Notes on 4-point interpolatory subdivision

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Abstract

These notes provide an introduction to the construction of functions through subdivision, focusing on 4-point interpolatory subdivision. The material is based on the papers [1, 2, 3, 4].

1 Introduction

Given a sequence of values f_i, for $i = 0, 1, 2, \ldots, n$, we want to find an interpolant, i.e., a function $f : [0, n] \to \mathbb{R}$ such that $f(i) = f_i$, for $i = 0, 1, \ldots, n$, with good smoothness and approximation properties. One way of doing this is to use interpolatory subdivision. One of the earliest and best known examples of interpolatory subdivision is the four-point scheme.

Suppose we extend the data values a little so that we start with f_i for $i = -2, -1, 0, \ldots, n, n+1, n+2$. We will compute an interpolant f as the limit of polygons. We start by setting $f_i^0 = f_i$ and the first and coarsest polygon is the continuous function $f^0 : [-2, n+2] \to \mathbb{R}$ which is linear on each interval $[i, i+1]$, $-2 \le i \le n+1$, and such that $f^0(i) = f_i^0$, $-2 \le i \le n+2$. In short, f^0 is the piecewise linear interpolant to the data (i, f_i^0), $-2 \le i \le n+2$. We then generate a sequence $f^1, f^2, \ldots,$ of finer and finer polygons through a four point rule. We choose some real value w. Then for each $k = 0, 1, 2, \ldots,$ we set

$$f_{2i}^{k+1} = f_i^k \quad -1 \le i \le 2^k n + 1, \tag{1}$$

$$f_{2i+1}^{k+1} = -w f_{i-1}^k + \left(\frac{1}{2} + w\right) f_i^k + \left(\frac{1}{2} + w\right) f_{i+1}^k - w f_{i+2}^k, -1 \le i \le 2^k n, \tag{2}$$

and let $f^k : [-2^{1-k}, n + 2^{1-k}] \to \mathbb{R}$ be the piecewise linear interpolant to the data $(2^{-k} i, f_i^k)$, $-2 \le i \le 2^k n + 2$. Figure 1 shows the first four polygons for an example data set, with $w = 1/16$.

Exercise 1 *Show that for all w, the scheme has linear precision, in the sense that if $f_i = p_1(i)$ for some linear polynomial p_1, then $f_i^k = p_1(2^{-k} i)$ for all i and k, i.e., the scheme reproduces linear polynomials p_1.*

Exercise 2 *Show that if $w = 1/16$ then the scheme has cubic precision, i.e., reproduces polynomials of degree* 3.

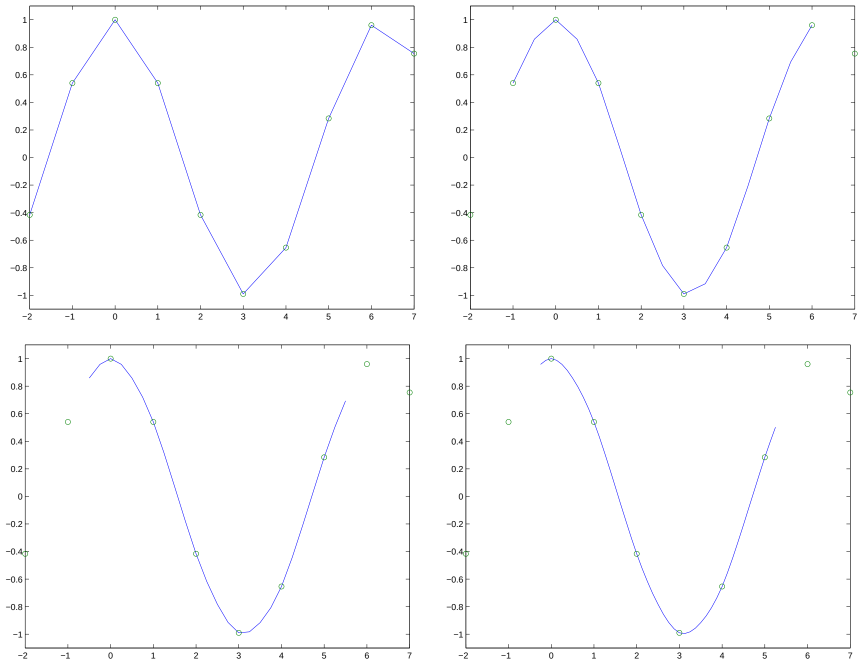

Figure 1: Scheme with $w = 1/16$. Top row: $k = 0, 1$, bottom row: $k = 2, 3$.

