Shortest paths and trees

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(from Lex Schrijver notes)

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Combinatorial Optimization
Basic Definition and examples

Combinatorial Optimization Problem

- Finite ground set E, weight function $w : E \to R$. (i.e. $w \in R^E$)
- Feasible solutions $\mathcal{F} = \{F_1, \ldots, F_m\}$, with $F_i \subseteq E$, $i = 1, \ldots, m$
- Combinatorial optimization problem (CO)

$$\max \{w(F): F \in \mathcal{F}\}, \quad \text{where} \quad w(F) = \sum_{e \in F} w(e)$$

- Let $S \subseteq \{0,1\}^E$ be the set of the incidence vectors of the sets in $\mathcal{F}$

$$S = \{\chi^F : F \in \mathcal{F}\}$$

Remark: $$w(F) = \sum_{e \in F} w(e) = w^T \chi^F$$

- Combinatorial optimization problem (rewritten)

$$\max \{w^T x : x \in S\}$$ 0-1 *linear program*

- Solving (CO) and 0-1 LP is difficult (*NP – hard*)

Example: project selection

- Projects A e B
- Profits w_A e w_B
- Costs $c_A = 5$, $c_B = 7$
- Budget constraint $\leq D = 10$

Project selection problem:

Find a selection of projects satisfying the budget constraint and maximizing profit.

$E = \{A, B\}$

Feasible Solutions $\mathcal{F} = \{\{\}, \{A\}, \{B\}\}$

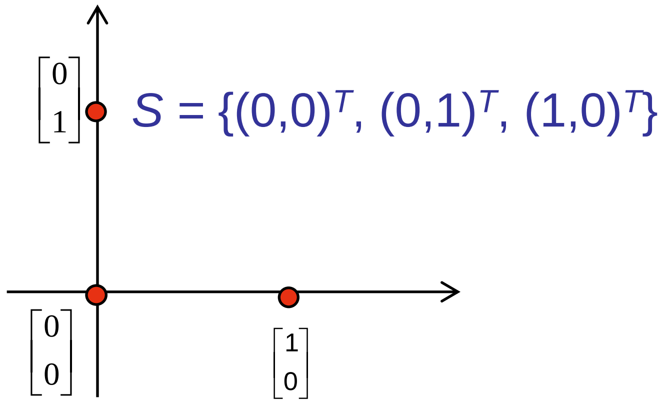
$c(\{\}) = 0$, $c(\{A\}) = 5$, $c(\{B\}) = 7$,

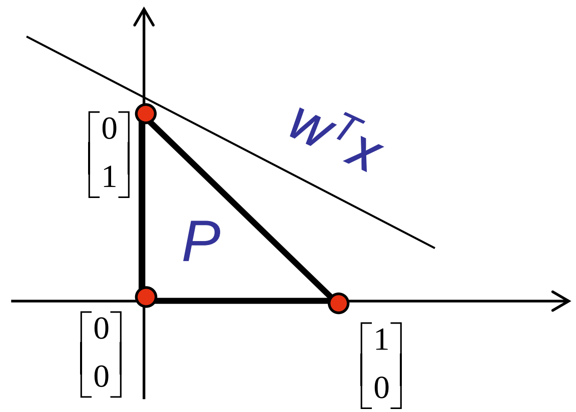
$c(\{A, B\}) = 12 > D$ $\{A, B\}$ not feasible

$$\max\ w_A x_A + w_B x_B$$

$$x \in S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

From CO to LP





$\max\ \{w^Tx: x \in S\}$

- $P = \mathrm{conv}(S)$ convex hull of the points in S. P is a <u>polytope</u>.
- Vertices of $P = \mathrm{ext}(P) = S$.

$$\max\ \{w^Tx: x \in S\} = \max\ \{w^Tx: x \in \mathrm{ext}(P)\} = \max\ \{w^Tx: x \in P\}$$

linear program!

- We can solve (CO) by linear programming!

Combinatorial Optimization Course

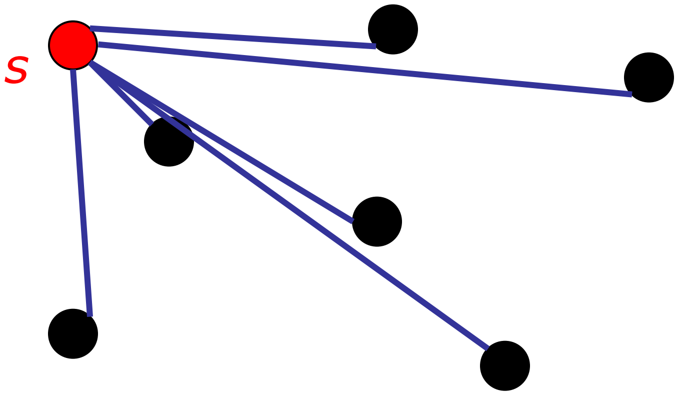
Outline of the course

- Basic combinatorial optimization problems: shortest path, minimum spanning tree, maximum flow, minimum cut.
- Connections with linear programming and polyhedral theory
- Integer Polyhedra
- Methods: heuristic algorithms
- Methods: exact approaches

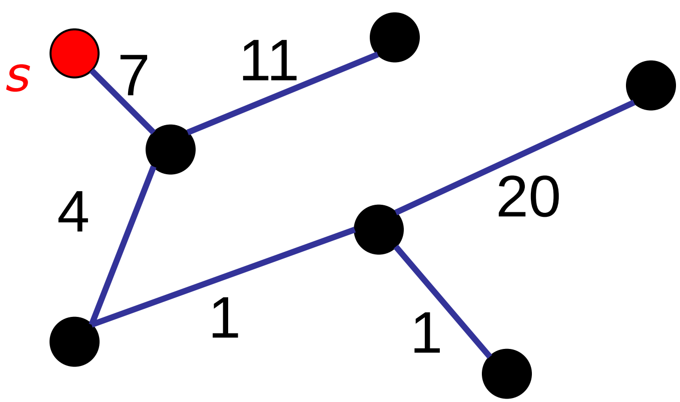
A huge number of relevant real-life applications can be modeled as COs

