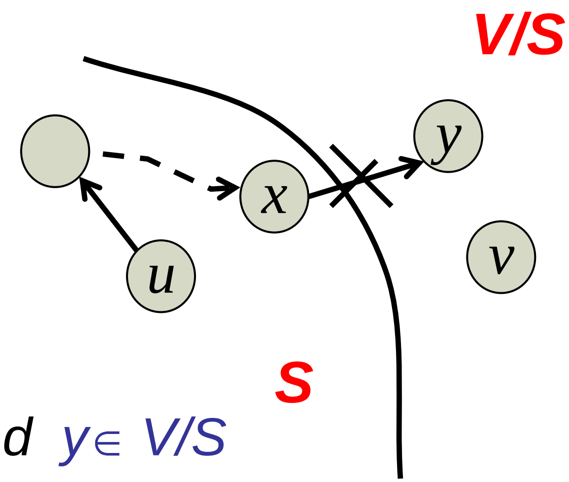
u-v cuts and *u-v paths*

Theorem (*u-v* connectivity and emtpy *u-v* cuts)

Let D be a directed graph. D contains no *u-v* path if and only if D contains an empty *u-v* cut.

Necessity. (no *u-v* path $\rightarrow$ empty *u-v* cut)

Let $S = \{w \in V$: there exists an *u-w* path in $D\}$



⇨ $v \in V \backslash S$ and $\delta^+(S)$ is an *u-v cut.*

$\delta^+(S)$ is empty   Suppose not.

Then there exists $(x,y) \in \delta^+(S)$, with $x \in S$ *and* $y \in V/S$

Since $x \in S$ there exists a *u-x path* P_x *in* D

Concatenating P_x and (x,y) provides a *u-y* path, contradiction

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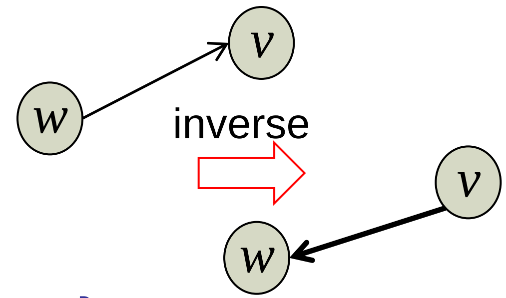
Auxiliary graph

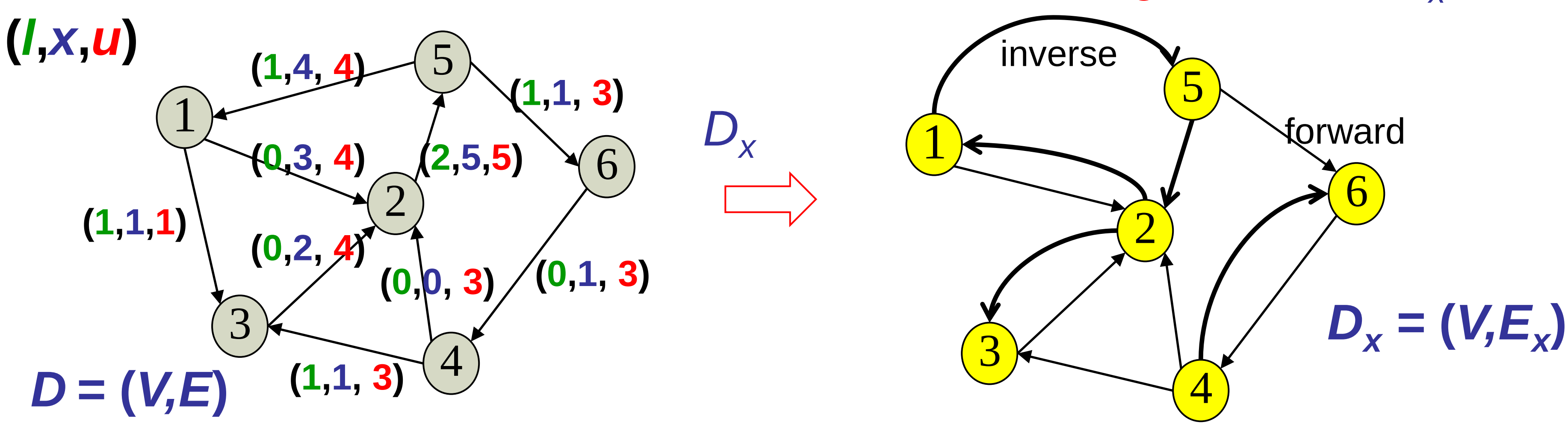
$e = (w,v)$ *inverse edge:* $e^{-1} = (v,w)$



