Computer Graphics and Geometric Modelling

Bezier curves and spline curves

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Bezier curves and spline curves

- Cubic Lagrange interpolation
- Cubic Hermite interpolation (A-M and H: Section 12.1.3)
- Spline curves (A-M and H: Section 12.1.2)
- Bezier curves on arbitrary intervals (A-M and H: Section 12.1.2)

Cubic Lagrange interpolation

Instead of using four Bezier control points to define a cubic polynomial curve, we might prefer to **interpolate**: define it using four points on the curve. Given four points $\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$, we can choose any distinct parameter values $t_0 < t_1 < t_2 < t_3$ and find the cubic curve $\mathbf{p}$ such that $\mathbf{p}(t_i) = \mathbf{q}_i$, $i = 0, 1, 2, 3$. For example we could take $t_0 = 0$, $t_1 = 1/3$, $t_2 = 2/3$, $t_3 = 1$. If we represent $\mathbf{p}$ in power form as

$$\mathbf{p}(t) = \mathbf{a}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \mathbf{a}_3 t^3,$$

we must solve the linear system

$$\begin{pmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \end{pmatrix} \begin{pmatrix} \mathbf{a}_0 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{pmatrix}.$$

The matrix is called the Vandermonde matrix.

Alternatively one could find $\mathbf{p}$ in Bernstein form

$$\mathbf{p}(t) = \mathbf{p}_0 B_{0,3}(t) + \mathbf{p}_1 B_{1,3}(t) + \mathbf{p}_2 B_{2,3}(t) + \mathbf{p}_3 B_{3,3}(t),$$

in which case we must solve the linear system

$$\begin{pmatrix} B_{0,3}(t_0) & B_{1,3}(t_0) & B_{2,3}(t_0) & B_{3,3}(t_0) \\ B_{0,3}(t_1) & B_{1,3}(t_1) & B_{2,3}(t_1) & B_{3,3}(t_1) \\ B_{0,3}(t_2) & B_{1,3}(t_2) & B_{2,3}(t_2) & B_{3,3}(t_2) \\ B_{0,3}(t_3) & B_{1,3}(t_3) & B_{2,3}(t_3) & B_{3,3}(t_3) \end{pmatrix} \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{q}_0 \\ \mathbf{q}_1 \\ \mathbf{q}_2 \\ \mathbf{q}_3 \end{pmatrix}.$$

The 'ideal' basis for Lagrange interpolation is the Lagrange basis:

$$L_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{3} \frac{t - t_j}{t_i - t_j},$$

because the solution is explicit:

$$\mathbf{p}(t) = \sum_{i=0}^{3} \mathbf{q}_i L_i(t),$$

but this may not be a good basis for further tasks.

Cubic Hermite interpolation

A good alternative to Lagrange interpolation is Hermite interpolation because it makes it easier to piece together several curves. We can define a cubic polynomial curve by its end points and end derivatives. Given points $\mathbf{q}_0$, $\mathbf{q}_1$ and vectors $\mathbf{m}_0$, $\mathbf{m}_1$, we seek the cubic curve $\mathbf{p} : [0, 1] \to \mathbb{R}^n$ such that

$$\mathbf{p}(0) = \mathbf{q}_0, \quad \mathbf{p}'(0) = \mathbf{m}_0, \quad \mathbf{p}(1) = \mathbf{q}_1, \quad \mathbf{p}'(1) = \mathbf{m}_1.$$

Using properties of Bezier curves we find that

$$\begin{aligned}\mathbf{p}(t) &= \mathbf{q}_0 B_{0,3}(t) + (\mathbf{q}_0 + \frac{\mathbf{m}_0}{3})B_{1,3}(t) + (\mathbf{q}_1 - \frac{\mathbf{m}_1}{3})B_{2,3}(t) + \mathbf{q}_1 B_{3,3}(t) \\ &= \mathbf{q}_0 H_0(t) + \mathbf{m}_0 H_1(t) + \mathbf{q}_1 H_2(t) + \mathbf{m}_1 H_3(t),\end{aligned}$$

where H_0, H_1, H_2, H_3 are the Hermite polynomials

$$H_0(t) = B_{0,3}(t) + B_{1,3}(t), \qquad H_1(t) = \frac{1}{3}B_{1,3}(t),$$

$$H_2(t) = B_{2,3}(t) + B_{3,3}(t), \qquad H_3(t) = -\frac{1}{3}B_{2,3}(t).$$

Splines

We can create complicated geometrical shapes by piecing together polynomial curves. A piecewise polynomial curve is called a **spline curve**. There are many ways of representing and generating a spline curve. One way is to piece together several cubic Hermite curves. For example, given three points $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$ and vectors $\mathbf{m}_0$, $\mathbf{m}_1$, $\mathbf{m}_2$ we could find the cubic Hermite curve $\mathbf{p}$ such that

$$\mathbf{p}(0) = \mathbf{q}_0, \quad \mathbf{p}'(0) = \mathbf{m}_0, \quad \mathbf{p}(1) = \mathbf{q}_1, \quad \mathbf{p}'(1) = \mathbf{m}_1.$$

and the cubic Hermite curve $\mathbf{q}$ such that

$$\mathbf{q}(1) = \mathbf{q}_1, \quad \mathbf{q}'(1) = \mathbf{m}_1, \quad \mathbf{q}(2) = \mathbf{q}_2, \quad \mathbf{q}'(2) = \mathbf{m}_2.$$