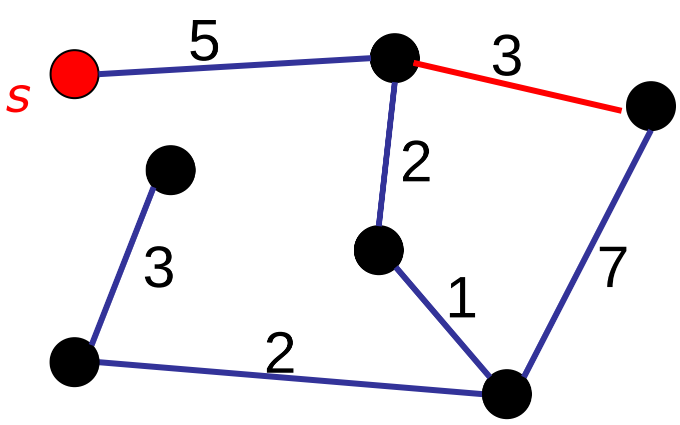
Example: connectivity

•Network design

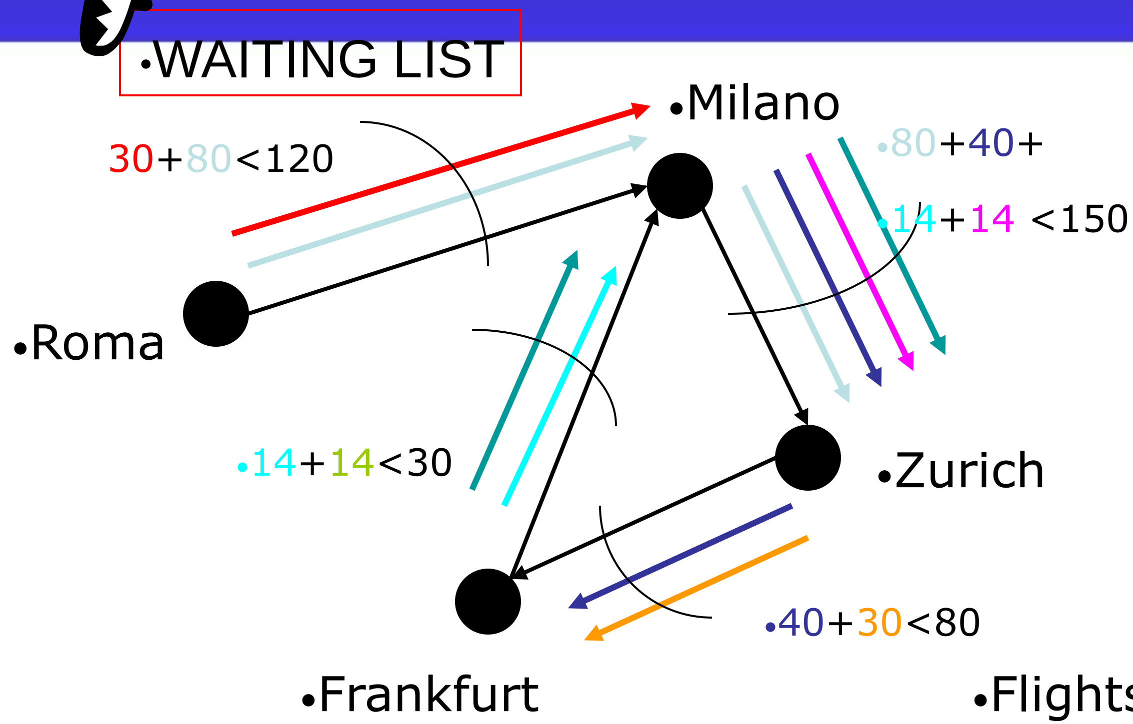


- You need to connect a source *s* (of water, packets, …) to a number of locations (*farms, computers*).
- Each connection (pipe, fiber) has a cost
- WANT: *find a minium cost network connecting all locations to the source*





Example: flow



Expected demand:

•DA \ •A	RM	MI	FR	ZU
RM	-	30	-	80
MI	-	-	40	14
FR	-	14	-	14
ZU	-	-	30	-

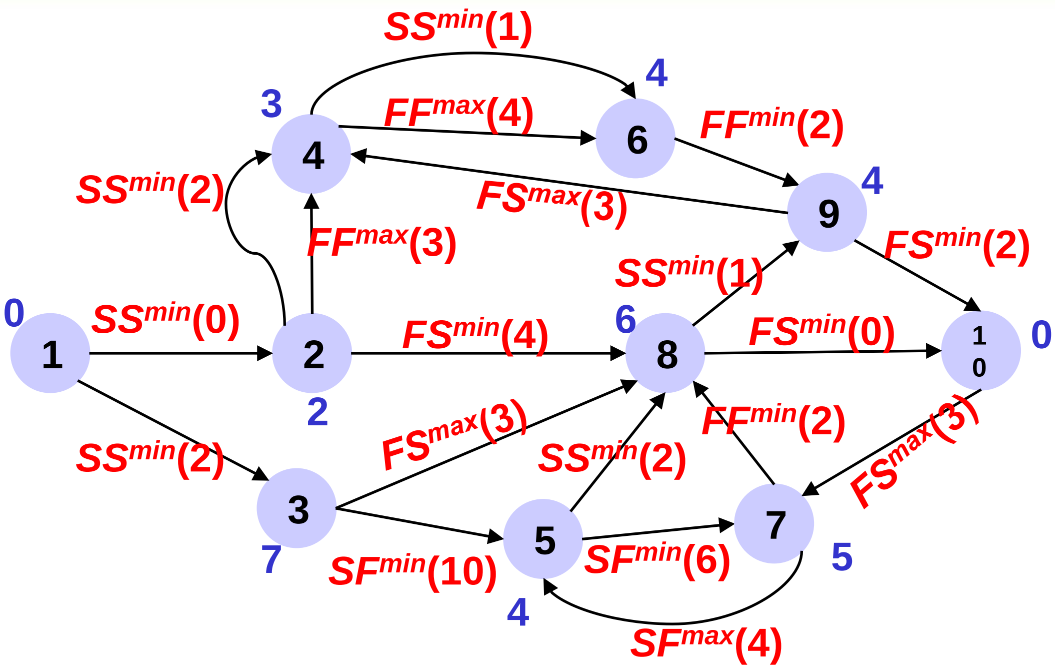
Passengers should

- ✓ *reach their destinations*
- ✓ *be assigned to departing flights*
- ✓ *…. Satisfying capacity constraints*

•Flights (available sits):

- •Roma-Milano: 120 sits
- •Milano-Zurich: 150 sits
- •Frankfurt -Milano: 30 sits
- •Zurich- Frankfurt : 80 sits

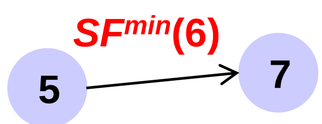
Example: Project Scheduling



Projects decompose into activities

Activities may require resources, which in turn may be limited

Precedence Relations exist between activities.



= "7 *must start at least 6 time units after 5 is terminated*"

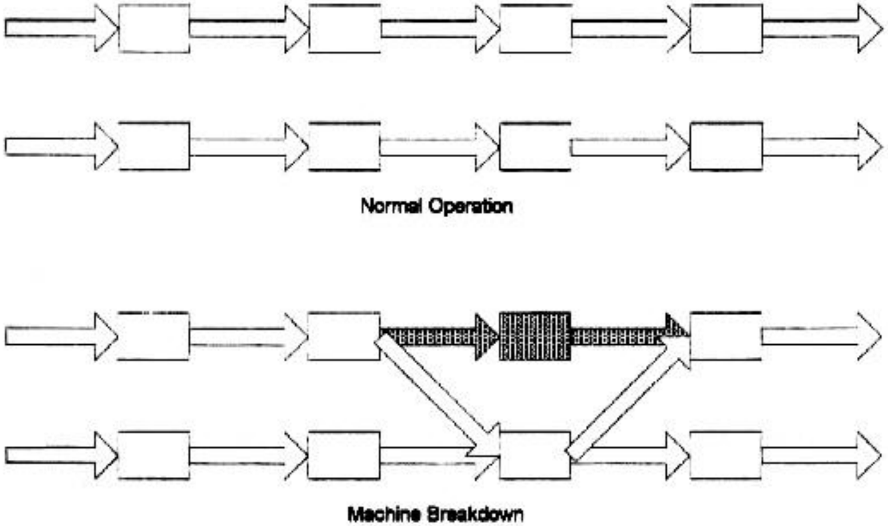
WANT: find a schedule of the activities satisfying all precedence constraints and minimizing the project completion time

Example: *Job-Shop Scheduling*

- A product (*job*) must be processed on different machines

- Processing a *job* on a machine is called *operation*

- Each machine can process at most *k* jobs at a time.



- WANT: find a schedule of the operations satisfying machine capacities and additional precedence constraints.

Example: *vehicle routing*

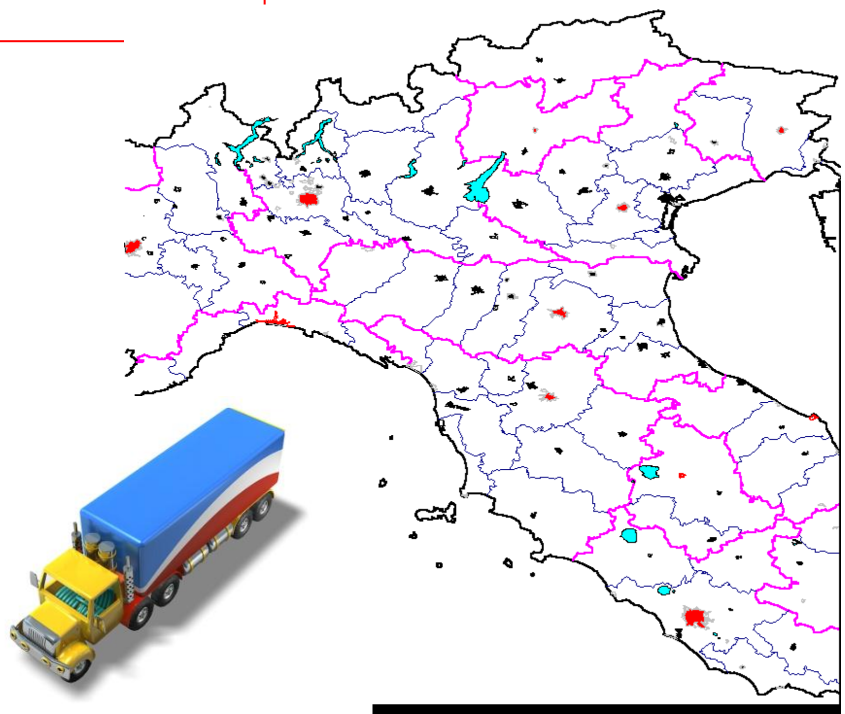
Transfer goods from *origins* to *destinations*