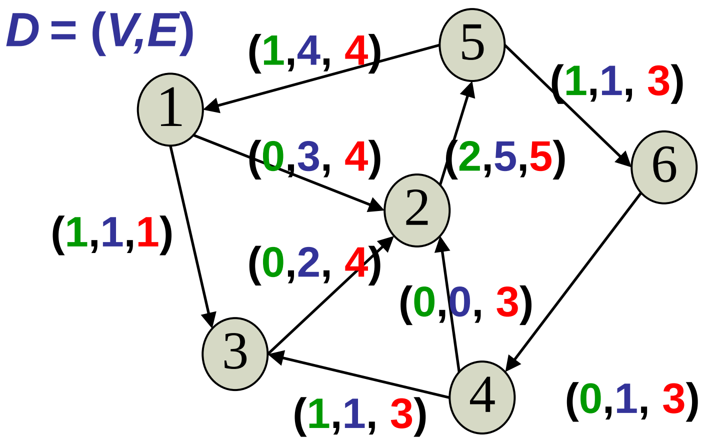
- Let $l, x, u \in R^E$, $l \leq x \leq u$
- Let $E_x = \{e \in E: x_e < u_e\} \cup \{e^{-1}: e \in E, l_e < x_e\}$

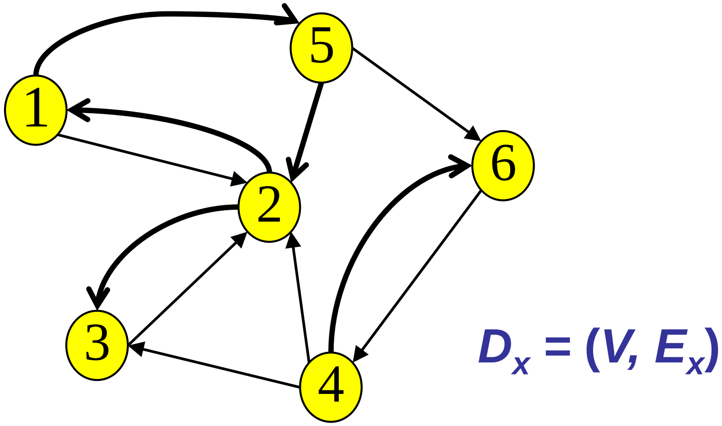
Auxiliary Graph $D_x = (V,E_x)$

forward edge $e \in E \cap E_x$:



Flows and Auxiliary graph





$E_x = \{e \in E: x_e < u_e\} \cup \{e^{-1}: e \in E, l_e < x_e\}$.

Let $\varepsilon > 0$.

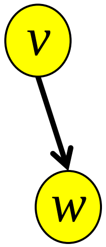
- Remark: if D_x contains a *forward edge* $e = (v,w)$: $x_{vw} < u_{vw}$
  ⟹ $z_{vw} = x_{vw} + \varepsilon$ is still feasible ($z_e = x_e$ *for* $e \in E / \{(v,w)\}$)

- Remark: if D_x contains an *inverse edge* $e = (v,w)$: $x_{wv} > l_{wv}$
  ⟹ $z_{wv} = x_{wv} - \varepsilon$ is still feasible ($z_e = x_e$ for $e \in E / \{(w,v)\}$)

- In both cases we say we are "sending ε units of flow" on the auxiliary edge (v,w).