Then the curve $\mathbf{r} : [0, 2] \to \mathbb{R}^n$, defined as

$$\mathbf{r}(t) = \begin{cases} \mathbf{p}(t) & 0 \leq t \leq 1, \\ \mathbf{q}(t) & 1 \leq t \leq 2, \end{cases}$$

is piecewise cubic and has C^1 continuity at $t = 1$. In short, $\mathbf{r}$ is a C^1 cubic spline curve.

Another way of generating a spline curve (of arbitrary degree) is to piece together several Bezier curves. For example, we could piece together two degree d Bezier curves

$$\mathbf{p}(t) = \sum_{i=0}^{d} \mathbf{p}_i B_{i,d}(t), \qquad \mathbf{q}(t) = \sum_{i=0}^{d} \mathbf{q}_i B_{i,d}(t),$$

to form a spline curve $\mathbf{r} : [0,2] \to \mathbb{R}^n$ of degree d, defined as

$$\mathbf{r}(t) = \begin{cases} \mathbf{p}(t) & 0 \le t \le 1, \\ \mathbf{q}(t-1) & 1 < t \le 2. \end{cases}$$

Continuity conditions

The curve $\mathbf{r}$ has continuity of order k at $t = 1$ if

$$\frac{d^j}{dt^j}\mathbf{p}(t)|_{t=1} = \frac{d^j}{dt^j}\mathbf{q}(t)|_{t=0}, \qquad j = 0, 1, \ldots, k.$$

From the derivative formula in the last lecture we have

$$\frac{d^j}{dt^j}\mathbf{p}(1) = \frac{d!}{(d-j)!}\Delta^j \mathbf{p}_{d-j}, \qquad \frac{d^j}{dt^j}\mathbf{q}(0) = \frac{d!}{(d-j)!}\Delta^j \mathbf{q}_0.$$

So $\mathbf{r}$ has C^k continuity at $t = 1$ if and only if

$$\Delta^j \mathbf{p}_{d-j} = \Delta^j \mathbf{q}_0, \qquad j = 0, 1, \ldots, k.$$

The conditions for C^0, C^1, and C^2 continuity are:

- (C^0) $\quad \mathbf{p}_d = \mathbf{q}_0$,
- (C^1) $\quad C^0$ and $\mathbf{p}_d - \mathbf{p}_{d-1} = \mathbf{q}_1 - \mathbf{q}_0$.
- (C^2) $\quad C^1$ and $\mathbf{p}_d - 2\mathbf{p}_{d-1} + \mathbf{p}_{d-2} = \mathbf{q}_2 - 2\mathbf{q}_1 + \mathbf{q}_0$.

Geometric continuity

For visual smoothness it is sufficient that a curve defined piecewise has G^k continuity, instead of C^k. A curve is said to have G^k continuity if it can be reparameterized so that it has C^k continuity. Equivalently, the curve should be C^k continuous with respect to arc length. The first two arc length derivatives of $\mathbf{r}$ are

$$\dot{\mathbf{r}}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \qquad \ddot{\mathbf{r}}(t) = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{\|\mathbf{r}'(t)\|^3}.$$

Thus $\mathbf{r}$ is G^1 continuous at $t = 1$ if and only if $\mathbf{p}(1) = \mathbf{q}(0)$ and

$$\frac{\mathbf{p}'(1)}{\|\mathbf{p}'(1)\|} = \frac{\mathbf{q}'(0)}{\|\mathbf{q}'(0)\|},$$

or equivalently $\mathbf{p}_d = \mathbf{q}_0$ and

$$\frac{\mathbf{p}_d - \mathbf{p}_{d-1}}{\|\mathbf{p}_d - \mathbf{p}_{d-1}\|} = \frac{\mathbf{q}_1 - \mathbf{q}_0}{\|\mathbf{q}_1 - \mathbf{q}_0\|}.$$

The curve $\mathbf{r}$ is G^2 continuous at $t = 1$ if it is G^1 at $t = 1$ and

$$\frac{\mathbf{p}'(1) \times \mathbf{p}''(1)}{\|\mathbf{p}'(1)\|^3} = \frac{\mathbf{q}'(0) \times \mathbf{q}''(0)}{\|\mathbf{q}'(0)\|^3}.$$