- Minimizing transportation costs
- Satisfying:
- - constraints on vehicle capacities
- - connectivity constraints
- …

Several parameters are involved

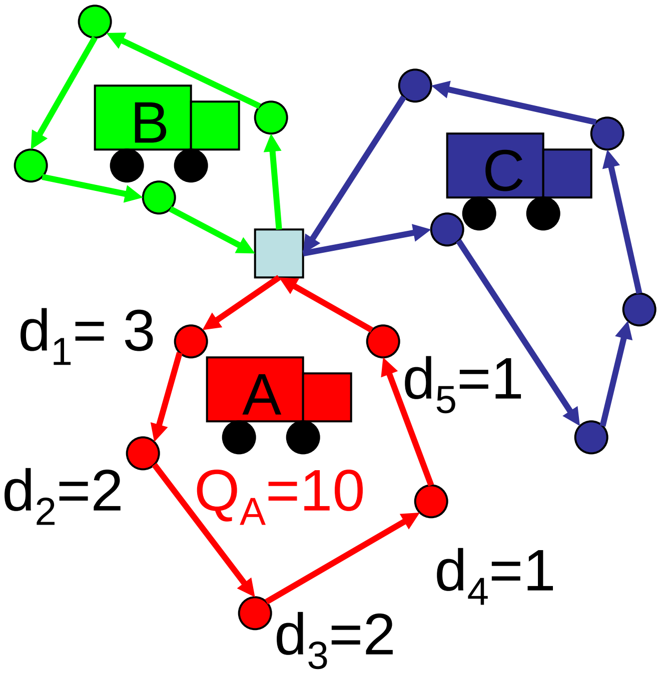
1. *Origin and destination position*
2. *Demand level*
3. *Fleet size*