Divergence and auxiliary graph

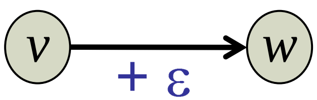
- Consider sending ε units of flow on edge (v,w) in D_x
- Obtain a new (feasible) flow z
- How does divergence change?



$$div_z(v) = \sum_{e \in \delta^+(v)} z(e) - \sum_{e \in \delta^-(v)} z(e)$$

If $(v,w) \in E_x$ **forward**

$(v,w) \in E$



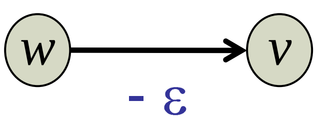
$\Rightarrow$ $z_{vw} = x_{vw} + \varepsilon$

$\mathrm{div}_z(v) = \mathrm{div}_x(v) + \varepsilon$

$\mathrm{div}_z(w) = \mathrm{div}_x(w) - \varepsilon$

If $(v,w) \in E_x$ **inverse**

$(w,v) \in E$



$\Rightarrow$ $z_{wv} = x_{wv} - \varepsilon$

$\mathrm{div}_z(v) = \mathrm{div}_x(v) - (-\varepsilon)$

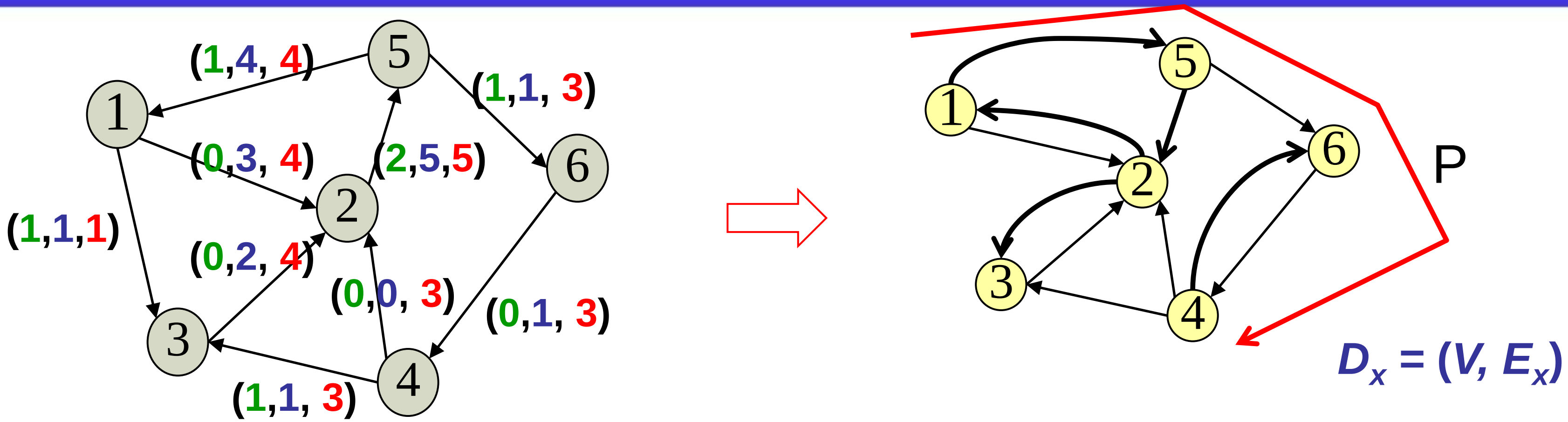
$\mathrm{div}_z(w) = \mathrm{div}_x(w) + (-\varepsilon)$