2 Smoothness

We hope that for some values of w, the sequence of polygons $(f^k)_k$ has a limit function over the interval $[0, n]$ as $k \to \infty$, and that the limit function has derivatives. Figure 2 shows limit functions with the values $w = 1/32$ and $w = 1/6$.

Theorem 1 *For* $|w| < 1/4$*, the sequence* $(f^k)_k$ *has a limit*

$$f(x) = \lim_{k\to\infty} f^k(x), \qquad 0 \le x \le n,$$

which is continuous in $[0, n]$.

In order to prove the theorem we will use a well known result from analysis that says that a sufficient condition for such convergence is that $(f^k)_k$ is a Cauchy sequence in the max norm

$$\|f\| := \max_{0\le x\le n} |f(x)|.$$

Thus we need to show that for any $\epsilon > 0$ there is some N such that for all $i, j \ge N$,

$$\|f^i - f^j\| \le \epsilon. \tag{3}$$

To this end we will use the following lemma.

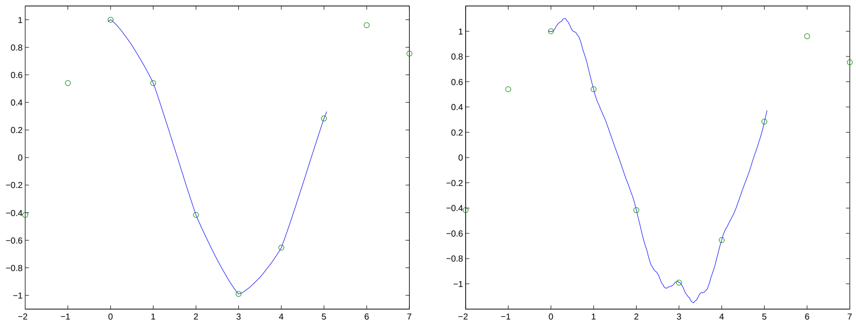

Figure 2: Limit function with (left) $w = 1/32$ and (right) $w = 1/6$.

Lemma 1 *If there are positive constants C and $\lambda < 1$ such that*

$$\|f^{k+1} - f^k\| \leq C\lambda^k, \qquad k = 0, 1, 2, \ldots, \tag{4}$$

then $(f^k)_k$ is a Cauchy sequence.

Proof. Observe that under condition (4), if $i > j$,

$$\begin{aligned}
\|f^i - f^j\| &\leq \|f^i - f^{i-1}\| + \|f^{i-1} - f^{i-2}\| + \cdots + \|f^{j+1} - f^j\| \\
&\leq C\lambda^{i-1} + C\lambda^{i-2} + \cdots + C\lambda^j \\
&= C\lambda^j(\lambda^{i-1-j} + \lambda^{i-2-j} + \cdots + \lambda + 1) \\
&\leq C\lambda^j(1 + \lambda + \lambda^2 + \cdots) \\
&= C\lambda^j/(1-\lambda) \leq C\lambda^N/(1-\lambda).
\end{aligned}$$

Thus (3) holds if we take N large enough that $C\lambda^N/(1-\lambda) \leq \epsilon$. □

We now prove Theorem 1:

Proof. Observe that the maximum value of $|f^{k+1}(x) - f^k(x)|$ is attained at a midpoint on level k, i.e., at a point of the form $x = 2^{-(k+1)}(2i+1)$, and at this point,

$$\begin{aligned}
f^{k+1}(x) - f^k(x) &= f^{k+1}_{2i+1} - (f^k_i + f^k_{i+1})/2 \\
&= -wf^k_{i-1} + wf^k_i + wf^k_{i+1} - wf^k_{i+2}, \\
&= w(f^k_i - f^k_{i-1}) - w(f^k_{i+2} - f^k_{i+1}),
\end{aligned}$$

