Triangulating polygons
Lecture B of INF3320, Autumn 2006

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1 One way to triangulate a polygon

Let P be a simple planar polygon with n vertices $v_0, v_1, \ldots, v_{n-1}$, as for example in Figure 1.

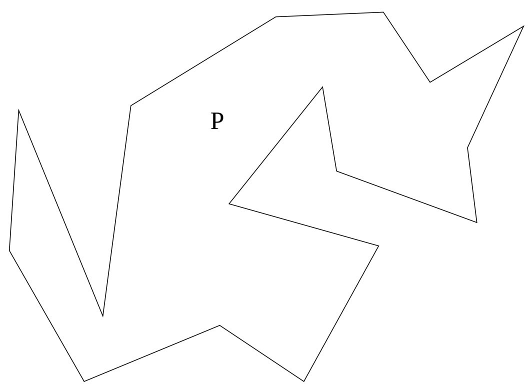

Figure 1: A planar polygon P.

One way to triangulate P is to check first if $n = 3$ in which case P is a triangle and is trivial to triangulate. Otherwise $n \geq 4$ and we look for an 'ear'. An ear is a triangle $E = [v_{i-1}, v_i, v_{i+1}]$ that lies entirely inside the polygon P. Equivalently, E is an ear if the line segment $[v_{i-1}, v_{i+1}]$ lies entirely inside P; see Figure 2. Once we have located an ear, we can form a reduced polygon P' with $n-1$ vertices by removing the vertex v_i from P, i.e., we replace the two edges $[v_{i-1}, v_i]$ and $[v_i, v_{i+1}]$ by the new edge $[v_{i-1}, v_{i+1}]$ to form P'. We triangulate P' and add to it the triangle E.

This triangulation method is simple and in fact always works although it is not very fast. However, it relies on the fact that there is always at least

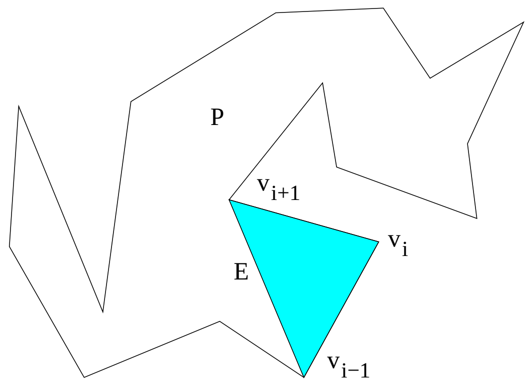

Figure 2: An ear E.

one ear in a planar polygon (with at least four vertices). We will prove this and at the same time prove a stronger statement: the Two Ears Theorem:

Theorem 1 *Every polygon P with $n \geq 4$ vertices has at least two non-overlapping ears.*

Proof. We can prove this by induction on n. If $n = 4$, P is a quadrilateral which is either convex or concave. In either case, we can easily locate two non-overlapping ears; see Figure 3.

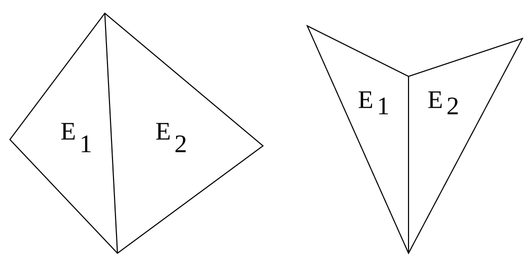

Figure 3: Convex and concave quads.

Otherwise, $n \geq 5$. Let v_i be a vertex whose interior angle is less than $\pi/2$. Consider two cases.

Case 1. The triangle $E = [v_{i-1}, v_i, v_{i+1}]$ is an ear of P; see Figure 4. Then if v_i is removed from P the reduced polygon P' has $n-1$ vertices and therefore at least four vertices. Therefore, by the induction step we can assume that P' has at least two non-overlapping ears E_1 and E_2. Since they are non-overlapping, at least one of them, say E_1, does not contain the edge $[v_{i-1}, v_{i+1}]$

of P'. Then E_1 is also an ear of P and E and E_1 are two non-overlapping ears of P.