- In both cases

$\mathrm{div}_z(v) = \mathrm{div}_x(v) + \varepsilon$ INCREASES in v

$\mathrm{div}_z(w) = \mathrm{div}_x(w) - \varepsilon$ DECREASES in w

Paths on the Auxiliary graph

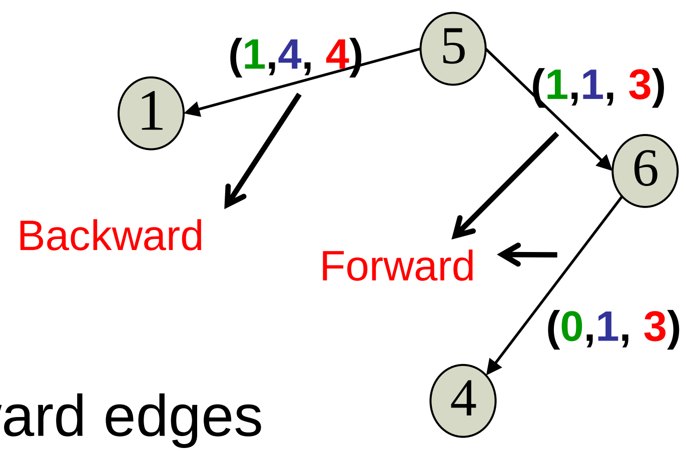


- $P = (v_1 , (v_1,v_2) ,v_2 ,\ldots, (v_{k-1},v_k) ,v_k)$ directed path on D_x

Forward edges $P^+ = \{e \in E: e \in E_x\}$

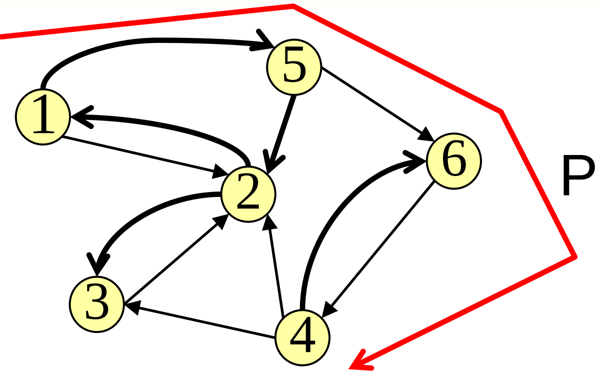
Backward edges $P^- = \{e \in E: e^{-1} \in E_x\}$

- What happens if we send $\varepsilon > 0$ units of flow on all the edges of P?



- You increase the flow by ε on the forward edges and you decrease it by ε on the backward edges

Paths on the Auxiliary graph



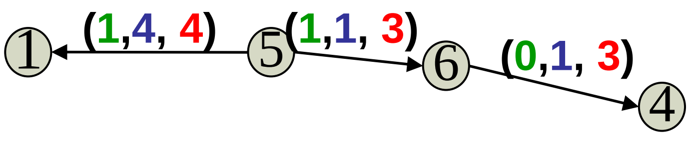
$D_x = (V, E_x)$

Auxiliary Graph $D_x = (V,E_x)$

Indentify directed path

$$P = (v_1\ ,\ (v_1,v_2)\ ,v_2\ ,\ldots,\ (v_{k-1},v_k)\ ,v_k)$$

Send $\varepsilon > 0$ units of flow on the edges of P



Original Graph $D = (V,E)$

Build from x a new flow z by

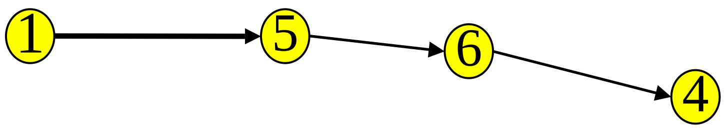
1. increasing x by ε on forward edges
2. decreasing x by ε on backward edges

- What happens to divergences?

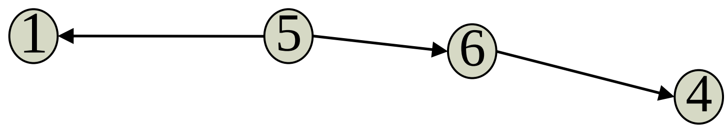
$$\operatorname{div}_x(v) = \sum_{e\in\delta^+(v)} x(e) - \sum_{e\in\delta^-(v)} x(e)$$

Divergence

$D_x = (V, E_x)$



$D = (V, E)$



$P = (v_1 , (v_1,v_2) ,v_2 ,\ldots, (v_{k-1},v_k) ,v_k)$

ε - augmentation on *P*

(v_i,v_{i+1}) ⇒ div(v_i) increases by ε div(v_{i+1}) decreases by ε

div(v_1) increases by ε

div(v_k) decreases by ε

div(v_i) stays unchanged $(2 \leq i \leq k\text{-}1)$

Hoffman's circulation theorem

Theorem 1.1 (Hoffman's circulation theorem)

Let $l,u : E \to R$ satisfying $l \le u$. *Then there* exists *a circulation* x in D such that $l \le x \le u$ if and only if

$$\sum_{e\in\delta^-(S)} l(e) \le \sum_{e\in\delta^+(S)} u(e) \qquad (S \subseteq V)$$

Also, if l and u are integral, then x can be taken integral

Proof. (*Necessity*)

Let x be a circulation

From $l \le x \le u$ ⇨ $$\sum_{e\in\delta^-(S)} l(e) \le \underbrace{\sum_{e\in\delta^-(S)} x(e) = \sum_{e\in\delta^+(S)} x(e)}_{\text{Circulation through a cut}} \le \sum_{e\in\delta^+(S)} u(e) \qquad (S \subseteq V)$$