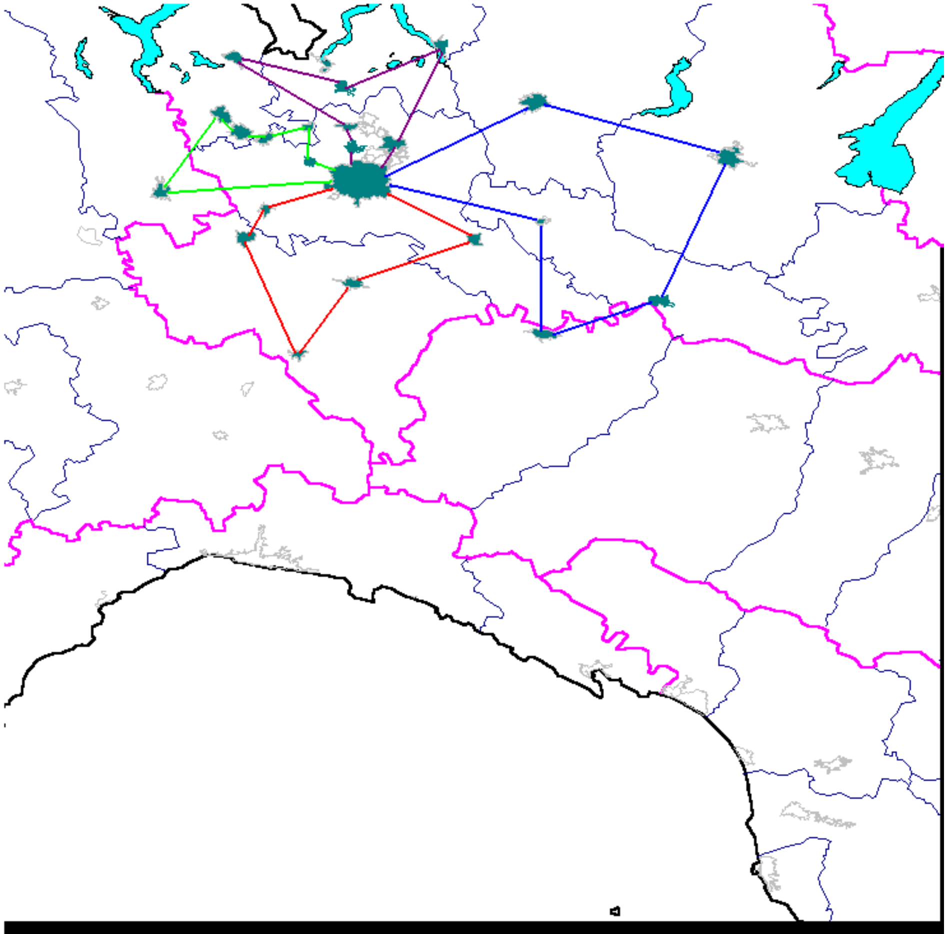
…



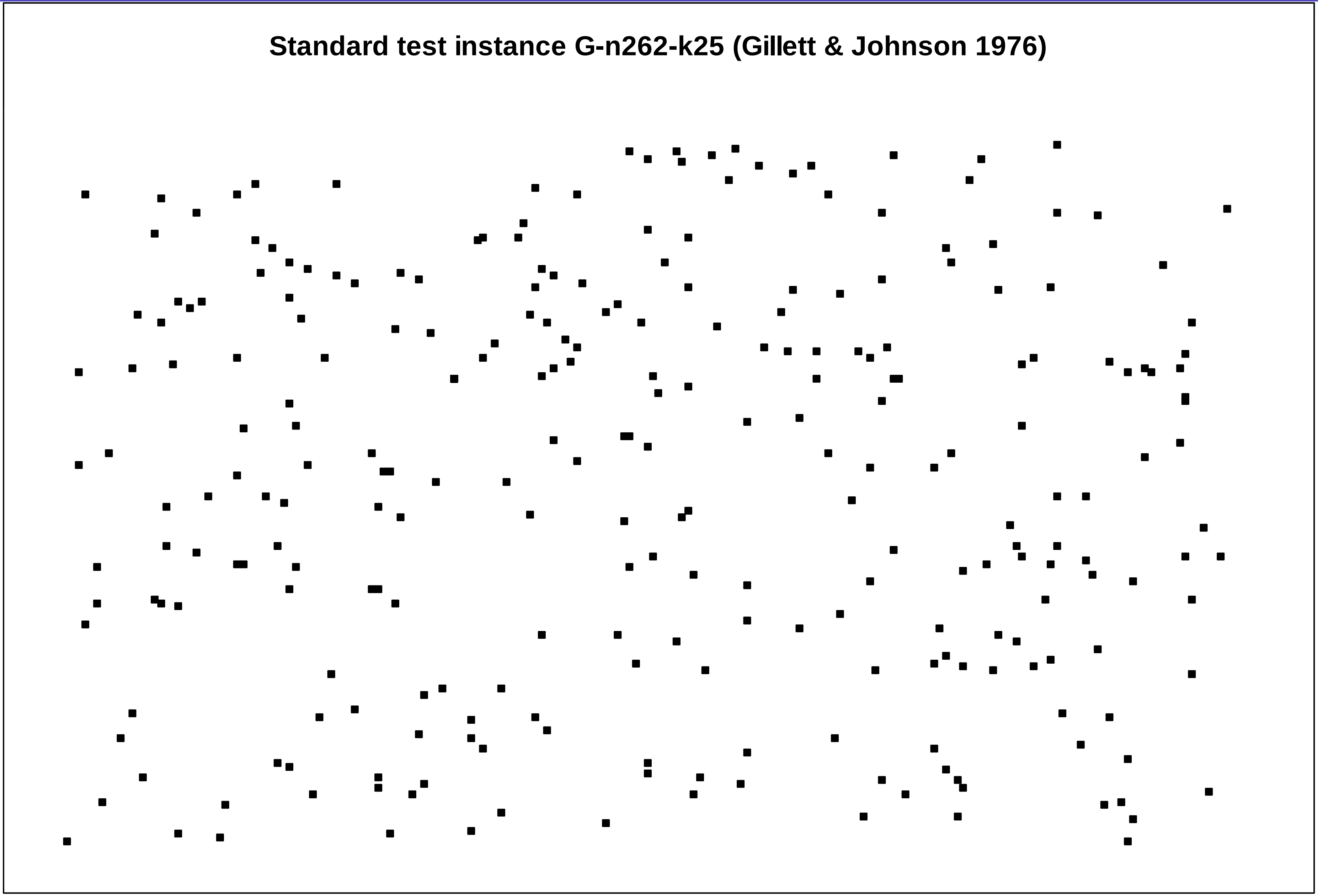
Example: *vehicle routing*

- *Each vehicle visits a subset of customers and returns to depot*

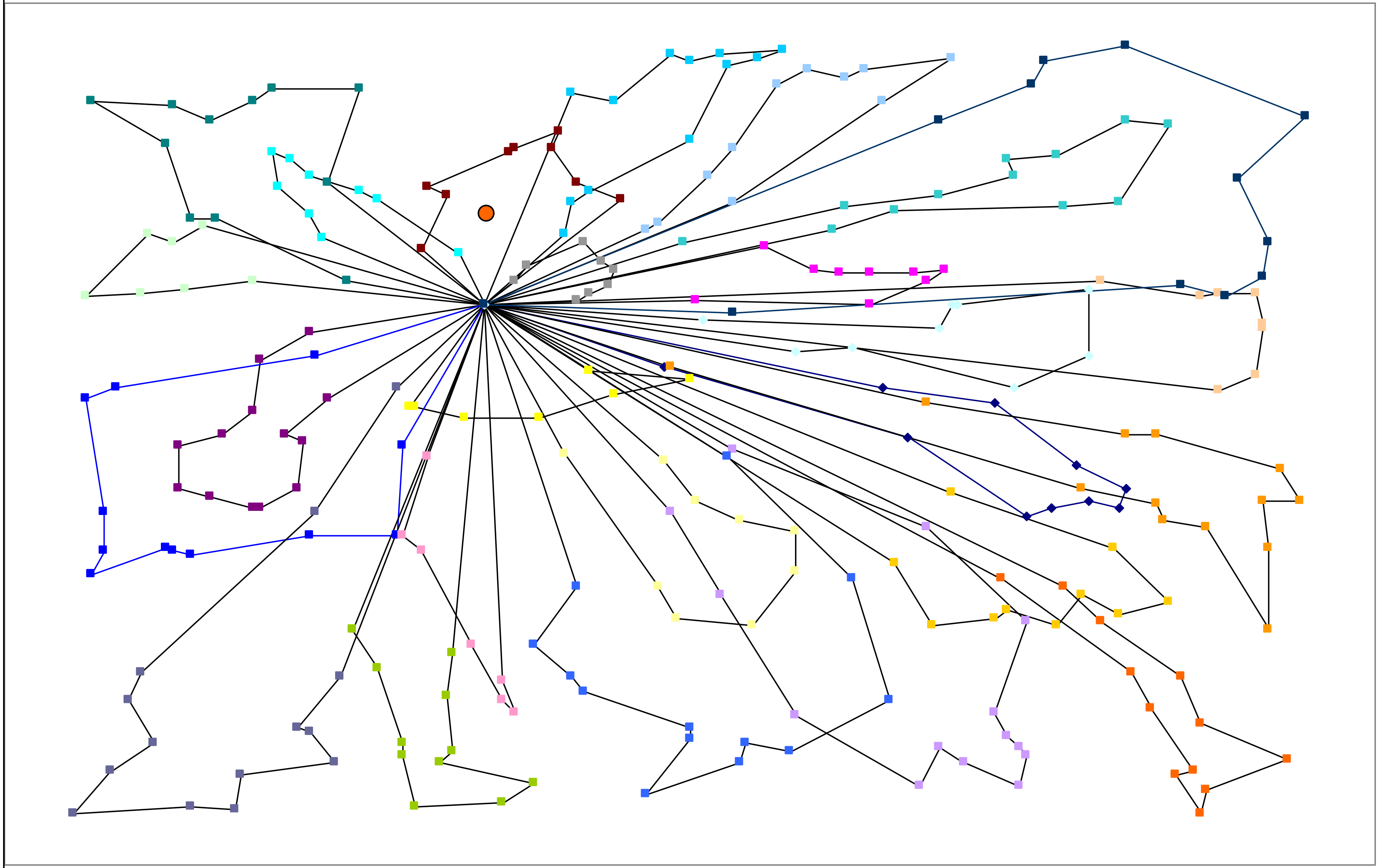




A famous instance

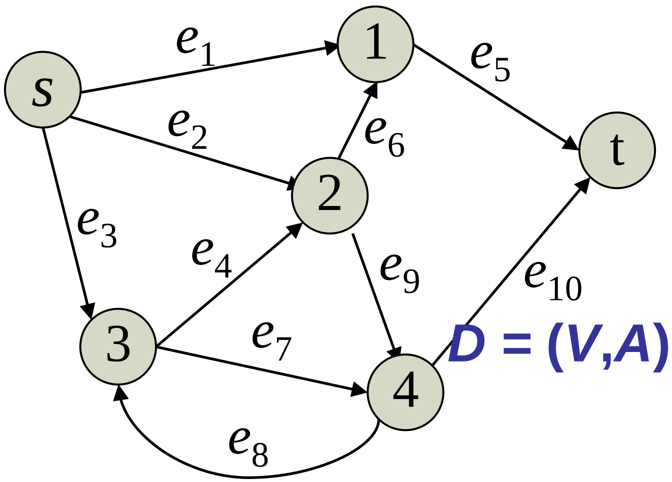


"The world record" for G-n262-k25: 5685 vs. 6119
(SINTEF 2003)



Shortest paths and trees

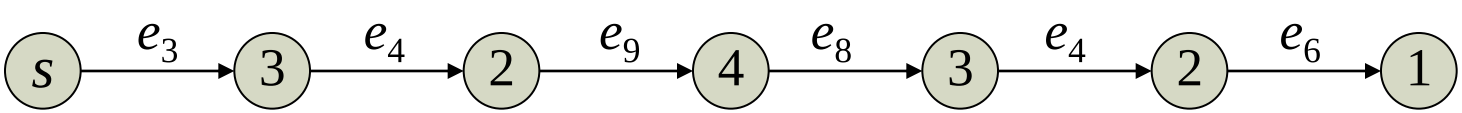
Walks and paths



- $D = (V,A)$ directed graph, $s,t \in V$
- V vertices
- A arcs

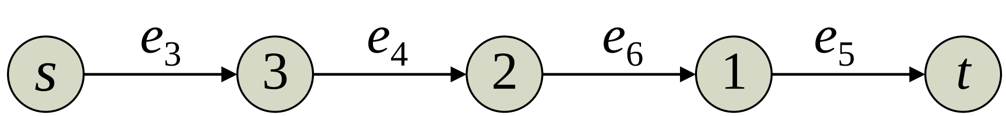
- *Walk*: alternating sequence of vertices and arcs:

$P = (v_0 , a_1 , v_1 , \ldots, a_m , v_m) : \ a_i = (v_{i-1} , v_i) \quad i = 1, \ldots, m$

