Max Flows and Minimum Cuts

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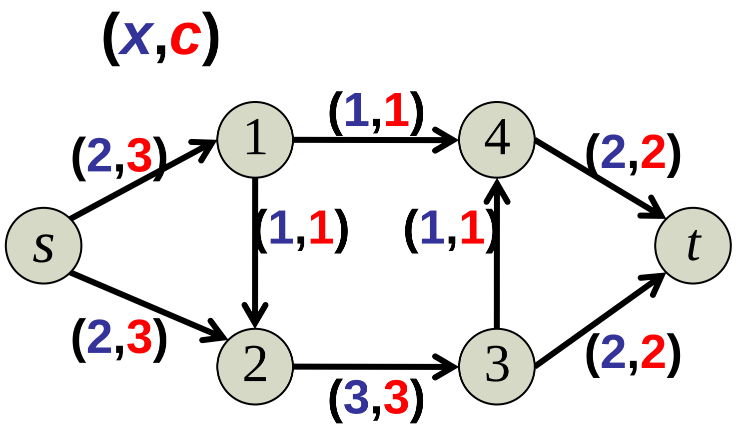
(from Geir Dahl notes)

University of Oslo, INF-MAT5360 - Autumn 2010 (Mathematical optimization)

st-flow

- Given a directed graph $D = (V,E)$
- 2 distinct vertices $s,t \in V$
- s source: no edges entering s
- t sink: no edges outgoing from t
- Edge capacity function $c : E \to R_+$



An *st-flow is a function* x: $E \to R$, *satisfying*

$$\sum_{e\in\delta^+(v)} x(e) = \sum_{e\in\delta^-(v)} x(e) \quad (v \in V \setminus \{s,t\})$$ flow conservation constraints

$$0 \le x \le c$$ non-negativity and capacity constraints

- Obs: $\mathrm{div}_x(v) = 0$ *for all* $v \neq s,t$.

Flow value

VALUE of an st-flow $val(x) = \sum_{e \in \delta^+(s)} x(e) = \mathrm{div}_x(s)$ = *flow outgoing from s*

Theorem (net st-flow through an st-cut)

Let $\delta^+(S)$ be a *st*-cut of $D = (V,E)$ *and* x be an *st*-flow. *Then*

$$\sum_{e \in \delta^+(S)} x(e) - \sum_{e \in \delta^-(S)} x(e) = val(x)$$

- Since $s \in S$ and $t \notin T$ we have $\sum_{v \in S} \mathrm{div}_x(v) = \mathrm{div}_x(s) = val(x)$
- The result follows from $\sum_{e \in \delta^+(S)} x(e) - \sum_{e \in \delta^-(S)} x(e) = \sum_{v \in S} \mathrm{div}(v)$ ∎

Corollary (the flow sent by s equals the flow into t)

Let $\delta^+(S)$ be a *st*-cut of $D = (V,E)$ *and* x be an *st*-flow. *Then*

$$val(x) = \sum_{e \in \delta^-(t)} x(e) = -\mathrm{div}_x(t)$$

The maximum *st*-flow problem

Maximum st-flow problem

Given a directed graph D = (V,E) with edge capacity $c{:}E \to R_+$, source s and sink t, find an *st-flow of maximum value.*

$$\max \sum_{e \in \delta^+(s)} x(e)$$ maximize flow value

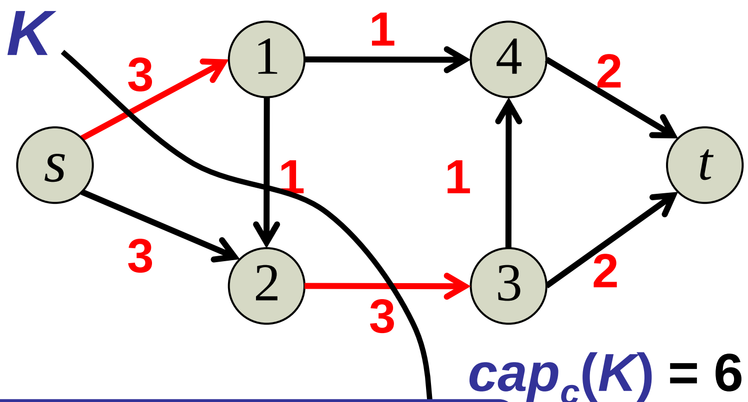
$$\sum_{e \in \delta^+(v)} x(e) = \sum_{e \in \delta^-(v)} x(e) \quad (v \in V \setminus \{s,t\})$$ flow conservation constraints

$$0 \le x \le c$$ non-negativity and capacity constraints

- LP- program: optimum exists. We can apply simplex method.
- More effective combinatorial algorithms.