The latter equation is equivalent to

$$\frac{\Delta \mathbf{p}_{d-1} \times \Delta^2 \mathbf{p}_{d-2}}{\|\Delta \mathbf{p}_{d-1}\|^3} = \frac{\Delta \mathbf{q}_0 \times \Delta^2 \mathbf{q}_0}{\|\Delta \mathbf{q}_0\|^3},$$

or

$$\frac{\Delta \mathbf{p}_{d-2} \times \Delta \mathbf{p}_{d-1}}{\|\Delta \mathbf{p}_{d-1}\|^3} = \frac{\Delta \mathbf{q}_0 \times \Delta \mathbf{q}_1}{\|\Delta \mathbf{q}_0\|^3}.$$

If the curve is C^1 at $t = 1$ then $\Delta \mathbf{p}_{d-1} = \Delta \mathbf{q}_0$ and the G^2 condition becomes

$$(\mathbf{q}_2 - \mathbf{p}_{d-2}) \times (\mathbf{q}_1 - \mathbf{p}_{d-1}) = 0.$$

Bezier curves on arbitrary intervals

A Bezier curve can be defined on any interval $[a, b]$ by again letting

$$\mathbf{p}(t) = \sum_{i=0}^{d} \mathbf{p}_i B_{i,d}(t),$$

but now using the more general Bernstein basis functions

$$B_{i,d}(t) = \binom{d}{i} \left(\frac{t-a}{b-a}\right)^i \left(\frac{b-t}{b-a}\right)^{d-i}.$$

This ensures the end conditions $\mathbf{p}(a) = \mathbf{p}_0$, $\mathbf{p}(b) = \mathbf{p}_d$. Letting $\lambda = (t-a)/(b-a)$, the i-th basis function can also be written as

$$B_{i,d}(t) = \binom{d}{i} \lambda^i (1-\lambda)^{d-i}.$$

The curve $\mathbf{p} : [a, b] \to \mathbb{R}^n$ is the same geometrically as the standard Bezier curve with control points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_d$ but the 'speed' of the curve is different and affects derivatives.

Since by the chain rule,

$$\frac{d}{dt} = \frac{d\lambda}{dt}\frac{d}{d\lambda} = \frac{1}{b-a}\frac{d}{d\lambda},$$

we find that

$$B'_{i,d}(t) = \frac{d}{b-a}(B_{i-1,d-1}(t) - B_{i,d-1}(t)),$$

and so

$$\mathbf{p}'(t) = \frac{d}{b-a}\sum_{i=0}^{d-1}(\mathbf{p}_{i+1} - \mathbf{p}_i)B_{i,d-1}(t),$$

and similarly,

$$\mathbf{p}^{(r)}(t) = \frac{1}{(b-a)^r}\frac{d!}{(d-r)!}\sum_{i=0}^{d-r}\Delta^r\mathbf{p}_i B_{i,d-r}(t).$$

Suppose we now piece together two degree d Bezier curves

$$\mathbf{p}(t) = \sum_{i=0}^{d} \mathbf{p}_i B_{i,d}(t), \qquad \mathbf{q}(t) = \sum_{i=0}^{d} \mathbf{q}_i \tilde{B}_{i,d}(t),$$

where

$$B_{i,d}(t) = \binom{d}{i} \lambda^i (1-\lambda)^{d-i}, \qquad \tilde{B}_{i,d}(t) = \binom{d}{i} \tilde{\lambda}^i (1-\tilde{\lambda})^{d-i},$$

and

$$\lambda = \frac{t-a}{b-a}, \qquad \tilde{\lambda} = \frac{t-b}{c-b},$$

giving the spline curve $\mathbf{r} : [a, c] \to \mathbb{R}^n$, defined as

$$\mathbf{r}(t) = \begin{cases} \mathbf{p}(t) & a \leq t \leq b, \\ \mathbf{q}(t) & b < t \leq c. \end{cases}$$

The curve $\mathbf{r}$ has continuity of order k at $t = b$ if

$$\frac{d^j}{dt^j}\mathbf{p}(t)|_{t=b} = \frac{d^j}{dt^j}\mathbf{q}(t)|_{t=b}, \qquad j = 0, 1, \ldots, k,$$

and this condition now becomes

$$\frac{1}{(b-a)^j}\Delta^j \mathbf{p}_{d-j} = \frac{1}{(c-b)^j}\Delta^j \mathbf{q}_0, \qquad j = 0, 1, \ldots, k.$$

The conditions for C^0, C^1, and C^2 continuity are:

- (C^0) $\quad \mathbf{p}_d = \mathbf{q}_0$,
- (C^1) $\quad C^0$ and $\frac{\mathbf{p}_d - \mathbf{p}_{d-1}}{b-a} = \frac{\mathbf{q}_1 - \mathbf{q}_0}{c-b}$.
- (C^2) $\quad C^1$ and $\frac{\mathbf{p}_d - 2\mathbf{p}_{d-1} + \mathbf{p}_{d-2}}{(b-a)^2} = \frac{\mathbf{q}_2 - 2\mathbf{q}_1 + \mathbf{q}_0}{(c-b)^2}$.