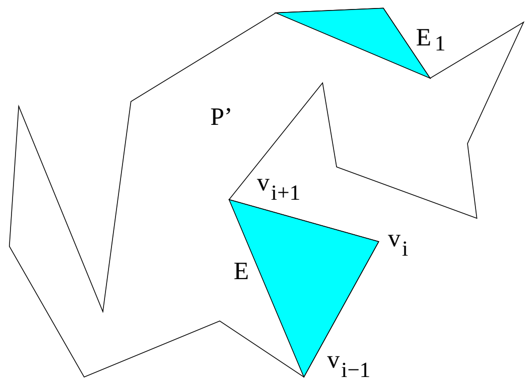

Figure 4: Case 1.

Case 2. The triangle $E = [v_{i-1}, v_i, v_{i+1}]$ is not an ear of P; see Figure 5. Then E contains at least one other vertex of P. Let w be a vertex of P such that the line through w, parallel to the line through v_{i-1} and v_{i+1}, is as close as possible to v_i. Then the line segment $[w, v_i]$ lies entirely inside P and therefore divides P into two polygons: P_1 containing the vertices $v_i, v_{i+1}, \ldots, w$ and P_2 containing the vertices $v_i, v_{i-1}, \ldots, w$. Now we consider two subcases.

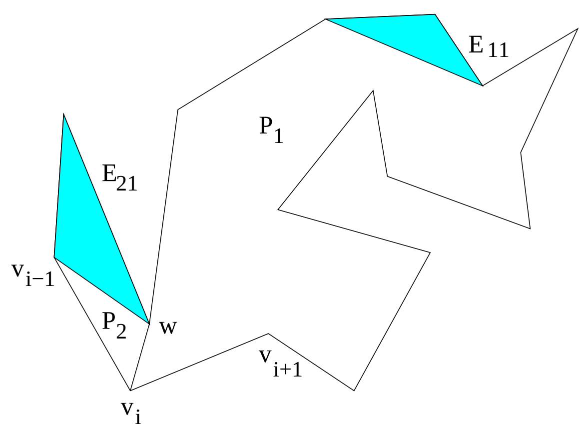

Figure 5: Case 2.

Case 2a. One of P_1 and P_2, say P_1, is a triangle. Then P_2 is not a triangle because $n \geq 5$ and so by induction P_2 has two non-overlapping ears E_1 and

E_2. Since they are non-overlapping, at least one of them, say E_1, does not contain the edge $[w, v_i]$ of P_2. Then E_1 is also an ear of P and so E_1 and the triangle P_1 are two non-overlapping ears of P.
Case 2b. Neither P_1 nor P_2 is a triangle. Then by induction, P_1 has two non-overlapping ears E_{11} and E_{12}, and at least one of them, say E_{11}, does not contain the edge $[w, v_i]$ of P_1. Similarly, by induction, P_2 has two non-overlapping ears E_{21} and E_{22}, and at least one of them, say E_{21}, does not contain the edge $[w, v_i]$ of P_2. Then E_{11} and E_{21} are two non-overlapping ears of P. □

This theorem is very nice, especially when we notice that for any number of vertices n there are polygons which have only two ears, as Figure 6 shows. So we cannot hope to find three or more ears in general.

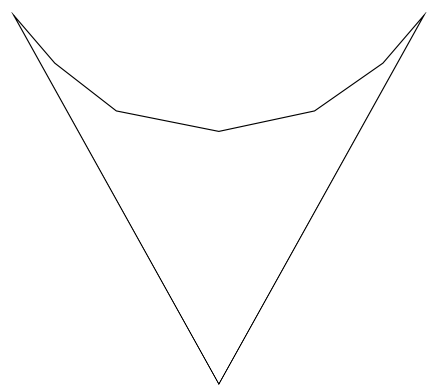

Figure 6: Polygon with only two ears.

2 Another way to triangulate a polygon

After going through the proof of the two ears theorem, we notice that there is another way of triangulating a polygon. Instead of locating an ear, it is enough to find a vertex v_i whose interior angle is less than $\pi/2$. Now if $E = [v_{i-1}, v_i, v_{i+1}]$ is an ear then as in the previous method, we remove v_i from P to form P', triangulate P' and add the triangle E. But if E is not an ear then we locate a vertex w as in Case 2 of the proof and we split P into P_1 and P_2 by the edge $[w, v_i]$. We then triangulate both P_1 and P_2 and combine them.