Minimum st-cut

- Let $K = \delta^{+}(S)$ be an *st-cut* ($s \in S$, $t \notin S$)



CAPACITY of K: $cap_c(K) = \sum_{e \in K} c(e)$

Minimum st-cut problem

Given a directed graph $D = (V,E)$ with edge capacity $c{:}E \rightarrow R_{+}$, source s and sink t, find an *st-cut with minimum capacity.*

- Combinatorial optimization problem with linear objective function.

Minimum st-cut

Lemma (weak max-flow min-cut property)

The following inequality holds:

$\max\{\mathrm{val}(x): x \text{ is } st\text{-}flow\} \leq \min\{\mathrm{cap}_c(K): K \text{ is } st\text{-}cut\}$

- Let $K = \delta^+(S)$ be a (minimum capacity) *st-cut* $(s \in S, t \notin S)$

$$val(x) = \sum_{e \in \delta^+(S)} x(e) - \sum_{e \in \delta^-(S)} x(e) \leq \sum_{e \in \delta^+(S)} x(e) \leq \sum_{e \in \delta^+(S)} c(e) = \mathrm{cap}_c(K)$$

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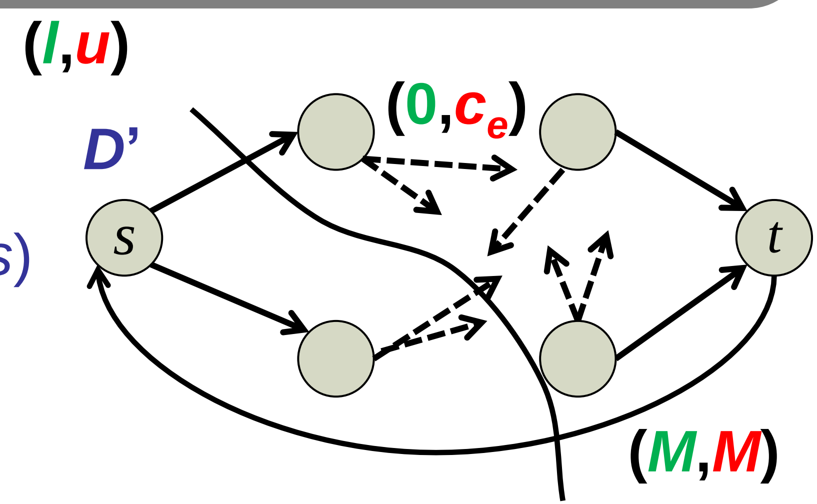
The max-flow min-cut theorem

Theorem (max-flow / min-cut)

The value of a maximum *st*-flow equals the minimum *st-cut* capacity:

$\max\{\mathrm{val}(x): x \text{ is } st\text{-}flow\} = \min\{\mathrm{cap}_c(K): K \text{ is } st\text{-}cut\}$

- Let $M = \min\{\mathrm{cap}_c(K): K \text{ is } st\text{-}cut\}$
- The flow value is no larger than M
- Build graph D' from D adding edge (t,s)
- Let $l(t,s) = u(t,s) = M$;
  $l(e) = 0,\ u(e) = c(e)$ for $e \in E$



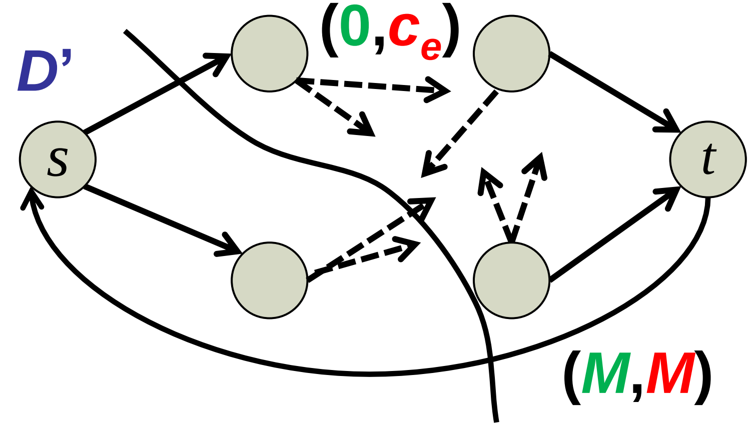
- We show that D', l and u satisfy Hoffman's conditions

$$\sum_{e\in\delta^-(S)} l(e) \le \sum_{e\in\delta^+(S)} u(e) \qquad (S \subseteq V)$$

and D' admits a circulation x'.