and so

$$|f^{k+1}(x) - f^k(x)| \leq 2|w| \max\{|f^k_i - f^k_{i-1}|, |f^k_{i+2} - f^k_{i+1}|\},$$

and therefore

$$\|f^{k+1} - f^k\| \leq C_1 \max_i |f^k_{i+1} - f^k_i|,$$

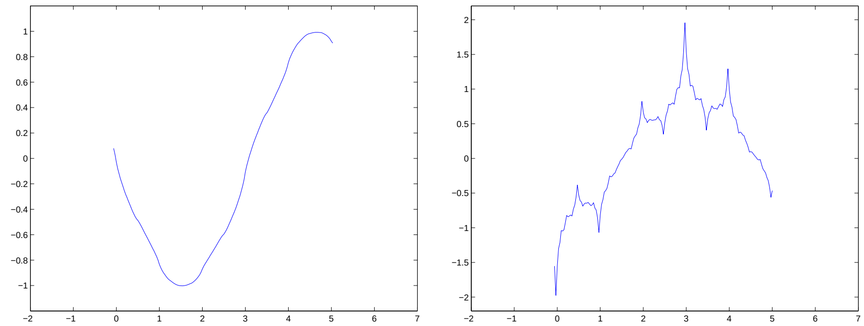

Figure 3: First and 'second' derivative (level $k = 5$) with $w = 1/16$.

where $C_1 = 2|w|$. Thus if we can show that there are constants C_2 and $\lambda < 1$ such that

$$\max_i |f_{i+1}^k - f_i^k| \leq C_2 \lambda^k, \qquad k = 0, 1, 2, \ldots,$$

then we can apply Lemma 1 with $C = C_1 C_2$. To this end observe that

$$f_{2i+1}^{k+1} - f_{2i}^{k+1} = w(f_i^k - f_{i-1}^k) + (f_{i+1}^k - f_i^k)/2 - w(f_{i+2}^k - f_{i+1}^k), \tag{5}$$

$$f_{2i+2}^{k+1} - f_{2i+1}^{k+1} = -w(f_i^k - f_{i-1}^k) + (f_{i+1}^k - f_i^k)/2 + w(f_{i+2}^k - f_{i+1}^k). \tag{6}$$

It follows that

$$\max_i |f_{i+1}^{k+1} - f_i^{k+1}| \leq (2|w| + 1/2) \max_i |f_{i+1}^k - f_i^k|,$$

and therefore that (2) holds with

$$\lambda = 2|w| + 1/2, \qquad C_2 = \max_i |f_{i+1}^0 - f_i^0|.$$

Thus the sequence $(f^k)_k$ is Cauchy if $\lambda < 1$ which is equivalent to the condition that $|w| < 1/4$. □

We next consider the smoothness of the limit function. Figure 3 shows the numerical first derivative (a plot of first order divided differences) at level $k = 5$ and the 'second derivative' (second order divided differences) at the same level.

Theorem 2 *For $|w| < 1/4$, let f be the limit function of Theorem 1. If $0 < w < 1/8$ then $f \in C^1[0, n]$.*

Proof. Define the first order divided difference

$$d_i^k := 2^k (f_{i+1}^k - f_i^k),$$

and let $d^k : [-2^{1-k}, n + 2^{-k}] \to \mathbb{R}$ be the piecewise linear interpolant to the data $(2^{-k} i, d_i^k)$, $-2 \leq i \leq 2^k n + 1$. We will show

(i) that the sequence of polygons $(d^k)_k$ has a continuous limit

$$d(x) = \lim_{k\to\infty} d^k(x), \qquad 0 \le x \le n,$$

and

(ii) that

$$f(x) = f(0) + \int_0^x d(y)\,dy, \qquad x \in [0, n], \tag{7}$$

which implies that f is differentiable with $f' = d$.