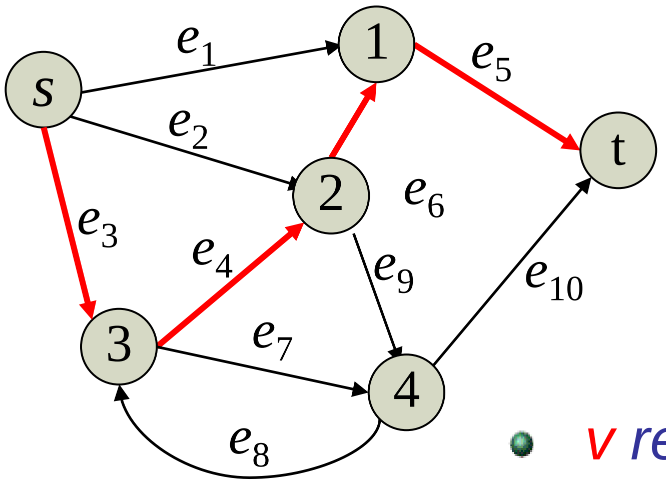


$P = (v_0 , a_1 , v_1 , \ldots, a_m , v_m)$ *goes* from v_0 to v_m

- *Path*: walk with no repeated vertices

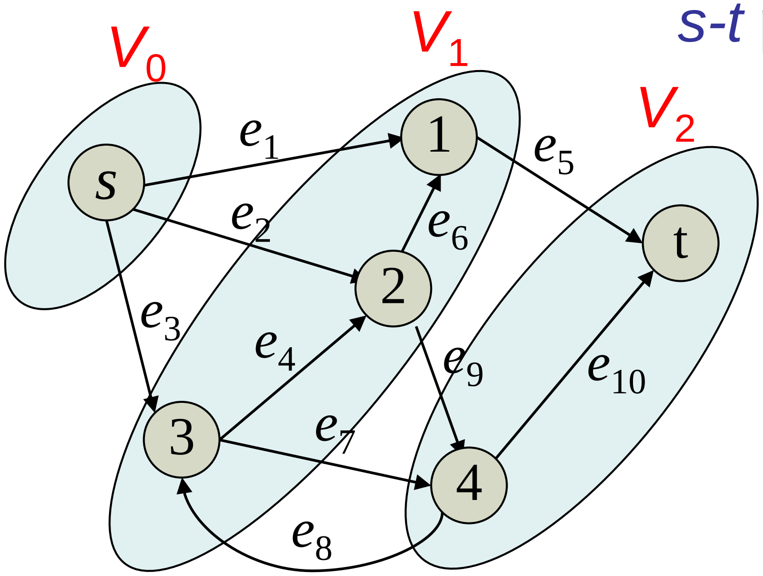


Length and distance



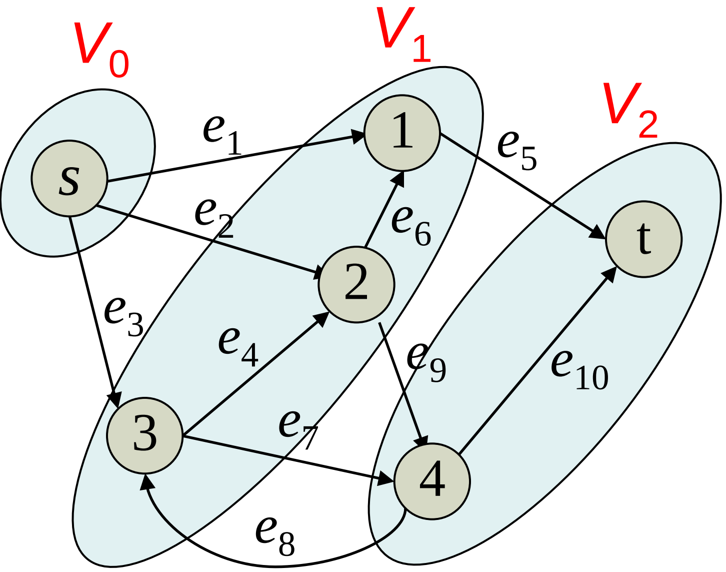
$D = (V,A)$

- *s-t walk (path)*: walk (path) with starting vertex *s* and end vertex *t*
- *Length* of walk *P* : number of arcs
- *v reachable from u:* there exists and *u-v* path in *D*
- *Distance* from *s* to *t*: minimum length of any *s-t* path (+∞ if *t* is not reachable from *s*)



- V_i: set of vertices at distance *i* from *s*

Finding shortest paths



- V_i: set of vertices at distance i from s

Recursive Rule:

V_{i+1}: set of vertices $v \in V \setminus (V_0 \cup V_1 \cup \ldots V_i)$
for which $(u,v) \in A$ for some $u \in V_i$

Shortest Path Algorithm:

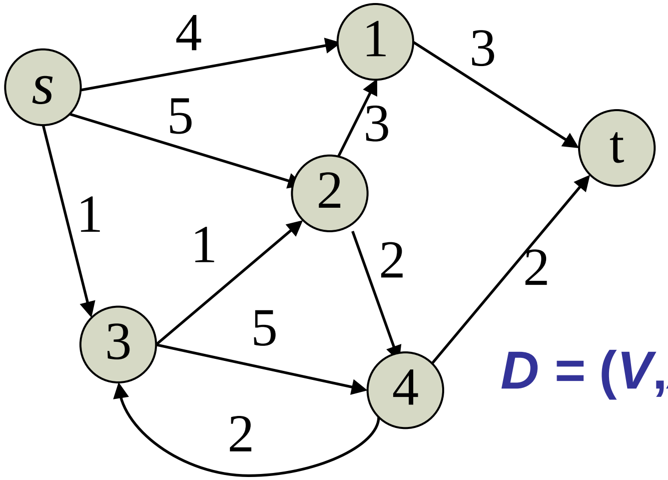
1. *Set* $V_0 = \{s\}$, $i = 0$.
2. *While* $V_i \neq \{\}$
   3. Compute V_{i+1} from V_i
   4. Set $i = i + 1$

EndWhile

- Finds the distance from s to <u>*all*</u> <u>vertices *reachable*</u> from s
- Finds $T = (V', A')$ *shortest path tree*
- At each iteration explores new arcs; at the end every arc is visited at most once

- Running Time: $O(|A|)$:

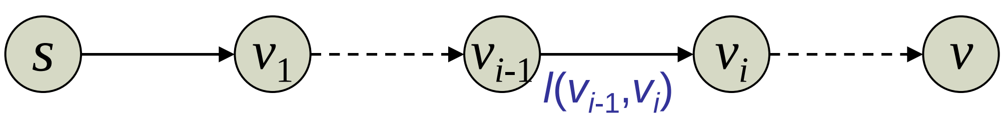
Graphs with non-negative arc lengths



$D = (V,A)$

- Length (weight) function l: $A \rightarrow Q_+$
- *Given walk* $P = (v_0, a_1, v_1, \ldots, a_m, v_m)$
- *Length of P*: $l(P) = \sum_{i=1}^{m} l(a_i)$
- *Distance from s to v* (w.r.t. l): *dist*(v) length of a minimum length s-v path in D ($+\infty$ if no s-v path exists)

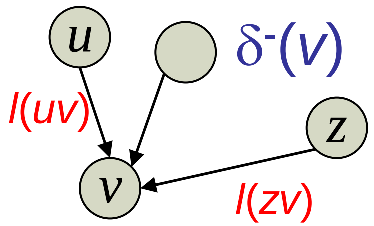
On a shortest s-v path $s = v_0, v_1, \ldots, v_k = v$



$\text{dist}(v_i) = \text{dist}(v_{i-1}) + l(v_{i-1}, v_i)$

(every sub-path is a shortest path)

Graphs with non-negative arc lengths



negative star

$\delta^-(v) = \{e \in A\text{: } e = (u,v) \text{ for some } u \in V\}$

positive star: $\delta^+(v) = \{e \in A\text{: } e = (v,u) \text{ for some } u \in V\}$

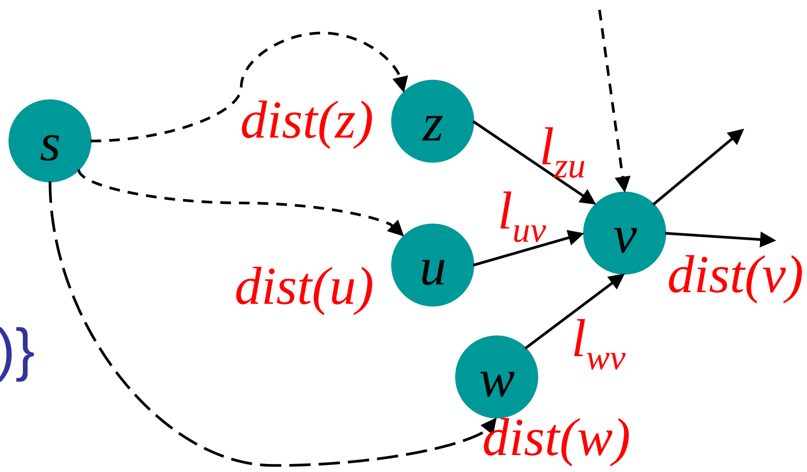
negative neighborhood: $N^-(v) = \{u \in V\text{: } uv \in A\}$

- Trivial Facts:

(i) dist(s) = 0

(ii) dist(v) = $\min$ {dist(u) + $l(uv)$: $uv \in \delta^-(v)$}

(iii) dist(v) $\leq$ dist(u) + $l(uv)$, $uv \in \delta^-(v)$



Dijkstra shortest path algorithm

Dijkstra Shortest Path Algorithm:

1. *Set* $U := V$, $f(s) := 0$, $f(v) := +\infty$ for $v \in U \setminus \{s\}$
2. *While* $U \neq \{\}$
   3. *Select* $u \in U$ minimizing $f(u)$. *Set* $U := U \setminus \{u\}$.
   4. *For* each $uv \in \delta^+(u)$
      5. If $f(v) > f(u)+l(uv)$ *Reset* $f(v) := f(u)+l(uv)$

   EndFor

EndWhile

Theorem 1.3

The final function f gives the distance from s.

Proof of Theorem 1.3

Proof.

- Claim 1: at any iteration $f(v) \geq \text{dist}(v)$, for each $v \in V$.