Circulation through a cut

■

Hoffman's theorem: sufficiency

Proof of Hoffman's theorem**.** (*Sufficiency*)

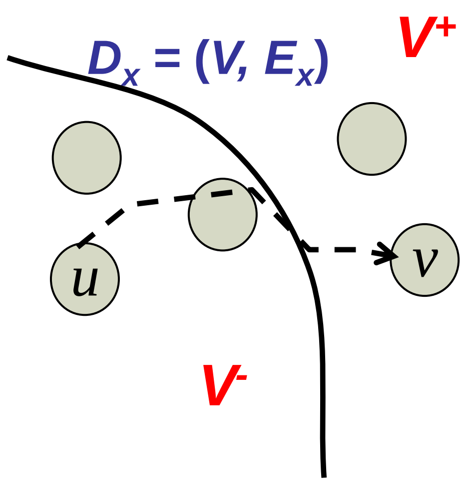
- Let x such that $l \le x \le u$ and $||div_x(v)||_1$ is minimized

  (x exists by *extreme value theorem*).

Let $V^- = \{v \in V : div_x(v) < 0\}$ $\quad V^+ = \{v \in V : \text{div}_x(v) > 0\}$

1. $V^- = \emptyset \Rightarrow V^+ = \emptyset$ $\quad (\text{since} \sum_{v \in V} \text{div}_x(v) = 0)$

   $\Rightarrow$ *x is a circulation.* ■



2. $V^- \ne \emptyset \Rightarrow V^+ \ne \emptyset$

   If $D_x = (V, E_x)$ **contains a path** from $u \in V^-$ to $v \in V^+$

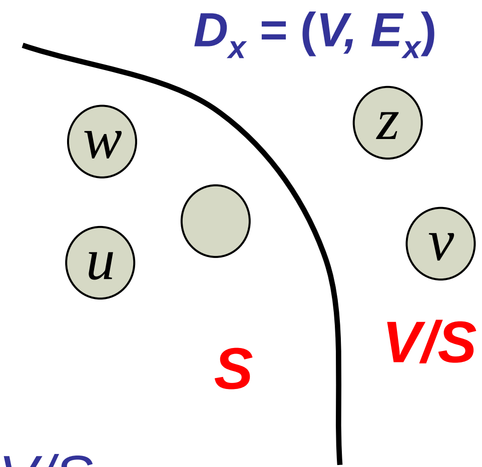
   Send $\varepsilon > 0$ units of flow from u to v *in* D_x

   div(u) increases by ε and div(v) decreases by ε

$\Rightarrow$ $||\text{div}\,(v)||_1$ decreases by 2ε, a contradiction.

Hoffman's theorem: sufficiency

Proof of Hoffman's theorem. (*Sufficiency*)



No *u*-*v* *path in* D_x from $u \in V^-$ to $v \in V^+$

⇒ D_x contains an empty *u*-*v* cut $\delta^+(S)$

such that $V^- \subseteq S$ and $V^+ \subseteq V \backslash S$

If $D = (V,E)$ contains an edge (w,z) with $w \in S$ to $z \in V/S$

⇒ $x_{wz} = u_{wz}$ (otherwise $(w,z) \in E_x$ ⇒ $\sum_{e \in \delta_D^+(S)} x(e) = \sum_{e \in \delta_D^+(S)} u(e)$

If $D = (V,E)$ contains an edge (z,w) with $w \in S$ to $z \in V/S$

⇒ $x_{zw} = l_{zw}$ (otherwise $(z,w)^{-1} = (w,z) \in E_x$) ⇒ $\sum_{e \in \delta_D^-(S)} x(e) = \sum_{e \in \delta_D^-(S)} l(e)$

$$\sum_{e \in \delta^+(S)} u(e) - \sum_{e \in \delta^-(S)} l(e) = \sum_{e \in \delta^+(S)} x(e) - \sum_{e \in \delta^-(S)} x(e) = \sum_{v \in S} \mathrm{div}_x(v) = \sum_{v \in V^-} \mathrm{div}_x(v) < 0$$