3 Another proof of the two ears theorem

Another way of proving the two ears theorem is shorter but relies on other known facts. It is known that any polygon P can be triangulated, for example by the algorithm in the last section. Figure 7 shows a possible triangulation of the polygon P of Figure 1. So let T be some triangulation of P and

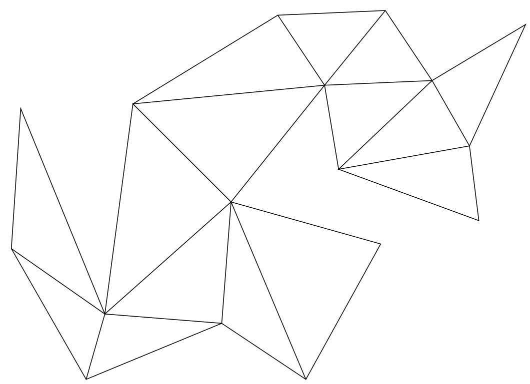

Figure 7: One possible triangulation T.

consider its dual graph G; see Figure 8. Each node in G corresponds to a

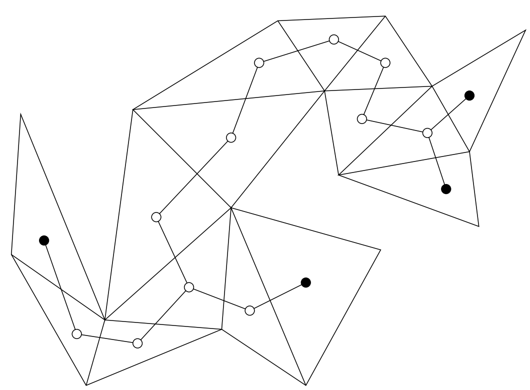

Figure 8: The dual graph G.

triangle in T and each edge in G corresponds to a pair of triangles in T that share a common edge in T. Then G has no cycles: no closed paths of edges because the triangulation T has no interior vertices. So the dual graph G is a so-called tree. The nodes of G that have only one incident edge (the black ones in the figure) are called leaves. Notice that any triangle in T that corresponds to a leaf of G is an ear of P. Hence the number of ears of P is greater or equal to the number of leaves of G. So it remains to show that G has at least two leaves. Well if $n \geq 4$, G has at least two nodes and therefore, being a tree, must have at least two leaves.

4 Triangulations of a convex polygon

Most polygons can be triangulated in many different ways. To get an idea of how many ways, consider convex polygons. There is only one possible triangulation of a triangle, there are two triangulations of a convex quadrilateral, and five triangulations of a convex pentagon; see Figure 9.

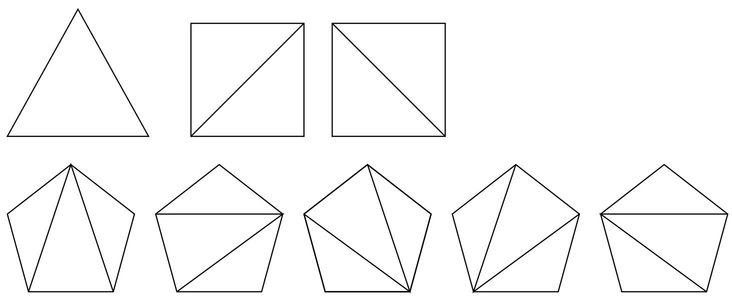

Figure 9: All possible triangulations of first three convex polygons.

It can be shown that if C_n denotes the number of ways of triangulating a convex polygon with $n+2$ vertices then C_n is the so-called Catalan number,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The first few numbers in the sequence are $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$. The sequence was first studied by Euler but is named after Eugene Charles Catalan, who discovered a connection to parenthesized expressions. The number C_n grows exponentially with n. More precisely,

$$C_n \approx \frac{4^n}{n^{3/2}\sqrt{\pi}}.$$