UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in	INF-MAT 5360/9360 — Mathematical optimization
Day of examination:	December 2., 2010
Examination hours:	14.30 – 18.30

This problem set consists of 6 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 12 questions with about the same weight.

Solution.

Problem 1

1a

Let $C \subseteq \mathbb{R}^n$ be a convex set and consider a line $L = \{x \in \mathbb{R}^n : x = a + tr,\ t \in \mathbb{R}\}$ where $a, r \in \mathbb{R}^n$ are given vectors. Is $C \cap L$ a convex set? Depending on your answer, give a proof or a counterexample.

Solution: True. Proof: L is also convex (may be shown directly from the definition of convexity) and the intersection of convex sets is again convex. Thus $C \cap L$ is convex.

1b

Let $a, b \in \mathbb{R}^n$ and consider the set

$$S = \{x \in \mathbb{R}^n : \|x - a\| \leq \|x - b\|\}$$

where $\|z\| = \sqrt{z^T z}$ is the Euclidean norm of a vector z. Show that S is a halfspace. (Hint: work on the inequalities in the definition of S). Give an example in the plane, i.e., when $n = 2$.

Solution: Since the norm is nonnegative, the following is equivalent: (i) $\|x-a\| \leq \|x-b\|$, and (ii) $\|x-a\|^2 \leq \|x-b\|^2$. Moreover, a calculation in (ii) gives: $x^T x - 2a^T x + a^T a \leq x^T x - 2b^T x + b^T b$, or $(b-a)^T x \leq (1/2)(\|b\|^2 - \|a\|^2)$. So S is the halfspace defined by this linear inequality (with normal vector $b - a$). Example for $n = 2$: let $a = (0, 0)$ and $b = (2, 0)$, then S is the halfspace given by $x_1 \leq 1$; all points closer to a than b (or equal distance).

(Continued on page 2.)

Problem 2

2a

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function and let $\alpha \in \mathbb{R}$. Show that the set

$$K = \{x \in \mathbb{R}^n : f(x) \leq \alpha\}$$

is convex.

Solution: Let $x_1, x_2 \in K$ and $\lambda \in [0,1]$. Then $f(x_1), f(x_2) \leq \alpha$ so by convexity of f

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda x_2 f(x_2) \leq (1-\lambda)\alpha + \lambda\alpha = \alpha$$

so $(1-\lambda)x_1 + \lambda x_2 \in K$, and this set is convex.

2b

Let $x \in \mathbb{R}^n$ be a convex combination of the vectors $z_1, z_2, \ldots, z_k \in \mathbb{R}^n$, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function. Show that

$$f(x) \leq \max\{f(z_1), \ldots, f(z_k)\}.$$

(Hint: Jensen's inequality)

Solution: Then $x = \sum_{j=1}^k \lambda_j z_j$ for some $\lambda_j \geq 0$ $(j \leq k)$ and $\sum_{j=1}^k \lambda_j = 1$. By Jensen's inequality

$$f(x) = f(\sum_{j=1}^k \lambda_j z_j) \leq \sum_{j=1}^k \lambda_j f(z_j) \leq \sum_{j=1}^k \lambda_j M = M \sum_{j=1}^k \lambda_j = M$$

where $M = \max\{f(z_1), \ldots, f(z_k)\}$. This proves the inequality.

2c

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron (so A is a real $m \times n$ matrix and $b \in \mathbb{R}^m$). Show that the recession cone of P is given by

$$\mathrm{rec}(P) = \{z \in \mathbb{R}^n : Az \leq O\}$$

where O is the zero vector.

Solution: Let $z \in \mathrm{rec}(P)$ and $x_0 \in P$. Then $x(\lambda) := x_0 + \lambda z \in P$ for each $\lambda \geq 0$. Now $Ax(\lambda) = Ax_0 + \lambda Az$. If $(Az)_i > 0$ for some i, then $(Ax(\lambda))_i > b_i$ when λ is large enough, but this contradicts that $x(\lambda) \in P$. This proves that $Az \leq O$. Conversely: assume $Az \leq O$. Then, for each $x_0 \in P$, $A(x_0 + \lambda z) = Ax_0 + \lambda Az \leq b + O = b$, so $z \in \mathrm{rec}(P)$. Thus, $\mathrm{rec}(P)$ has the desired form.

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2d

Let $C \subset \mathbb{R}^3$ be the unit cube, i.e.,

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_i \leq 1 \ \ (i \leq 3)\}$$

Determine, with reference to general theory, each face F of C such that $\dim(F) = 1$ and F contains the point $(1, 1, 1)$.

