Exercises for INF3320 and INF4320

Bézier curves and spline curves

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Revision: 30. oktober 2006

1. Show that if $t_0 < t_1 < t_2$, the Vandermonde matrix

$$V = \begin{pmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{pmatrix}$$

is non-singular by showing that its determinant can be written as

$$(t_1 - t_0)(t_2 - t_0)(t_2 - t_1).$$

Solution: *Using the first column to expand the determinant, we get*

$$\det V = (t_1 t_2^2 - t_2 t_1^2) - (t_0 t_2^2 - t_2 t_0^2) + (t_0 t_1^2 - t_1 t_0^2).$$

Meanwhile

$$\begin{aligned}(t_1 - t_0)(t_2 - t_0)(t_2 - t_1) &= (t_1 t_2 - t_1 t_0 - t_0 t_2 + t_0^2)(t_2 - t_1) \\ &= (t_1 t_2^2 + t_2 t_0^2 + t_0 t_1^2) - (t_2 t_1^2 + t_0 t_2^2 + t_1 t_0^2).\end{aligned}$$

2. Recall the cubic Lagrange basis functions

$$L_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{3} \frac{t - t_j}{t_i - t_j}, \qquad i = 0, 1, 2, 3.$$

What is the value of L_i at the points $t = t_0, t_1, t_2, t_3$?

Solution:

$$L_i(t_j) = \begin{cases} 1 & \text{if } j = i; \\ 0 & \text{if } j \neq i. \end{cases}$$

We often express this property using the Kronecker delta, as $L_i(t_j) = \delta_{ij}$.

3. What are the values and first derivatives of the Hermite polynomials H_0, H_1, H_2, H_3, defined in the lecture, at the end points $t = 0$ and $t = 1$?

Solution:

$$\begin{aligned}
&H_0(0) = 1, \quad H_0'(0) = 0, \quad H_0(1) = 0, \quad H_0'(1) = 0, \\
&H_1(0) = 0, \quad H_1'(0) = 1, \quad H_1(1) = 0, \quad H_1'(1) = 0, \\
&H_2(0) = 0, \quad H_2'(0) = 0, \quad H_2(1) = 1, \quad H_2'(1) = 0, \\
&H_3(0) = 0, \quad H_3'(0) = 0, \quad H_3(1) = 0, \quad H_3'(1) = 1.
\end{aligned}$$

4. Let $\mathbf{p}$ and $\mathbf{q}$ be two Bezier curves of degree d,

$$\mathbf{p}(t) = \sum_{i=0}^{d} \mathbf{p}_i B_{i,d}(t), \qquad \mathbf{q}(t) = \sum_{i=0}^{d} \mathbf{q}_i \tilde{B}_{i,d}(t),$$

where

$$B_{i,d}(t) = \binom{d}{i} \lambda^i (1-\lambda)^{d-i}, \qquad \tilde{B}_{i,d}(t) = \binom{d}{i} \tilde{\lambda}^i (1-\tilde{\lambda})^{d-i},$$

and

$$\lambda = \frac{t-a}{b-a}, \qquad \tilde{\lambda} = \frac{t-b}{c-b},$$

and $a < b < c$ and define a spline curve $\mathbf{r} : [a, c] \to \mathbb{R}^n$ by

$$\mathbf{r}(t) = \begin{cases} \mathbf{p}(t) & a \le t \le b, \\ \mathbf{q}(t) & b < t \le c. \end{cases}$$

What are the conditions on the control points for G^1 and G^2 continuity?

Solution: *The first two arc length derivatives of* $\mathbf{r}$ *(its tangent and curvature vectors) are*

$$\dot{\mathbf{r}}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \qquad \ddot{\mathbf{r}}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3}.$$

The first two derivatives of $\mathbf{p}$ *and* $\mathbf{q}$ *at* $t = b$ *are:*

$$\mathbf{p}'(b) = \frac{d}{b-a} \Delta \mathbf{p}_{d-1}, \qquad \mathbf{q}'(b) = \frac{d}{c-b} \Delta \mathbf{q}_0,$$

$$\mathbf{p}''(b) = \frac{d(d-1)}{(b-a)^2} \Delta^2 \mathbf{p}_{d-2}, \qquad \mathbf{q}''(b) = \frac{d(d-1)}{(c-b)^2} \Delta^2 \mathbf{q}_0.$$

Thus $\mathbf{r}$ *is* G^1 *continuous at* $t = b$ *if and only if* $\mathbf{p}(1) = \mathbf{q}(0)$ *and*

$$\frac{\mathbf{p}'(1)}{\|\mathbf{p}'(1)\|} = \frac{\mathbf{q}'(0)}{\|\mathbf{q}'(0)\|},$$

or equivalently $\mathbf{p}_d = \mathbf{q}_0$ *and*

$$\frac{\Delta \mathbf{p}_{d-1}}{\|\Delta \mathbf{p}_{d-1}\|} = \frac{\Delta \mathbf{q}_0}{\|\Delta \mathbf{q}_0\|}.$$

Notice that the interval lengths $b - a$ *and* $c - b$ *cancelled out.*

The curve $\mathbf{r}$ *is* G^2 *continuous at* $t = b$ *if it is* G^1 *at* $t = b$ *and*

$$\frac{\mathbf{p}'(b) \times \mathbf{p}''(b)}{\|\mathbf{p}'(b)\|^3} = \frac{\mathbf{q}'(b) \times \mathbf{q}''(b)}{\|\mathbf{q}'(b)\|^3},$$

and again the interval lengths $b-a$ and $c-b$ cancel out, giving

$$\frac{\Delta \mathbf{p}_{d-1} \times \Delta^2 \mathbf{p}_{d-2}}{\|\Delta \mathbf{p}_{d-1}\|^3} = \frac{\Delta \mathbf{q}_0 \times \Delta^2 \mathbf{q}_0}{\|\Delta \mathbf{q}_0\|^3},$$

or

$$\frac{\Delta \mathbf{p}_{d-2} \times \Delta \mathbf{p}_{d-1}}{\|\Delta \mathbf{p}_{d-1}\|^3} = \frac{\Delta \mathbf{q}_0 \times \Delta \mathbf{q}_1}{\|\Delta \mathbf{q}_0\|^3}.$$

Thus the conditions for G^1 and G^2 are independent of the intervals $[a, b]$ and $[b, c]$. This is to be expected because tangent and curvature vectors are independent of the parameterization of the curve.