- *Suppose not.* True at initialization. Then there is a first *Reset* and a vertex w such that:

(i) $f(w) < \text{dist}(w)$; (ii) $f(w) = f(u)+l(uw)$; (iii) $f(u) \geq \text{dist}(u)$

$\text{dist}(w) \leq \text{dist}(u) + l(uw) \rightarrow \text{dist}(w) \leq f(u) + l(uw) = f(w)$, contradiction

Proof of Theorem 1.3

- Claim 2: at any iteration $f(v) = dist(v)$, for each $v \in V \setminus U$
- We show: when the algorithm *Selects* $u \in V \setminus U$ then $f(u) = \mathrm{dist}(u)$.

Suppose not. Then, $f(u) > \mathrm{dist}(u)$ for some u (when selected)

- $s = v_0, v_1, \ldots, v_k = u$ *shortest* s-u *path.*
- *Let* i smallest such that $v_i \in U$, $v_{i-1} \notin U$.

$s \in U \to i = 0$, $f(s) = 0 = \mathrm{dist}(s)$, contradiction.

$s \notin U \to i > 0$, $v_i \in U \to i \leq k$

$i > 0 \quad \to f(v_{i-1}) = \mathrm{dist}(v_{i-1})$ (by induction, since $v_{i-1} \in V \setminus U$)

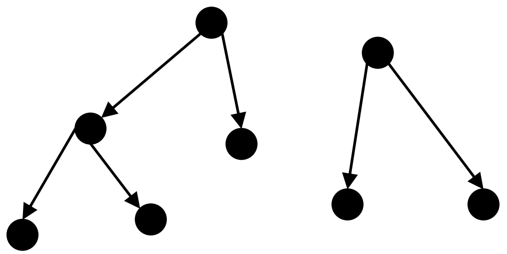
$f(v_i) \leq f(v_{i-1}) + l(v_{i-1}, v_i)$ $\qquad$ ($v_{i-1} \in V \setminus U$)

$= \mathrm{dist}(v_{i-1}) + l(v_{i-1}, v_i) = \mathrm{dist}(v_i)$ $\qquad$ (shortest path)

$\to f(v_i) = \mathrm{dist}(v_i) \leq \mathrm{dist}(u) < f(u)$ contradicting the choice of u. ■

Complexity of Dijkstra Algorithm

- The *While* iteration is repeated $|V|$
- The *Select* operation requires at most $|V|$ checks
- The contribution to overall complexity is then $O(|V|^2)$
- Every arc is visited exactly once
- Overall complexity $O(|V|^2) + O(|A|)$. This complexity can be improved when $|A| < |V|^2$
- Improve the *Select* by using *heaps* to store $f(u)$, $u \in U$

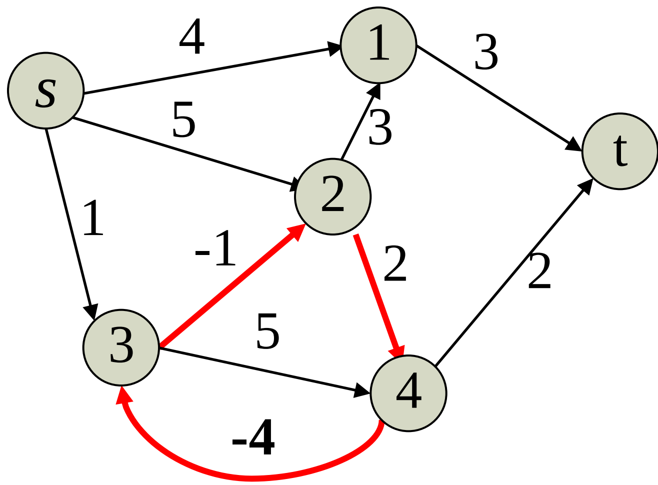


Heap: routed forest (U,F), $uv \in F \rightarrow f(u) \leq f(v)$

Routed Forest: every vertex has indegree at most 1.

Arbitrary arc lengths

- Dijkstra algorithm can be applied only when arcs have non-negative lengths (*conservative*)
- Otherwise a *shortest walk* may not exist (if *D* contains a negative length di-cycle).
- Observe that if *D* contains a path from *s* to *v*, then it contains a *shortest path* from *s* to *v*.

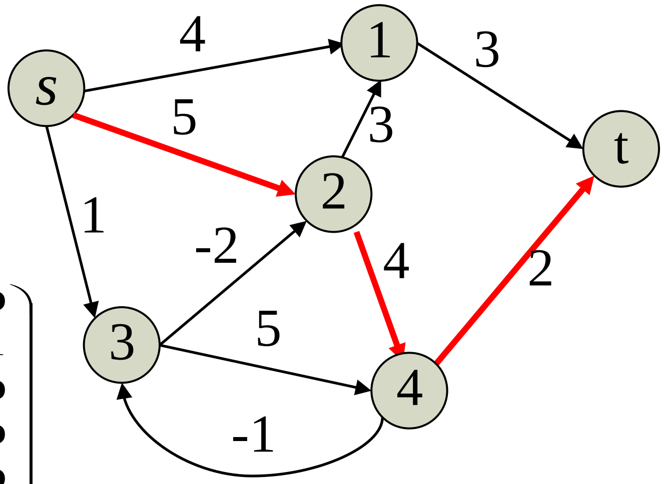


- Finding the shortest path with arbitrary arc lengths is difficult (*NP-hard*)
- Easy if *D* contains no negative length di-cycles, but we need a different algorithm (e.g. Bellman-Ford, or Floyd-Warshall)

The s-t path polyhedron

$\boldsymbol{x}^P \in \{0,1\}^A$ *incidence vector of* *an s-t path* P

$$\begin{cases} \boldsymbol{x}^P_{uv} = 1 & uv \in P \\ \boldsymbol{x}^P_{uv} = 0 & uv \notin P \end{cases}$$



$P = (s,(s,2),2,(2,4),4,(4,t),t)$

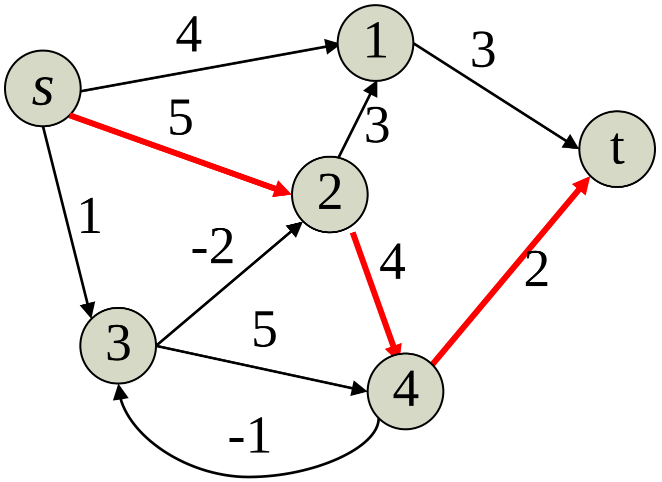
$$\boldsymbol{x}^P = \begin{array}{c} s1 \\ s2 \\ s3 \\ 1t \\ 21 \\ 24 \\ 32 \\ 34 \\ 43 \\ 4t \end{array} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

- a vector $\boldsymbol{x}^P \in \{0,1\}^A$ is the incidence vector of an *s-t path* of D if and only if it satisfies a number of equalities

The s-t path polyhedron

$\boldsymbol{x}^P \in \{0,1\}^A$ <u>*incidence vector of*</u> *an s-t path P*

$$\begin{cases} \boldsymbol{x}^P_{uv} = 1 & uv \in P \\ \boldsymbol{x}^P_{uv} = 0 & uv \notin P \end{cases}$$



No arc incoming s. One arc outgoing from s

$$\sum_{us \in \delta^-_D(s)} \boldsymbol{x}^P_{us} = 0 \qquad \sum_{su \in \delta^+_D(s)} \boldsymbol{x}^P_{su} = 1$$

$$\sum_{ut \in \delta^-_D(t)} \boldsymbol{x}^P_{ut} = 1 \qquad \sum_{tu \in \delta^+_D(t)} \boldsymbol{x}^P_{tu} = 0$$

One arc incoming t. No arc outgoing from t

$$\sum_{uv \in \delta^-_D(v)} \boldsymbol{x}^P_{uv} = \sum_{vu \in \delta^+_D(v)} \boldsymbol{x}^P_{vu}$$