Minimum st-cut

$$\sum_{e\in\delta^-(S)} l(e) \le \sum_{e\in\delta^+(S)} u(e) \qquad (S \subseteq V)$$ Hoffman's condition



1. If $(t,s) \in \delta^+(S)$ ⇨ $\sum_{e\in\delta^-(S)} l(e) = 0 \le \sum_{e\in\delta^+(S)} c(e) = \sum_{e\in\delta^+(S)} u(e)$

2. If $(t,s) \in \delta^-(S)$ ⇨ $\sum_{e\in\delta^-(S)} l(e) = M$

⇨ $s \in S,\ t \notin S$ ⇨ $K = \delta^+(S)$ is an *st*-cut

⇨ $\sum_{e\in\delta^+(S)} u(e) = \sum_{e\in\delta^+(S)} c(e) = \text{cap}_c(K) \ge M$ satisfying Hoffman's condition

⇨
- There exists a circulation x in D' *with* $l \le x \le u \rightarrow x(t,s) = M$
- *x circulation* ⇨ $\sum_{e\in\delta^+(s)} x(e) = \sum_{e\in\delta^-(s)} x(e) = x(t,s) = M$
- The restriction of x to the edges of D provides an *st-flow* with value M

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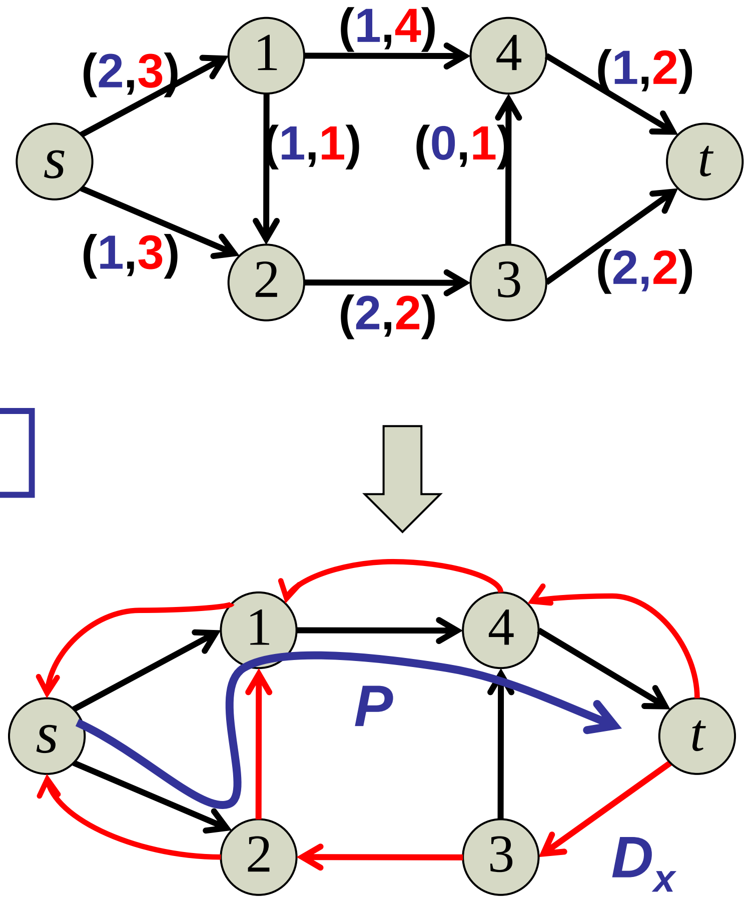
Augmention paths

$D = (V,E)$, *capacity* $c \in R^E_+$, *st-flow* $x \in R^E$

Auxiliary Graph $D_x = (V,E_x)$

$E_x = \{e \in E: x_e < u_e\} \cup \{e^{-1}: e \in E,\ 0 < x_e\}$



x-AUGMENTING PATH: *st-path* P *in* D_x

Forward edges $P^+ = \{e \in E: e \in E_x\}$

Backward edges $P^- = \{e \in E: e^{-1} \in E_x\}$

The augmenting path theorem

Theorem (augmenting path theorem)

An st-flow x is maximum if and only if D_x contains no x-augmenting path.