⇒ $\sum_{e \in \delta^+(S)} u(e) < \sum_{e \in \delta^-(S)} l(e)$ a contradiction. ■

(The integrality proof is left as an exsercise)

Existence of flows

Theorem 1.1 (Existence of flows)

Let $D = (V,E)$, let $b: V \to R$, *be a* supply function *and* $c : E \to R_+$ an edge capacity function. *Then there exists a flow* x with divergence b satisfying $0 \le x \le c$ if and only if

$$\sum_{v \in V} b(v) = 0$$

$$\sum_{v \in S} b(v) \le \sum_{e \in \delta^+(S)} c(e) \qquad (S \subseteq V)$$

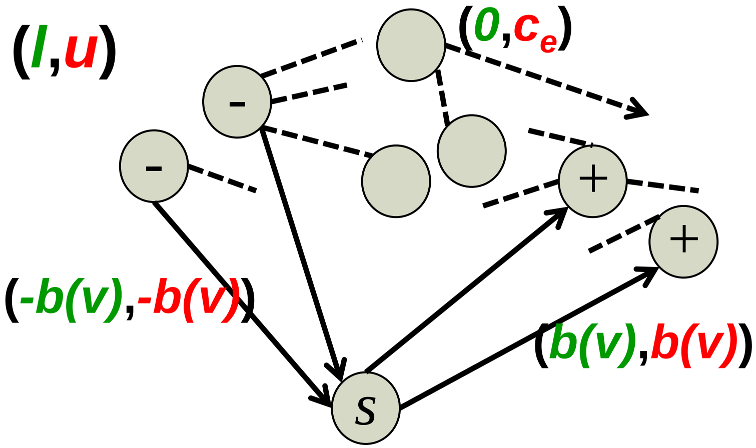
Proof.

Necessity of $\sum_{v \in V} b(v) = 0$ derives from $\sum_{v \in V} \operatorname{div}_x(v) = 0$

- Let $V^- = \{v \in V: b(v) < 0\}$ *and* $V^+ = \{v \in V: b(v) > 0\}$.
- Let $V' = V \cup \{s\}$ (s is a *"new"* vertex)
- Define a new graph $D' = (V', E')$ *where* $E' = E \cup E''$

Proof of existence of flows

$D' = (V', E')$ where $E' = E \cup E''$



$E'' = \{(s,v): v \in V^+\} \cup \{(v,s): v \in V^-\}$

$l, u : E' \rightarrow R$

$$\begin{cases} l(e) = 0,\ u(e) = c(e) & (e \in E) \\ l(s,v) = u(s,v) = b(v) & (v \in V^+) \\ l(v,s) = u(v,s) = -b(v) & (v \in V^-) \end{cases}$$

- Let x *be a circulation in* D' *with* $l \leq x \leq u$

⇨ $x(s,v) = b(v)\ \ (v \in V^+)$ and $x(v,s) = -b(v)\ \ (v \in V^-)$

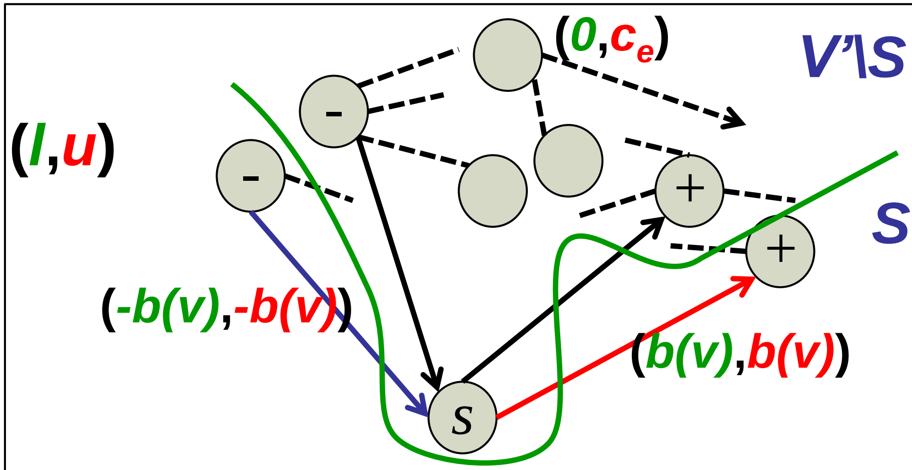
⇨ restriction of x *to* E *satisfies flow balance and* $0 \leq x(e) \leq c(e)$