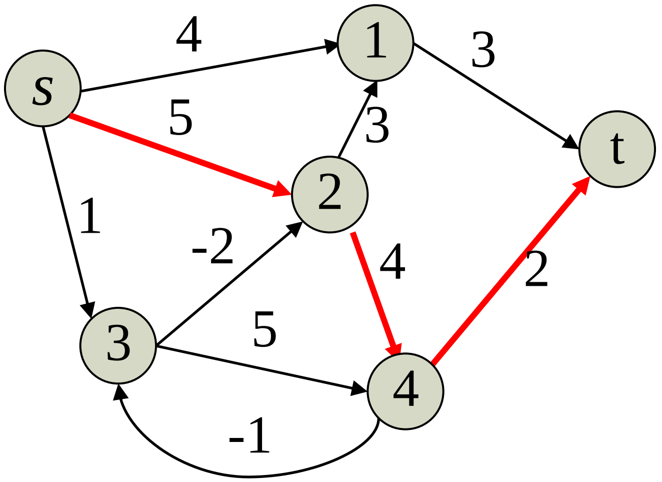
In every $v \notin \{s,t\}$
number of incoming arcs =
= number of outgoing arcs

The s-t path polyhedron

$$\sum_{us \in \delta_D^-(s)} x^P_{us} - \sum_{su \in \delta_D^+(s)} x^P_{su} = -1 \qquad s$$

$$\sum_{ut \in \delta_D^-(t)} x^P_{ut} - \sum_{tu \in \delta_D^+(t)} x^P_{tu} = 1 \qquad t$$

$$\sum_{uv \in \delta_D^-(v)} x^P_{uv} - \sum_{vu \in \delta_D^+(v)} x^P_{vu} = 0 \qquad v \in V\text{-}\{s,t\}$$



$M \in \{-1,0,1\}^{V \times A}$ be the vertex-arc incidence matrix of D

$b = (-1, 1, 0, \ldots, 0)^T$

- The s-t path polyhedron: $Q_{st} = \{x \in \Re^A : \; Mx=b, \; x \geq 0\}$

The *s-t* path polyhedron

- The *s-t* path polyhedron: $Q_{st} = \{x \in \mathfrak{R}^A: \; Mx=b, \; x \geq 0\}$

Theorem

The vertices of Q_{st} are precisely the incidence vectors of the *s-t* paths in D.

- Consider the following LP :

$$\begin{array}{ll} (SP) & \min\ l^T x \\ & Mx=b \\ & x \geq 0 \end{array}$$

- If (*SP*) has an optimal solution, then it has an optimal solution which is the incidence vector of an *s-t* path

Dual to the *s-t* path problem

$$\text{(SP)} \quad \min\ l^T x \quad Mx = b \quad x \geq 0$$

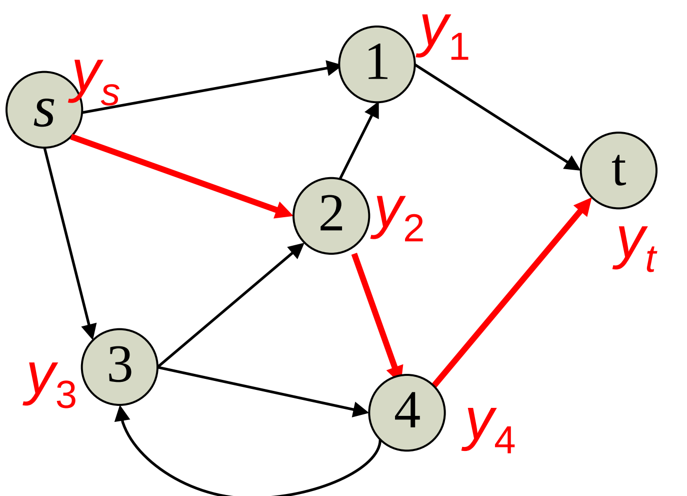
- if D has an s-t path (SP) is non-empty

Assumption

D contains an s-v path for every $v \in V$

- Associate to (SP) its dual problem, by introducing $y \in \Re^V$:

$$\textbf{(DSP)}\ \max\ y_t - y_s \qquad y_v - y_u \leq l_{uv} \ \text{ for all } \ uv \in A$$



Minimum length s-t walk existence

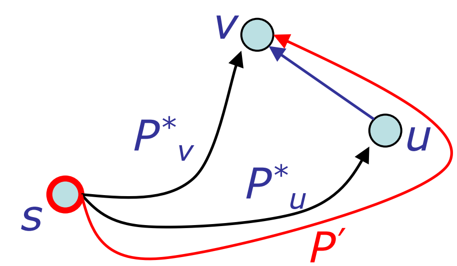
- Since (*SP*) is non-empty, (*SP*) has an optimal solution if and only if it is not unbounded

- (*SP*) is not unbounded if and only if (*DSP*) is non-empty.

> ***Theorem***
>
> (*DSP*) is non-empty iff *D* does not contain a negative length directed cycle.

Proof of existence theorem

- **Proof**: **If part** (D does not contain a negative length dicycle)
- Let P^*_u be a shortest path from s to u in D, $u \in V$.
- Let $y'_u = l(P^*_u)$, for $u \in V$. Then y' is dual feasible. Suppose not.
- Let uv such that $y'_v - y'_u > l_{uv}$



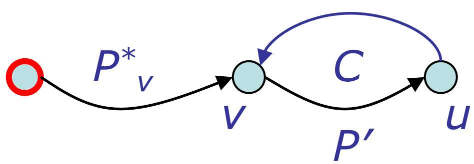
$\Rightarrow$ $l(P^*_v) - l(P^*_u) > l_{uv}$ $\Rightarrow$ $l(P^*_v) > l_{uv} + l(P^*_u)$

- If v does not belong to $P^*_u = (s,\ldots,u)$ $\Rightarrow$

$P' = (s,\ldots,u,uv,v)$ is s-v path with $l(P') = l(P^*_u) + l_{uv} < l(P^*_v)$, *contradiction*

- v belongs to $P^*_u = (s,\ldots,v,\ \ldots,u)$. Let P^*_v s-v subpath, P' u-v subpath

$\Rightarrow$ $C = P' \cup \{uv\}$ *is a cycle*

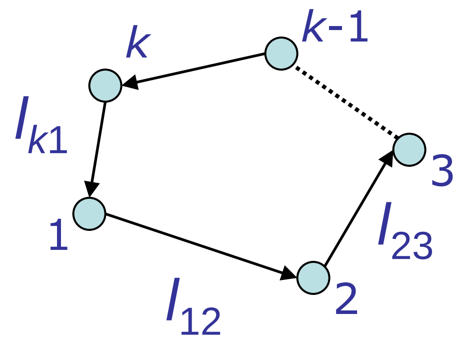


$l(P^*_v) > l(P^*_v) + l(P') + l_{uv}$ $\Rightarrow$ $0 > l(P') + l_{uv} = l(C)$ *C Negative dicycle ! contradiction*

Proof of existence theorem

- **Proof**: **Only-If part** (if y' feasible, no negative dicycles in D)
- Let y'= be a feasible dual solution
- Let $C=(1,(1,2),2,\ldots,k,(k,1),1)$ be a negative length dicycle: $l(C) < 0$

  - y' feasible implies

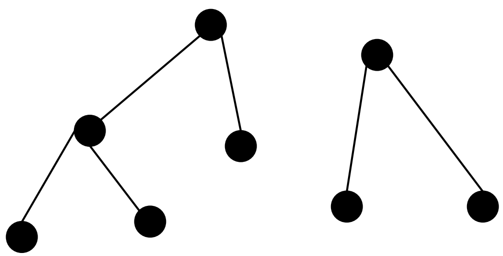


$$y'_2 - y'_1 \leq l_{12}$$
$$y'_3 - y'_2 \leq l_{23}$$
$$\vdots$$
$$y'_1 - y'_k \leq l_{k1} \qquad +$$

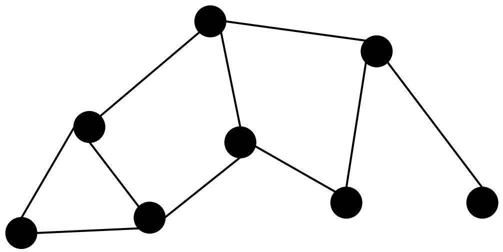
$0 \leq l(C) < 0$ *contradiction!* ■

Trees and spanning trees

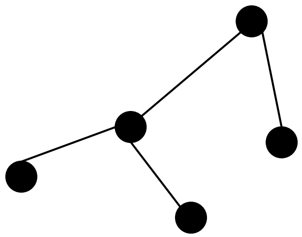
***G* = (*V*,*E*)** <u>undirected graph</u>