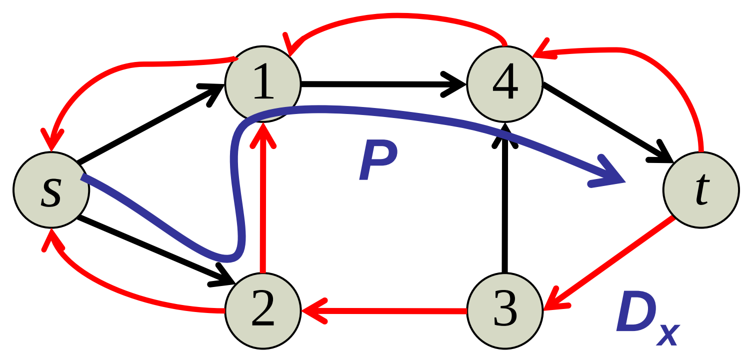
Proof. (*Necessity*) By contradiction.

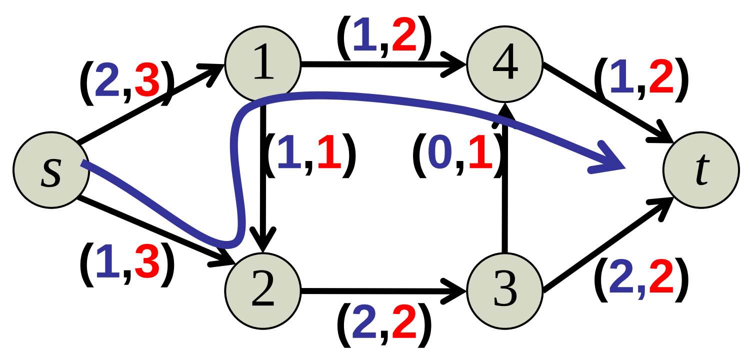
Suppose x *is maximum and* P *st-path of* D_x

Let $\varepsilon^+ = \min\{c(e)-x(e): e \in P^+\} > 0$

Let $\varepsilon^- = \min\{x(e): e \in P^-\} > 0$

$\varepsilon = \min\{\varepsilon^-, \varepsilon^+\} > 0$





Send ε *units of flow on* P*. Obtain flow* x'

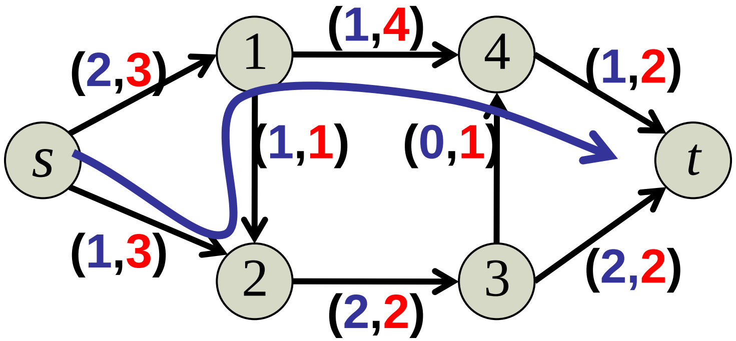
$$x'(e) = \begin{cases} x(e) & \text{if } e \notin P \\ x(e) + \varepsilon & \text{if } e \in P^+ \\ x(e) - \varepsilon & \text{if } e \in P^- \end{cases}$$

The augmenting path theorem

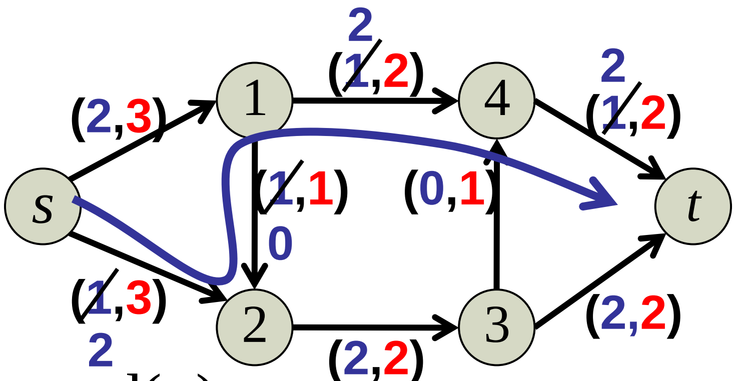
Proof. (*Necessity*)

Send ε *units of flow on P. Obtain flow x'*

$$x'(e) = \begin{cases} x(e) & \text{if } e \notin P \\ x(e) + \varepsilon & \text{if } e \in P^+ \\ x(e) - \varepsilon & \text{if } e \in P^- \end{cases}$$



x' is a feasible st-flow (show it!)