Solution: Since C is a polyhedron, each face F is an exposed face, and it is obtained by setting certain inequalities in the defining inequalities (for C) to equality (see Section 4.4 in "An Intro. to Convexity"). Since F has dimension 1, we must set two such inequalities to equality, and since F contains $(1, 1, 1)$ we can only use the inequalities $x_i \leq 1$ for this. The desired faces are therefore $F_1 = \{x \in P : x_2 = 1,\ x_3 = 1\} = \text{conv}\{(0, 1, 1), (1, 1, 1)\}$, $F_2 = \{x \in P : x_1 = 1,\ x_3 = 1\} = \text{conv}\{(1, 0, 1), (1, 1, 1)\}$ and $F_3 = \{x \in P : x_1 = 1,\ x_2 = 1\} = \text{conv}\{(1, 1, 0), (1, 1, 1)\}$.

Problem 3

3a

Give an example of a 2×2 matrix $B = [b_{ij}]$ where (a) $b_{ij} \in \{-1, 0, 1\}$ for $1 \leq i, j \leq 2$ and (b) B is *not* totally unimodular. Also, give a proof of the following fact: the node-edge incidence matrix of a directed graph is totally unimodular. (This is a result in the lecture notes.)

Solution: An example is

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then $\det A = 2$ so A is not TU. The proof: see lecture notes on comb.opt., Proposition 2.12.

Problem 4

Let $F(G) = \{x \in \mathbb{R}^E : Ax \leq b\}$ be the forest polytope associated with the undirected graph $G = (V, E)$ in Figure 1.a). An inequality of type $x_e \geq 0$ $(e \in E)$ is said to be a *trivial inequality*.

4a

Let $\mathcal{A}$ be a separation oracle for $F(G)$ and let $\hat{x} \in \mathbb{R}^E$ be the point indicated in the picture (i.e. $x_{13} = 2/3$, $x_{12} = x_{14} = 1/3$, $x_{23} = 0$, $x_{24} = x_{34} = 1$). If the input to $\mathcal{A}$ is $\hat{x}$, what will it be its output?

Solution: The oracle returns the violated constraint $x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34} \leq 3$, associated with the vertex set $S = \{1, 2, 3, 4\}$ (remark $E(S) = E$).

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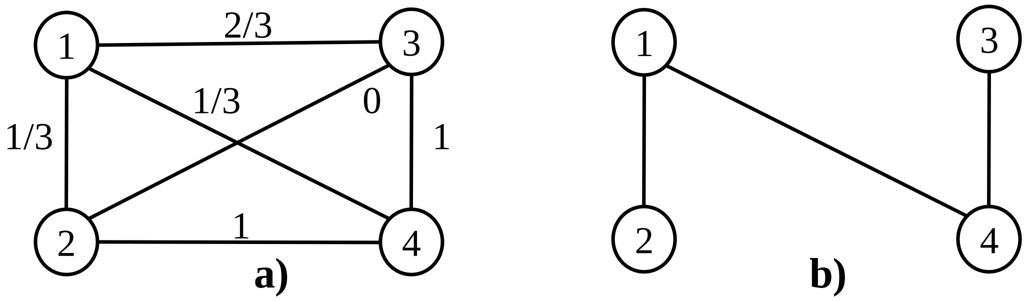

Figure 1:

4b

Consider the spanning tree H of G given in Figure 1.b). The incidence vector of H is $x^H_{12} = x^H_{14} = x^H_{34} = 1$, $x^H_{13} = x^H_{23} = x^H_{24} = 0$. Show that x^H is a vertex of $F(G)$, without using trivial inequalities in your proof.

Solution: x^H satisfies with equality the following $|E|$ inequalities in the definition of P: $x_{12} \leq 1$ $(S = \{1,2\})$, $x_{14} \leq 1$ $(S = \{1,4\})$, $x_{34} \leq 1$ $(S = \{3,4\})$, $x_{12} + x_{14} + x_{24} \leq 2$, $(S = \{1,2,4\})$, $x_{13} + x_{14} + x_{34} \leq 2$ $(S = \{1,3,4\})$, $x_{12} + x_{13} + x_{14} + x_{23} + x_{24} + x_{34} \leq 3$ $(S = \{1,2,3,4\})$.

Problem 5

Let $G = (V, E)$ be an undirected simple graph (no loops, no multiple edges). A *stable set* of vertices is a set $S \subseteq V$ of pairwise non-adjacent vertices of G, i.e. for all $i, j \in S$ we have $[i, j] \notin E$. For example, in Figure 2, the set $S = \{2, 3, 5\}$ is a stable set.