- *G* is a *forest* if it does not contain a cycle



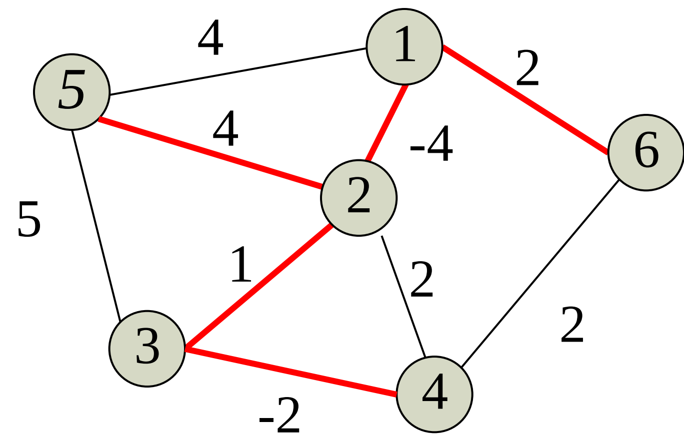
- $u,v \in V$ *connected* if *G* contains an *u-v* path
- *G* *connected* every pair $u,v \in V$ is connected



- *Tree*: *connected* forest

- Every pair of vertices in a tree is connected by a unique path (prove it).

Spanning trees



- $G = (V,E)$ connected undirected graph .
  - A tree $H = (W, T)$ is *spanning* $G = (V,E)$ iff $W = V$ and $T \subseteq E$
- Length (weight) function $l: E \rightarrow R$
- Length of $H = (W, T)$: $l(T) := \sum_{e \in T} l(e)$

Minimum Spanning Tree Problem

Given a connected undirected graph G, and length function l, find a spanning tree in G of minimum length

- When no confusion arises, forests and trees will be represented by sets of edges.

Dijkstra-Prim spanning tree algorithm

- Maintains a tree on a subset of vertices and grows it at each iteration until it becomes spanning

U-cut: $U \subseteq V$ $\delta(U)$: $\{uv \in E: u \in U, v \in V/U\}$

Dijkstra-Prim minimum spanning tree algorithm:

1. *Choose* $v_1 \in V$. *Set* $U_1 = \{v_1\}$. *Set* $T_1 = \{\}$.
2. *While* $U_k \neq V$
3. *Chose* $e_{k+1} \in \delta(U_k)$ with minimum length
4. Reset $T_{k+1} = T_k \cup \{e_{k+1}\}$; *Reset* $U_{k+1} = U_k \cup e_{k+1}$
5. Reset $k = k+1$

EndWhile

Greedy spanning tree algorithm (Kruskal)

- Maintains a forest and grows it at each iteration until it becomes a (spanning) tree

Greedy minimum spanning tree algorithm (Kruskal):

1. Set $T_0 = \{\}$.

2. *For* $k=1,\dots,|V|-1$

3. *Chose* e_k such that:

$T_{k-1} \cup \{e_k\}$ is a forest and $l(e_k)$ is minimum

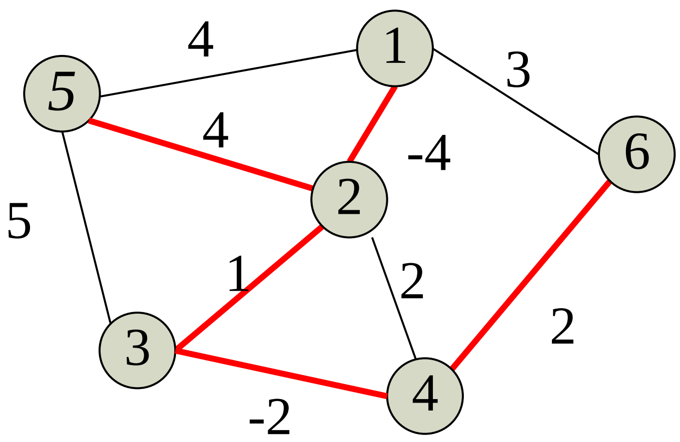
4. Reset $T_k = T_{k-1} \cup \{e_{k+1}\}$;

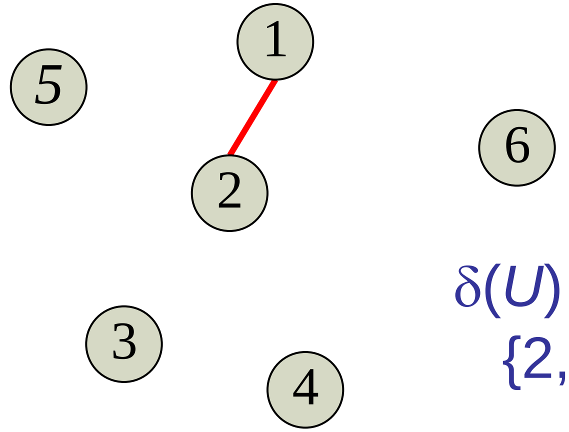
EndFor

- The proof of correctness for both algorithms is based on the properties of the *greedy forests*.

Greedy forests

- A forest F is greedy if there exists a minimum-length spanning tree T of G that contains F.





$U = \{1,2\}$

$\delta(U) = \big\{\{1,5\},\{1,6\},$
$\{2,3\}\ \{2,4\},\ \{2,5\}\big\}$

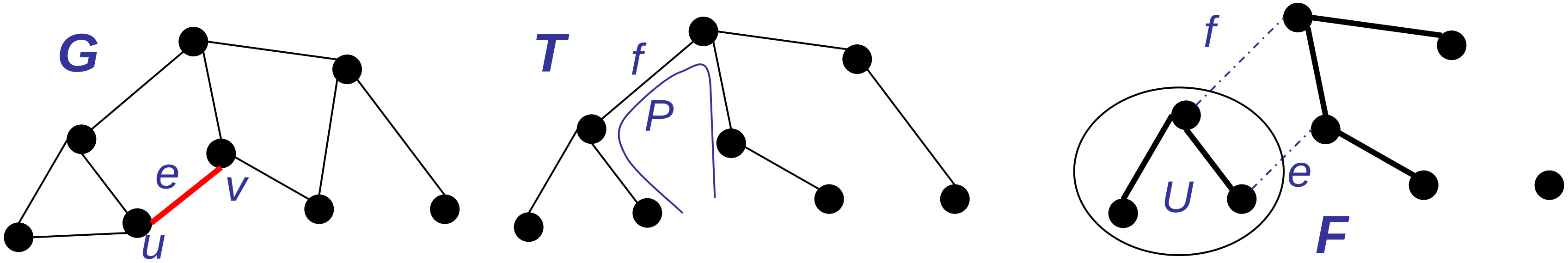
$e = \{2,3\}$

Theorem 1.11

Let F be a greedy forest, let U be one of its components, and let $e \in \delta(U)$. If e has minimum length among all edges in $\delta(U)$, then $F \cup \{e\}$ is again a greedy forest.

Proof of Theorem 1.11

- T minimum-length spanning tree that contains F.



- P unique path between end vertices of $e \in \delta(U)$
- P contains at least an edge $f \in \delta(U)$
- $T' = T \backslash \{f\} \cup \{e\}$ is a spanning tree (prove it).
- $l(e) \leq l(f) \rightarrow l(T') \leq l(T)$ and T' is a minimum length spanning tree
- $F \cup \{e\}$ does not contain cycles (forest)
- $F \cup \{e\} \subseteq T'$ implies $F \cup \{e\}$ greedy forest

Correctness of the Dijkstra-Prim method

Corollary 1.11a

The Dijkstra-Prim method and the Kruskal method yield a spanning tree of minimum length.

- At the first stage of the algorithms T_0 is a greedy forest
- At each subsequent stage k, T_k is a greedy forest.
- After $|V|-1$ steps the algorithms terminate with a spanning tree.

Exercises

- Show that every pair of vertices in a tree is connected by a unique path.

- Show that if $G = (V,T)$ is a tree, then $|T| = |V|-1$.

- Show that if T is a spanning tree of $G = (V,E)$ and $f \in E \backslash T$, then $T \cup f$ contains a unique cycle C (called *fundamental*). Show that if $e \in C$, then $T \backslash \{e\} \cup f$ is a spanning tree.

- Let D be a directed graph, and M be the corresponding vertex-arc incidence matrix. Show that a set of independent columns of M corresponds to the edges of a forest.

- Show formally that the $0,1$ solutions of the s-t path polyhedron are precisely the incidence vectors of the *s-t paths.*