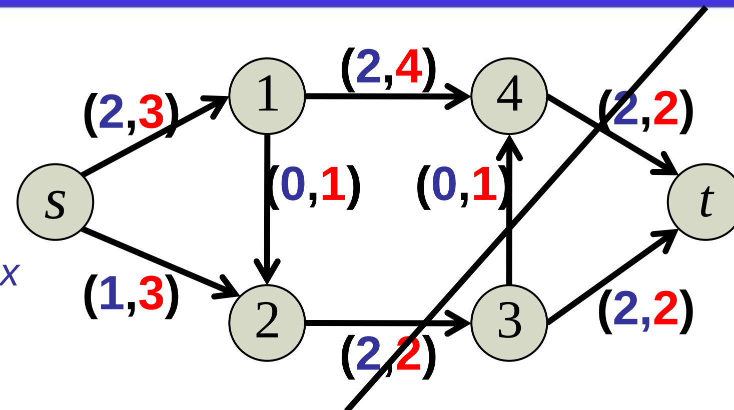
Since P is an st-path, we have:

$$\mathrm{val}(x') = \sum_{e \in \delta^+(s)} x'(e) = \sum_{e \in \delta^+(S)} x(e) + \varepsilon > \sum_{e \in \delta^+(S)} x(e) = \mathrm{val}(x)$$

and *x* is not maximum, a contradiction. ∎

The augmenting path theorem

Proof. (*Sufficiency*)



Suppose no *st-path* in D_x

Then there is an empty *st*-cut $K=\delta^+(S)$ in D_x

(i.e. $S = \{w \in V$: there is a *sw-path* in $D_x\}$

If $D = (V,E)$ contains an edge (w,z) with $w \in S$ to $z \in V \setminus S$

⇨ $x_{wz} = c_{wz}$ (otherwise $(w,z) \in E_x$) ⇨ $\sum_{e \in \delta_D^+(S)} x(e) = \sum_{e \in \delta_D^+(S)} c(e)$

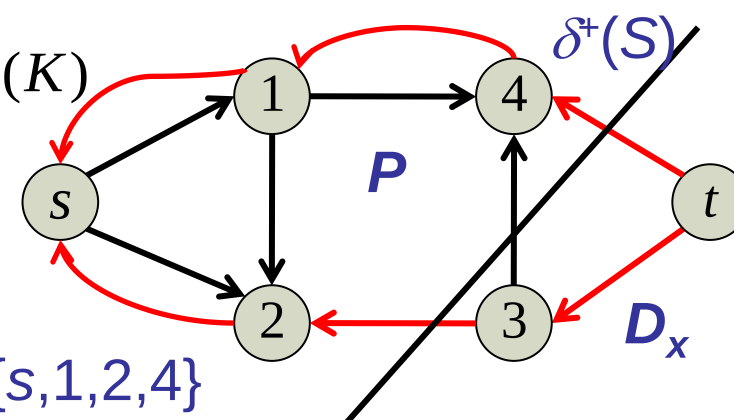
If $D = (V,E)$ contains an edge (z,w) with $w \in S$ to $z \in V/S$

⇨ $x_{zw} = 0$ (otherwise $(z,w)^{-1} = (w,z) \in E_x$) ⇨ $\sum_{e \in \delta_D^-(S)} x(e) = 0$

⇨ $$val(x) = \sum_{e \in \delta^+(S)} x(e) - \sum_{e \in \delta^-(S)} x(e) = \sum_{e \in \delta^+(S)} c(e) = cap_c(K)$$

⇨ x is a *maximum flow* and

K is a *minimum cut*



$S=\{s,1,2,4\}$

Ford-Fulkerson algorithm

1. Start with the zero flow $x = 0$
2. Look for an *x-augmenting path* P in D_x
3. if P exists, then find the maximum possible increase ε along P and augment x. Goto 2.
4. if no such P exists then x is a maximum flow.

   A minimum st-cut is $\delta^+(S(x))$ where $S(x)=\{w \in V$: there is a *sw-path* in $D_x\}$.

- The algorthm can take C iterations, where C is the maximum capacity of an edge.
- There are more efficient versions (polynomial in $|V|$ and $|E|$).

Exercises

- Show that for any *st*-flow x, we have $val(x) = \sum_{e \in \delta^-(t)} x(e)$
- Use the *augmenting path theorem* to give an alternative proof to the max-flow / min-cut theorem
- Show: if x is a maximum st-flow of $D = (V,E)$, and all capacity are integral, then x can be taken integral.
- Show that the augmented flow x' in the necessity proof of the augmenting path theorem is a (feasible) *st-flow.*
- Two directed edges of the form (u,v) , (v,u) are said *anti-parallel*. Show how to transform an instance of max-flow problem D=(V, E), $c \in R^E$ into an equivalent instance D' = (V', E') , c' $\in R^{E'}$ with no anti-parallel edges.