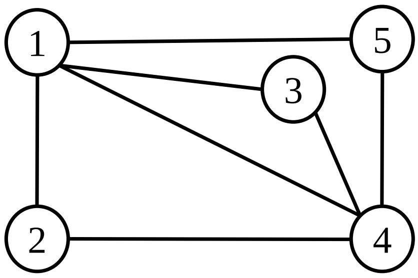

Figure 2:

Let $Q(G) \subseteq \{0,1\}^V$ be the set of incidence vectors of stable sets of G. It is not difficult to see that a $(0,1)$-vector $x \in \mathbb{R}^V$ lies in $Q(G)$ if and only if it satisfies $x_u + x_v \leq 1$ for all $[u, v] \in E$. In other words, the polyhedron

(Continued on page 5.)

$P(G) = \{x \in \mathbb{R}^V : x_u \geq 0$ for all $u \in V,\ x_u + x_v \leq 1$ for all $[u, v] \in E\}$ is a *formulation* of $Q(G)$.

Consider now three distinct and pairwise adjacent vertices v, w, z of G, i.e. $\{[v, w], [w, z], [v, z]\} \subseteq E$.

5a

Show that the inequality $x_v + x_w + x_z \leq 1$ is valid for the convex hull of $Q(G)$.

Solution: The inequality can be obtained as a Gomory cut from the constraints defining $P(G)$. In particular, for every edge $[i, j] \in E$, denote by u_{ij} the multiplier associated with the constraint $x_i + x_j \leq 1$. Then we let $u_{vw} = u_{vz} = u_{wz} = 1/2$, and all other multipliers be equal to 0.

5b

Show that the *clique* inequality $x_v + x_w + x_z \leq 1$ is not valid for $P(G)$.

Solution: Consider the point $\hat{x}_v = 1/2$ $(v \in V)$. It is not difficult to see that $\hat{x} \in P(G)$. But the clique inequality $x_v + x_w + x_z \leq 1$ is violated by $\hat{x}$.

Problem 6

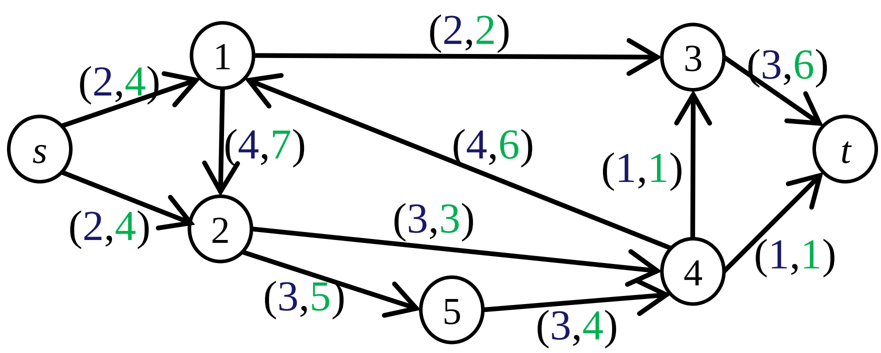

Figure 3:

Consider the graph in Figure 3 where flow x_e and capacity c_e are shown next to each edge e (in this order).

6a

Show that the given flow x is a maximum st-flow.

Solution: We have $val(x^*) = 4$. Consider the st-cut $K = \delta^+(\{s, 1, 2, 4, 5\}) = \{(1, 3), (4, 3,), (4, t)\}$. We have $c(K) = 4$, and $c(K) = val(x^*)$, implying that K is a minimum st-cut and x^* a maximum st-flow.

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Problem 7

(**For the course INF-MAT9360**)

An internet network can be modelled by means of a directed graph $G = (V, E)$, where the set of nodes V correspond to the routers, while a directed edge (u, v) represents a direct link from router u to router v, so that one can send traffic from u to v. Consider two distinct routers s and t.

7a

We want to check if it is possible to send traffic from s to t even if at most k links fail. Find a suitable max-flow problem which solves this question.

Solution: For every $(u, v) \in E$, define the capacity $c(u, v) = 1$. Find a maximum (integer) st-flow x^* with such capacities. If $val(x^*) = q > k$ then the answer is YES, otherwise the answer is NO. First, suppose $q > k$: then we can decompose x^* into flows on st-paths $P^1, \ldots, P^r$, with $r \geq 1$ (flows on cycles can be neglected). Since edge capacities are unitary and x^* is integer, every path carries exactly 1 unit of flow, and no two paths share a common edge: the paths are disjoint. This implies that we have exactly q distinct paths ($r = q$) and we are done as no choice of k edges can "hit" all these paths. If $q \leq k$, the max-flow value is q so the min-cut capacity is also q. But then there is a cut with at most k edges, so if they all fail, one cannot send traffic from s to t.

Good luck!