Starting with (i), we will show that $(d^k)_k$ is a Cauchy sequence. Notice that the maximum difference $|d^{k+1}(x) - d^k(x)|$ occurs at a point x of the form $2^{-(k+1)}(2i)$ or $2^{-(k+1)}(2i+1)$. Thus

$$\|d^{k+1} - d^k\| \le \max\{|d_{2i}^{k+1} - d_i^k|, |d_{2i+1}^{k+1} - (d_i^k + d_{i+1}^k)/2|\}. \tag{8}$$

Notice that from (5) there is a subdivision scheme for the d_i^k:

$$d_{2i}^{k+1} = 2wd_{i-1}^k + d_i^k - 2wd_{i+1}^k, \tag{9}$$

$$d_{2i+1}^{k+1} = -2wd_{i-1}^k + d_i^k + 2wd_i^k. \tag{10}$$

Substituting these into (8) it immediately follows that there is some constant (depending only on w) such that

$$\|d^{k+1} - d^k\| \le C \max_i |d_{i+1}^k - d_i^k|.$$

Thus it remains to bound the differences on the right. Taking differences of the d_i^k gives

$$d_{2i+1}^{k+1} - d_{2i}^{k+1} = 4w(d_i^k - d_{i-1}^k) + 4w(d_{i+1}^k - d_i^k), \tag{11}$$

$$d_{2i+2}^{k+1} - d_{2i+1}^{k+1} = -2w(d_i^k - d_{i-1}^k) + (1-4w)(d_{i+1}^k - d_i^k) - 2w(d_{i+2}^k - d_{i+1}^k). \tag{12}$$

It follows that

$$\max_i |d_{i+1}^{k+1} - d_i^{k+1}| \le \max_i |d_{i+1}^k - d_i^k|$$

for $0 \le w \le 1/8$. But this only shows that

$$\max_i |d_{i+1}^k - d_i^k| \le C, \qquad k = 0, 1, 2, \ldots,$$

for these w, which is not good enough to show that $(d^k)_k$ is a Cauchy sequence. Yet by applying (11) twice, a longer calculation shows that for $0 < w < 1/8$, there is some $\beta < 1$ such that

$$\max_i |d_{i+1}^{k+1} - d_i^{k+1}| \le \beta \max_i |d_{i+1}^{k-1} - d_i^{k-1}|.$$

Thus for these w,

$$\max_i |d_{i+1}^k - d_i^k| \le K\lambda^k, \qquad k = 0, 1, 2, \ldots,$$

with $\lambda = \sqrt{\beta} < 1$ and this shows that $(d^k)_k$ is a Cauchy sequence.

Considering (ii), since both sides of (7) are are continuous in x, it is sufficient to show that the equation holds for all dyadic points, i.e., points of the form

$$x = p + \sum_{j=1}^{m} a_j 2^{-j},$$

where $p \in \{0, 1, \ldots, n-1\}$ and $a_1, \ldots, a_m \in \{0, 1\}$. Then, for any $k \ge m$, we can express x as $x = 2^{-k} q_k$, where

$$q_k = 2^k p + \sum_{j=1}^{m} a_j 2^{k-j}.$$

So by the trapezoidal rule,

$$\int_0^x d^k(y)\, dy = 2^{-k}\left(\frac{d_0^k}{2} + \sum_{i=1}^{q_k - 1} d_i^k + \frac{d_{q_k}^k}{2}\right) = \frac{f_{q_k}^k + f_{q_k+1}^k}{2} - \frac{f_0^k + f_1^k}{2}.$$

We now let $k \to \infty$ in this equation. The left hand side converges to $\int_0^x d(y)\, dy$ and since $f_0^k = f(0)$ and $f_{q_k}^k = f(x)$, and since $f_1^k \to f(0)$ and $f_{q_k+1}^k \to f(x)$, the right hand side converges to $f(x) - f(0)$ and this establishes (7). □

It has been shown that no value of w gives a limit function that is twice differentiable for general data values f_i. However, with $w = 1/16$, the limit function is 'close' to C^2 in the following sense. First we need to define what we mean by Hölder continuity. Recall that a function $g : [a, b] \to \mathbb{R}$ is said to be Hölder continuous with exponent α, where $0 < \alpha < 1$, if there is a constant $C > 0$ such that

$$\frac{|g(y) - g(x)|}{|y - x|^\alpha} \le C, \qquad \text{for } a \le x < y \le b,$$

in which case we write $f \in C^\alpha[a, b]$. Hölder continuity in the limiting case $\alpha = 1$ is the same as Lipschitz continuity. We also write $f \in C^{k+\alpha}[a, b]$ for $k = 1, 2, \ldots$ and $\alpha \in (0, 1)$ if $f^{(k)} \in C^\alpha[a, b]$.