- Hoffman's conditions for x *circulation in* D' $\displaystyle\sum_{e \in \delta^-(S)} l(e) \leq \sum_{e \in \delta^+(S)} u(e) \quad (S \subseteq V)$

$$\sum_{e \in \delta^-(S) \cap E''} l(e) + \sum_{e \in \delta^-(S) \cap E} l(e) \leq \sum_{e \in \delta^+(S) \cap E''} u(e) + \sum_{e \in \delta^+(S) \cap E} u(e) \Rightarrow \sum_{e \in \delta^-(S) \cap E''} l(e) \leq \sum_{e \in \delta^+(S) \cap E''} u(e) + \sum_{e \in \delta^+(S) \cap E} c(e)$$

Proof of existence of flows

- Hoffman's conditions. Let $S \subseteq V'$. $\quad \sum_{e\in\delta^-(S)\cap E''} l(e) \le \sum_{e\in\delta^+(S)\cap E''} u(e) + \sum_{e\in\delta^+(S)\cap E} c(e)$



1. $s \in V/S$.

$\delta^-(S) \cap E'' = \{sv: v \in V^+ \cap S\}$

$\delta^+(S) \cap E'' = \{vs: v \in V^- \cap S\}$

$$\Rightarrow \sum_{v\in V^+\cap S} b(v) \le \sum_{v\in V^-\cap S} -b(v) + \sum_{e\in\delta^+(S)\cap E} c(e)$$

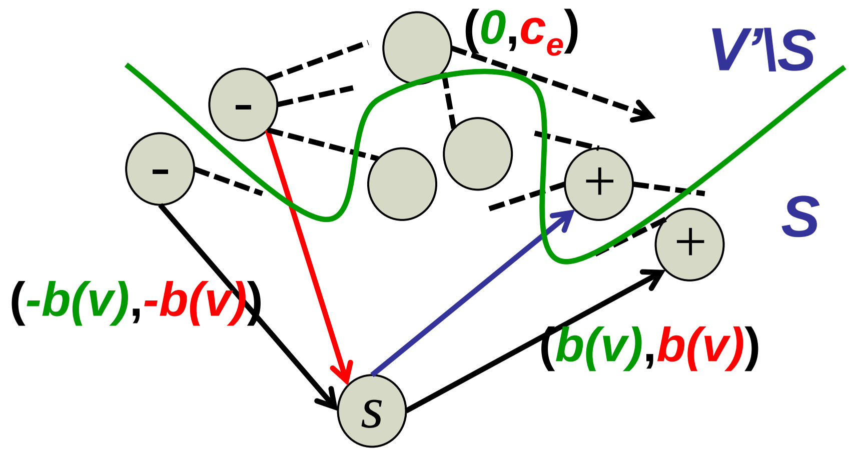
2. $s \in S$.

$\delta^-(S) \cap E'' = \{vs: v \in V^- \setminus S\}$

$\delta^+(S) \cap E'' = \{sv: v \in V^+ \setminus S\}$

$$\Rightarrow \sum_{v\in V^-\setminus S} -b(v) \le \sum_{v\in V^+\setminus S} b(v) + \sum_{e\in\delta^+(S)\cap E} c(e)$$

$$\Rightarrow \sum_{v\in V\setminus S} -b(v) \le \sum_{e\in\delta^+(S)\cap E} c(e)$$



The claim follows from $\sum_{v\in S} b(v) + \sum_{v\in V\setminus S} b(v) = 0$ ∎

Exercises

- Show that for any flow *x*, we have $\sum_{v \in V} div_x(v) = 0$

- Show: if *x* is a flow of *D = (V,E)*, then $\sum_{e \in \delta^+(S)} x(e) - \sum_{e \in \delta^-(S)} x(e) = \sum_{v \in S} \mathrm{div}(v)$

- Show: if *x* is a circulation of *D = (V,E)*, and lower and upper bounds *l* and *u* are integral, then *x* can be taken integral.

- Two directed edges of the form (*u*,*v*) , (*v*,*u*) are said *anti-parallel*. Show how to transform an instance of max-flow problem into an equivalent instance with no anti-parallel edges.