Exercise 3 *Show that the function* $g(x) = x^{1/2}$ *is Hölder continuous in* $[0, 1]$ *with any exponent* α *satisfying* $0 < \alpha \le 1/2$.

It can be shown that when $w = 1/16$, the derivative of the limit function f of the scheme (1) is Hölder continuous in $[0, n]$ for all exponents α, $0 < \alpha < 1$. We can express this by writing $f' \in C^{1-\epsilon}[0, n]$ for any small $\epsilon > 0$, and so $f \in C^{2-\epsilon}[0, n]$.

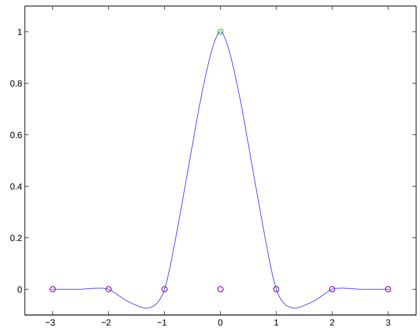

Figure 4: Basis function with $w = 1/16$.

3 Approximation order

We now establish the rate of approximation of the four-point scheme in the case $w = 1/16$. When the data is $f_i = \delta_{ij}$ for some j we call the limit function B_j. Figure 4 shows the function B_0. For some integer n let $h = 1/n$ and suppose we sample a function g at the points ih, $i = -2, \ldots, n+2$, giving the data $f_i^h = g(ih)$. Let $f^h : [0,1] \to \mathbb{R}$ denote the limit function of the subdivision scheme adapted to the grid spacing h, so that $f^h(ih) = g(ih)$, and let $\|f\| := \max_{0 \le x \le 1} |f(x)|$.

Theorem 3 *If $g \in C^4[-2h, 1+2h]$ and $w = 1/16$ there is a constant $C > 0$ such that*

$$\|g - f^h\| \le Ch^4 \|g^{(4)}\|.$$

Proof. Let B_j^h denote the limit function of the scheme with spacing h of the data $f_i^h = \delta_{ij}$. Then by the linearity of the scheme,

$$f^h(x) = \sum_{i=-2}^{n+2} f_i^h B_i^h(x). \tag{13}$$

Note also that $\|B_i^h\| = \|B_i^1\| = \|B_0^1\| = C_1$ for some constant $C_1 > 0$. Suppose $x \in [jh, (j+1)h]$. Then since the support of B_i^h is contained in $[(i-2)j, (i+3)h]$, the global sum (13) becomes a local one,

$$f^h(x) = \sum_{i=j-2}^{j+3} f_i^h B_i^h(x). \tag{14}$$

We want next to replace f_i^h in this sum by its Taylor expansion

$$f_i^h = g(ih) = g(x) + \sum_{r=1}^{3} (ih - x)^r g^{(r)}(x)/r! + (ih - x)^4 g^{(4)}(c_i)/4!, \tag{15}$$

where c_i is some point between x and ih. Notice that since the scheme with $w = 1/16$ has cubic precision,

$$\sum_{i=j-2}^{j+3} (ih - y)^r B_i^h(x) = (y - x)^r, \qquad r = 1, 2, 3,$$

for any y and in particular

$$\sum_{i=j-2}^{j+3} (ih - x)^r B_i^h(x) = 0, \qquad r = 1, 2, 3.$$

Thus when we substitute (15) into (14) we end up with

$$f^h(x) = g(x) + \sum_{i=j-2}^{j+3} (ih - x)^4 g^{(4)}(c_j) B_i^h(x)/4!.$$

Since $|ih - x| \leq 3h$ we deduce that

$$|f^h(x) - g(x)| \leq 6(3h)^4 \|g^{(4)}\| C_1/4!.$$

□

Note that the above method of proof can be applied to many interpolation and approximation problems. In general, if an approximation method:

(i) is linear

(ii) has bounded basis functions of local support, and

(iii) has polynomial precision of degree d,

then the approximation order is $O(h^{d+1})$ provided the function being approximated has a bounded $(d+1)$-st derivative.

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